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Outline

Static and dynamic properties

Example: Static properties

Mutual exclusion

Safety properties are often static.

They can be reduced to reachability.

Example: Dynamic properties

Every elevator request should be eventually granted.

The elevator should not cross a level for which a call is pending without stopping.

Temporal Structures

Definition: Flows of time

A *flow of time* is a strict order $(\mathbb{T}, <)$ where $\mathbb{T}$ is the nonempty set of *time points* and $<$ is an irreflexive transitive relation on $\mathbb{T}$.

Example: Flows of time

- $(\{0, \ldots, n\}, <)$: Finite runs of sequential systems.
- $(\mathbb{N}, <)$: Infinite runs of sequential systems.
- Trees: Finite or infinite run-trees of sequential systems.
- Mazurkiewicz traces: runs of distributed systems (partial orders).
- and also $(\mathbb{Z}, <)$ or $(\mathbb{Q}, <)$ or $(\mathbb{R}, <)$, $(\omega^2, <)$, ...

Definition: Temporal Structures

Let AP be a set of atoms (atomic propositions).

A *temporal structure* over a class $\mathcal{C}$ of time flows and AP is a triple $(\mathbb{T}, <, h)$ where $(\mathbb{T}, <)$ is a time flow in $\mathcal{C}$ and $h : \mathrm{AP} \to 2^{\mathbb{T}}$ is an assignment.

If $p \in \mathrm{AP}$ then $h(p) \subseteq \mathbb{T}$ gives the time points where p holds.

Linear behaviors and specifications

Let $M = (S, T, I, \mathrm{AP}, \ell)$ be a Kripke structure.

Definition: Runs as temporal structures

An infinite run $\sigma = s_0 \to s_1 \to s_2 \to \cdots$ with $s_i \to s_{i+1} \in T$ of M defines a *linear* temporal structure $\ell(\sigma) = (\mathbb{N}, <, h)$ where $h(p) = \{i \in \mathbb{N} \mid p \in \ell(s_i)\}$.

Such a temporal structure can be seen as an infinite word over $\Sigma = 2^{\mathrm{AP}}$: $\ell(\sigma) = \ell(s_0)\ell(s_1)\ell(s_2)\cdots = (\mathbb{N}, <, w)$ with $w(i) = \ell(s_i) \in \Sigma$.

Linear specifications only depend on runs.

Example: The printer manager is fair.

On each run, whenever some process requests the printer, it eventually gets it.

Remark:

Two Kripke structures having the same linear temporal structures satisfy the same linear specifications.

Branching behaviors and specifications

Let $M = (S, T, I, \mathrm{AP}, \ell)$ be a Kripke structure.

Definition: Run-trees as temporal strucutres

Run-tree = unfolding of the transition system.

Let D be a finite set with $|D|$ the outdegree of the transition relation T.

Unordered tree $t : D^* \to \Sigma$ (partial map).

Associated temporal structure $(\mathrm{dom}(t), <, h)$ where $<$ is the strict prefix relation over D^* and $h(p) = \{i \in \mathrm{dom}(t) \mid p \in t(i)\}$.

Example: Each process has the possibility to print first.

First-order Specifications

Definition: Syntax of $\mathrm{FO}(<)$

Let $P, Q, \ldots$ be unary predicates twinned with atoms $p, q, \ldots$ in AP.
Let $\mathrm{Var} = \{x, y, \ldots\}$ be first-order variables.

$$\varphi ::= \bot \mid P(x) \mid x = y \mid x < y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x\, \varphi$$

Definition: Semantics of $\mathrm{FO}(<)$

Let $w = (\mathbb{T}, <, h)$ be a temporal structure.
Predicates $P, Q, \ldots$ twinned with $p, q, \ldots$ are interpreded as $h(p), h(q), \ldots$
Let $\nu : \mathrm{Var} \to \mathbb{T}$ be an assignment of first-order variables.

$$\begin{array}{lll} w, \nu \models P(x) & \text{if} & \nu(x) \in h(p) \\ w, \nu \models x = y & \text{if} & \nu(x) = \nu(y) \\ w, \nu \models x < y & \text{if} & \nu(x) < \nu(y) \\ w, \nu \models \exists x\, \varphi & \text{if} & w, \nu[x \mapsto t] \models \varphi \text{ for some } t \in \mathbb{T} \end{array}$$

where $\nu[x \mapsto t]$ maps x to t and $y \neq x$ to $\nu(y)$.

Previous specifications can be written in $\mathrm{FO}(<)$.

First-order vs Temporal

First-order logic

- $\mathrm{FO}(<)$ has a good expressive power.
  ... but $\mathrm{FO}(<)$-formulae are not easy to write and to understand.
- $\mathrm{FO}(<)$ is decidable.
  ... but satisfiability and model checking are non elementary.

Temporal logics

- no variables: time is implicit.
- quantifications and variables are replaced by modalities.
- Usual specifications are easy to write and read.
- Good complexity for satisfiability and model checking problems.
- Good expressive power.

Linear Temporal Logic (LTL) over $(\mathbb{N}, <)$ introduced by Pnueli (1977) as a convenient specification language for verification of systems.

Temporal Specifications

Definition: Syntax of $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$

$\varphi ::= \bot \mid p\ (p \in \mathrm{AP}) \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{F}\,\varphi \mid \mathsf{P}\,\varphi \mid \mathsf{G}\,\varphi \mid \mathsf{H}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi \mid \varphi\,\mathsf{S}\,\varphi \mid \mathsf{X}\,\varphi \mid \mathsf{Y}\,\varphi$

Definition: Semantics: $w = (\mathbb{T}, <, h)$ temporal structure and $i \in \mathbb{T}$

$w, i \models p$	if	$i \in h(p)$
$w, i \models \mathsf{F}\,\varphi$	if	$\exists j\ i < j$ and $w, j \models \varphi$
$w, i \models \mathsf{G}\,\varphi$	if	$\forall j\ i < j \to w, j \models \varphi$
$w, i \models \varphi\,\mathsf{U}\,\psi$	if	$\exists k\ i < k$ and $w, k \models \psi$ and $\forall j\ (i < j < k) \to w, j \models \varphi$
$w, i \models \mathsf{X}\,\varphi$	if	$\exists j\ i < j$ and $w, j \models \varphi$ and $\neg\exists k\ (i < k < j)$
$w, i \models \mathsf{P}\,\varphi$	if	$\exists j\ i > j$ and $w, j \models \varphi$
$w, i \models \mathsf{H}\,\varphi$	if	$\forall j\ i > j \to w, j \models \varphi$
$w, i \models \varphi\,\mathsf{S}\,\psi$	if	$\exists k\ i > k$ and $w, k \models \psi$ and $\forall j\ (i > j > k) \to w, j \models \varphi$
$w, i \models \mathsf{Y}\,\varphi$	if	$\exists j\ i > j$ and $w, j \models \varphi$ and $\neg\exists k\ (i > k > j)$

Previous specifications can be written in $\mathrm{TL}(\mathrm{AP})$.

Temporal Specifications

Relations between modalities

$$\begin{aligned} \mathsf{F}\,\varphi &= \top\,\mathsf{U}\,\varphi \\ \mathsf{G}\,\varphi &= \neg\,\mathsf{F}\,\neg\varphi \\ \mathsf{X}\,\varphi &= \bot\,\mathsf{U}\,\varphi \end{aligned}$$

Definition: Derived modalities

$$\varphi\,\mathsf{W}\,\psi \stackrel{\text{def}}{=} (\mathsf{G}\,\varphi) \vee (\varphi\,\mathsf{U}\,\psi) \qquad \text{Weak Until}$$

$$\varphi\,\mathsf{R}\,\psi \stackrel{\text{def}}{=} (\mathsf{G}\,\psi) \vee (\psi\,\mathsf{U}\,(\varphi \wedge \psi)) \qquad \text{Release}$$

Definition: non-strict versions of modalities

$$\begin{aligned} \mathsf{F}'\,\varphi &\stackrel{\text{def}}{=} \varphi \vee \mathsf{F}\,\varphi \\ \mathsf{G}'\,\varphi &\stackrel{\text{def}}{=} \varphi \wedge \mathsf{G}\,\varphi \\ \varphi\,\mathsf{U}'\,\psi &\stackrel{\text{def}}{=} \psi \vee (\varphi \wedge \varphi\,\mathsf{U}\,\psi) \\ \varphi\,\mathsf{R}'\,\psi &\stackrel{\text{def}}{=} \psi \wedge (\varphi \vee \varphi\,\mathsf{R}\,\psi) \end{aligned}$$

Temporal Specifications

Example: Specifications on the time flow $(\mathbb{N}, <)$

- Safety: $\mathsf{G}'\,\mathrm{good}$
- MutEx: $\neg\,\mathsf{F}'(\mathrm{crit}_1 \wedge \mathrm{crit}_2)$
- Liveness: $\mathsf{G}\,\mathsf{F}\,\mathrm{active}$
- Response: $\mathsf{G}'(\mathrm{request} \to \mathsf{F}\,\mathrm{grant})$
- Response': $\mathsf{G}'(\mathrm{request} \to (\neg\mathrm{request}\,\mathsf{U}\,\mathrm{grant}))$
- Release: $\mathrm{reset}\,\mathsf{R}\,\mathrm{alarm}$
- Strong fairness: $\mathsf{G}\,\mathsf{F}\,\mathrm{request} \to \mathsf{G}\,\mathsf{F}\,\mathrm{grant}$
- Weak fairness: $\mathsf{F}\,\mathsf{G}\,\mathrm{request} \to \mathsf{G}\,\mathsf{F}\,\mathrm{grant}$

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Temporal Specifications

Proposition: For discrete linear time flows $(\mathbb{T}, <)$

$$\begin{aligned}
\mathsf{F}\,\varphi &= \mathsf{X}\,\mathsf{F}'\,\varphi \\
\mathsf{G}\,\varphi &= \mathsf{X}\,\mathsf{G}'\,\varphi \\
\varphi\,\mathsf{U}\,\psi &= \mathsf{X}(\varphi\,\mathsf{U}'\,\psi) \\
\neg\,\mathsf{X}\,\varphi &= \mathsf{X}\,\neg\varphi \vee \neg\,\mathsf{X}\,\top \\
\neg(\varphi\,\mathsf{U}\,\psi) &= (\mathsf{G}\,\neg\psi) \vee (\neg\psi\,\mathsf{U}\,(\neg\varphi \wedge \neg\psi)) \\
&= \neg\psi\,\mathsf{W}\,(\neg\varphi \wedge \neg\psi) \\
&= \neg\varphi\,\mathsf{R}\,\neg\psi
\end{aligned}$$

Definition: discrete linear time flows

A linear time flow $(\mathbb{T}, <)$ is discrete if $\mathsf{F}\,\top \to \mathsf{X}\,\top$ and $\mathsf{P}\,\top \to \mathsf{Y}\,\top$ are valid formulae.

$(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are discrete.
$(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are not discrete.

Expressivity

Definition: Equivalence

Let $\mathcal{C}$ be a class of time flows.

Two formulae $\varphi, \psi \in \mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ are equivalent over $\mathcal{C}$ if for all temporal structures $w = (\mathbb{T}, <, h)$ over $\mathcal{C}$ and all time points $t \in \mathbb{T}$ we have

$$w, t \models \varphi \quad \text{iff} \quad w, t \models \psi$$

Two formulae $\varphi \in \mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ and $\psi(x) \in \mathrm{FO}_{\mathrm{AP}}(<)$ are equivalent over $\mathcal{C}$ if for all temporal structures $w = (\mathbb{T}, <, h)$ over $\mathcal{C}$ and all time points $t \in \mathbb{T}$ we have

$$w, t \models \varphi \quad \text{iff} \quad w \models \psi(t)$$

Remark: $\mathrm{LTL}(\mathrm{AP}, \mathsf{S}, \mathsf{U}) \subseteq \mathrm{FO}_{\mathrm{AP}}(<)$

$\forall \varphi \in \mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$, $\exists \psi(x) \in \mathrm{FO}_{\mathrm{AP}}(<)$ such that φ and $\psi(x)$ are equivalent.

Expressivity

Theorem: Expressive completeness [6, Kamp 68]

For complete linear time flows,

$$\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U}) = \mathrm{FO}_{\mathrm{AP}}(<)$$

Definition: complete linear time flows

A linear time flow $(\mathbb{T}, <)$ is complete if every nonempty and bounded subset of $\mathbb{T}$ has a least upper bound and a greatest lower bound.

$(\mathbb{N}, <)$, $(\mathbb{Z}, <)$ and $(\mathbb{R}, <)$ are complete.
$(\mathbb{Q}, <)$ and $(\mathbb{R} \setminus \{0\}, <)$ are not complete.

Remark:

Elegant algebraic proof of $\mathrm{TL}(\mathrm{AP}, \mathsf{U}) = \mathrm{FO}_{\mathrm{AP}}(<)$ over $(\mathbb{N}, <)$ due to Wilke 98.

Stavi connectives: Time flows with gaps

Definition: Stavi Until: $\overline{\mathsf{U}}$

Let $w = (\mathbb{T}, <, h)$ be a temporal structure and $i \in \mathbb{T}$. Then, $w, i \models \varphi \,\overline{\mathsf{U}}\, \psi$ if

$$\begin{aligned}
&\exists k\ i < k \\
&\ \wedge \exists j\,(i < j < k \wedge w, j \models \neg\varphi) \\
&\ \wedge \exists j\,(i < j < k \wedge \forall \ell\,(i < \ell < j \to w, \ell \models \varphi) \\
&\ \wedge \forall j \left[i < j < k \to \begin{bmatrix} \exists k'\,[j < k' \wedge \forall j'\,(i < j' < k' \to w, j' \models \varphi)] \\ \vee\,[\forall \ell\,(j < \ell < k \to w, \ell \models \psi) \wedge \exists \ell\,(i < \ell < j \wedge w, \ell \models \neg\varphi)] \end{bmatrix} \right]
\end{aligned}$$

Similar definition for the Stavi Since $\overline{\mathsf{S}}$.

Theorem: [9, Gabbay, Hodkinson, Reynolds]

$\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U}, \overline{\mathsf{S}}, \overline{\mathsf{U}})$ is expressively complete for $\mathrm{FO}_{\mathrm{AP}}(<)$ over the class of all linear time flows.

Exercise: Isolated gaps

Show that $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ is $\mathrm{FO}_{\mathrm{AP}}(<)$-complete over the time flow $(\mathbb{R} \setminus \mathbb{Z}, <)$.

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Separation

Definition:

Let $w = (\mathbb{T}, <, h)$ and $w' = (\mathbb{T}, <, h')$ be temporal structures over the same time flow, and let $t \in \mathbb{T}$ be a time point.

- w, w' agree on t if $h(t) = h'(t)$
- w, w' agree on the past of t if $h(s) = h'(s)$ for all $s < t$
- w, w' agree on the future of t if $h(s) = h'(s)$ for all $s > t$

Definition: Pure formulae

Let $\mathcal{C}$ be a class of time flows. A formula φ over some logic $\mathcal{L}$ is pure past (resp. pure present, pure future) over $\mathcal{C}$ if for all temporal structures $w = (\mathbb{T}, <, h)$ and $w' = (\mathbb{T}, <, h')$ over $\mathcal{C}$ and all time points $t \in \mathbb{T}$ such that w, w' agree on the past of t (resp. on t, on the future of t) we have

$$w, t \models \varphi \quad \text{iff} \quad w', t \models \varphi$$

Separation

Definition: Separation

A logic $\mathcal{L}$ is separable over a class $\mathcal{C}$ of time flows if each formula $\varphi \in \mathcal{L}$ is equivalent to some (finite) boolean combination of pure formulae.

Theorem: [7, Gabbay, Pnueli, Shelah & Stavi 80]

$\mathrm{TL}(\mathrm{AP}, \mathrm{S}, \mathrm{U})$ is separable over discrete and complete linear orders.

- $(\mathbb{N}, <)$ is the unique (up to isomorphism) discrete and complete linear order with a first point and no last point.
- $(\mathbb{Z}, <)$ is the unique (up to isomorphism) discrete and complete linear order with no first point and no last point.
- Any discrete and complete linear order is isomorphic to a sub-flow of $(\mathbb{Z}, <)$.

Theorem: Gabbay, Reynolds, see [10]

$\mathrm{TL}(\mathrm{AP}, \mathrm{S}, \mathrm{U})$ is separable over $(\mathbb{R}, <)$.

Separation and Expressivity

Theorem: [8, Gabbay 89] (already stated by Gabbay in 81)

Let $\mathcal{C}$ be a class of linear time flows.

Let $\mathcal{L}$ be a temporal logic able to express F and P.

Then, $\mathcal{L}$ is separable over $\mathcal{C}$ iff it is expressively complete over $\mathcal{C}$.

Initial equivalence

Definition: Initial Equivalence

Let $\mathcal{C}$ be a class of time flows having a minimum (denoted 0).
Two formulae $\varphi, \psi \in \mathrm{TL}(\mathrm{AP}, \mathrm{S}, \mathrm{U})$ are initially equivalent over $\mathcal{C}$ if for all temporal structures $w = (\mathbb{T}, <, h)$ over $\mathcal{C}$ we have

$$w, 0 \models \varphi \quad \text{iff} \quad w, 0 \models \psi$$

Two formulae $\varphi \in \mathrm{TL}(\mathrm{AP}, \mathrm{S}, \mathrm{U})$ and $\psi(x) \in \mathrm{FO}_{\mathrm{AP}}(<)$ are initially equivalent over $\mathcal{C}$ if for all temporal structures $w = (\mathbb{T}, <, h)$ over $\mathcal{C}$ we have

$$w, 0 \models \varphi \quad \text{iff} \quad w \models \psi(0)$$

Corollary: of the separation theorem

For each $\varphi \in \mathrm{TL}(\mathrm{AP}, \mathrm{S}, \mathrm{U})$ there exists $\psi \in \mathrm{TL}(\mathrm{AP}, \mathrm{U})$ such that φ and ψ are initially equivalent over $(\mathbb{N}, <)$.

Initial equivalence

Example: $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ versus $\mathrm{TL}(\mathrm{AP}, \mathsf{U})$

$$\mathsf{G}'(\mathrm{grant} \to (\neg\mathrm{grant}\ \mathsf{S}\ \mathrm{request}))$$

is initially equivalent to

$$(\mathrm{request}\ \mathsf{R}'\ \neg\mathrm{grant}) \wedge \mathsf{G}(\mathrm{grant} \to (\mathrm{request} \vee (\mathrm{request}\ \mathsf{R}\ \neg\mathrm{grant})))$$

Theorem: (Laroussinie & Markey & Schnoebelen 2002)

$\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ may be exponentially more succinct than $\mathrm{TL}(\mathrm{AP}, \mathsf{U})$ over $(\mathbb{N}, <)$.

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Temporal depth

Definition: Temporal depth of $\varphi \in \mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$

$$\begin{aligned}
\mathrm{td}(p) &= 0 && \text{if } p \in \mathrm{AP}\\
\mathrm{td}(\neg\varphi) &= \mathrm{td}(\varphi)\\
\mathrm{td}(\varphi \vee \psi) &= \max(\mathrm{td}(\varphi), \mathrm{td}(\psi))\\
\mathrm{td}(\varphi\ \mathsf{S}\ \psi) &= \max(\mathrm{td}(\varphi), \mathrm{td}(\psi)) + 1\\
\mathrm{td}(\varphi\ \mathsf{U}\ \psi) &= \max(\mathrm{td}(\varphi), \mathrm{td}(\psi)) + 1
\end{aligned}$$

Lemma:

Let $B \subseteq \mathrm{AP}$ be finite and $k \in \mathbb{N}$.
There are (up to equivalence) finitely many formulae in $\mathrm{TL}(B, \mathsf{S}, \mathsf{U})$ of temporal depth at most k.

k-equivalence

Definition:

Let $w_0 = (\mathbb{T}_0, <, h_0)$ and $w_1 = (\mathbb{T}_1, <, h_1)$ be two temporal structures.
Let $i_0 \in \mathbb{T}_0$ and $i_1 \in \mathbb{T}_1$. Let $k \in \mathbb{N}$.

We say that (w_0, i_0) and (w_1, i_1) are k-equivalent, denoted $(w_0, i_0) \equiv_k (w_1, i_1)$, if they satisfy the same formulae in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ of temporal depth at most k.

Lemma: $\equiv_k$ is an equivalence relation of finite index.

Example:

Let $a = \{p\}$ and $b = \{q\}$. Let $w_0 = babaababaa$ and $w_1 = baababaaba$.

$$(w_0, 3) \equiv_0 (w_1, 4)$$
$$(w_0, 3) \equiv_1 (w_1, 4) \ ?$$
$$(w_0, 3) \equiv_1 (w_1, 6) \ ?$$

Here, $\mathbb{T}_0 = \mathbb{T}_1 = \{0, 1, 2, \ldots, 9\}$.

EF-games for $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$

The EF-game has two players: Spoiler (Player I) and Duplicator (Player II).

The game board consists of 2 temporal structures:
$w_0 = (\mathbb{T}_0, <, h_0)$ and $w_1 = (\mathbb{T}_1, <, h_1)$.

There are two tokens, one on each structure: $i_0 \in \mathbb{T}_0$ and $i_1 \in \mathbb{T}_1$.

A configuration is a tuple (w_0, i_0, w_1, i_1)
or simply (i_0, i_1) if the game board is understood.

Let $k \in \mathbb{N}$.

The k-round EF-game from a configuration proceeds with (at most) k moves.

There are 2 available moves for $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$: Until or Since (see below).

Spoiler chooses which move is played in each round.

Spoiler wins if

- Either duplicator cannot answer during a move (see below).
- Or a configuration such that $(w_0, i_0) \not\equiv_0 (w_1, i_1)$ is reached.

Otherwise, duplicator wins.

Until and Since moves

Definition: (Strict) Until move

- Spoiler chooses $\varepsilon \in \{0, 1\}$ and $k_\varepsilon \in \mathbb{T}_\varepsilon$ such that $i_\varepsilon < k_\varepsilon$.
- Duplicator chooses $k_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ such that $i_{1-\varepsilon} < k_{1-\varepsilon}$.
  Spoiler wins if there is no such $k_{1-\varepsilon}$.
  Either spoiler chooses (k_0, k_1) as next configuration of the EF-game,
  or the move continues as follows
- Spoiler chooses $j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ with $i_{1-\varepsilon} < j_{1-\varepsilon} < k_{1-\varepsilon}$.
- Duplicator chooses $j_\varepsilon \in \mathbb{T}_\varepsilon$ with $i_\varepsilon < j_\varepsilon < k_\varepsilon$.
  Spoiler wins if there is no such j_ε.
  The next configuration is (j_0, j_1).

Similar definition for the (strict) Since move.

Winning strategy

Definition: Winning strategy

Duplicator has a winning strategy in the k-round EF-game starting from (w_0, i_0, w_1, i_1) if he can win all plays starting from this configuration.
This is denoted by $(w_0, i_0) \sim_k (w_1, i_1)$.

Spoiler has a winning strategy in the k-round EF-game starting from (w_0, i_0, w_1, i_1) if she can win all plays starting from this configuration.

Example:

Let $a = \{p\}$, $b = \{q\}$, $c = \{r\}$. Let $w_0 = aaabbc$ and $w_1 = aababc$.

$$(w_0, 0) \sim_1 (w_1, 0)$$
$$(w_0, 0) \not\sim_2 (w_1, 0)$$

Here, $\mathbb{T}_0 = \mathbb{T}_1 = \{0, 1, 2, \ldots, 5\}$.

EF-games for $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$

Lemma: Determinacy

The k-round EF-game for $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ is determined:
For each initial configuration, either spoiler or duplicator has a winning strategy.

Theorem: Soundness and completeness of EF-games

For all $k \in \mathbb{N}$ and all configurations (w_0, i_0, w_1, i_1), we have

$$(w_0, i_0) \sim_k (w_1, i_1) \text{ iff } (w_0, i_0) \equiv_k (w_1, i_1)$$

Example:

Let $a = \{p\}$, $b = \{q\}$, $c = \{r\}$.
Then, $aaabbc, 0 \models p \,\mathsf{U}\,(q \,\mathsf{U}\, r)$ but $aababc, 0 \not\models p \,\mathsf{U}\,(q \,\mathsf{U}\, r)$.
Hence, $p \,\mathsf{U}\,(q \,\mathsf{U}\, r)$ cannot be expressed with a formula of temporal depth at most 1.

Exercise:

On finite linear time flows, "even length" cannot be expressed in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$.

Moves for Future and Past modalities

Definition: (Strict) Future move

- Spoiler chooses $\varepsilon \in \{0, 1\}$ and $j_\varepsilon \in \mathbb{T}_\varepsilon$ such that $i_\varepsilon < j_\varepsilon$.
- Duplicator chooses $j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ such that $i_{1-\varepsilon} < j_{1-\varepsilon}$.
  Spoiler wins if there is no such $j_{1-\varepsilon}$.
  The new configuration is (j_0, j_1).

Similar definition for (strict) Past move.

Example:

$p \,\mathsf{U}\, q$ is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{P}, \mathsf{F})$ over linear flows of time.
Let $a = \emptyset$, $b = \{p\}$ and $c = \{q\}$.
Let $w_0 = (abc)^n a (abc)^n$ and $w_1 = (abc)^n (abc)^n$.
If $n > k$ then, starting from $(w_0, 3n, w_1, 3n)$, duplicator has a winning strategy in the k-round EF-game using Future and Past moves.

Moves for Next and Yesterday modalities

Notation: $i \lessdot j \stackrel{\text{def}}{=} i < j \wedge \neg\exists k\,(i < k < j)$.

Definition: Next move

- Spoiler chooses $\varepsilon \in \{0, 1\}$ and $j_\varepsilon \in \mathbb{T}_\varepsilon$ such that $i_\varepsilon \lessdot j_\varepsilon$.
- Duplicator chooses $j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ such that $i_{1-\varepsilon} \lessdot j_{1-\varepsilon}$.
  Spoiler wins if there is no such $j_{1-\varepsilon}$.
  The new configuration is (j_0, j_1).

Similar definition for Yesterday move.

Exercise:

Show that $p \,\mathsf{U}\, q$ is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{Y}, \mathsf{P}, \mathsf{X}, \mathsf{F})$ over linear flows of time.

Non-strict Until and Since moves

Definition: non-strict Until move

- Spoiler chooses $\varepsilon \in \{0, 1\}$ and $k_\varepsilon \in \mathbb{T}_\varepsilon$ such that $i_\varepsilon \le k_\varepsilon$.
- Duplicator chooses $k_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ such that $i_{1-\varepsilon} \le k_{1-\varepsilon}$.
  Either spoiler chooses (k_0, k_1) as new configuration of the EF-game,
  or the move continues as follows
- Spoiler chooses $j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ with $i_{1-\varepsilon} \le j_{1-\varepsilon} < k_{1-\varepsilon}$.
- Duplicator chooses $j_\varepsilon \in \mathbb{T}_\varepsilon$ with $i_\varepsilon \le j_\varepsilon < k_\varepsilon$.
  Spoiler wins if there is no such j_ε.
  The new configuration is (j_0, j_1).

- If duplicator chooses $k_{1-\varepsilon} = i_{1-\varepsilon}$ then the new configuration must be (k_0, k_1).
- If spoiler chooses $k_\varepsilon = i_\varepsilon$ then duplicator must choose $k_{1-\varepsilon} = i_{1-\varepsilon}$, otherwise he loses.

Similar definition for the non-strict Since move.

Exercise:

1. Show that strict until is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}', \mathsf{U}')$ over $(\mathbb{R}, <)$.
2. Show that strict until is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}', \mathsf{U}')$ over $(\mathbb{N}, <)$.