1 CTL and CTL*

Fix AP = \{p, q, r\}. The goal is to see whether the CTL* formula

\[ \varphi_1 = E((p U q) U r) \]

can be expressed in CTL. Consider the following CTL formulæ:

\[ \varphi_2 = E((p \lor q) U r) \]

\[ \varphi_3 = E((p \lor q) U (r \land E(p U q))) \]

Let \( \psi_1, \psi_2 \) be two CTL* state formulæ. Recall that \( \psi_1 \) implies \( \psi_2 \) (resp. \( \psi_1 \) and \( \psi_2 \) are equivalent) if for all models \( M \) and all states \( s \) of \( M \), we have \( M, s \models \psi_1 \) implies \( M, s \models \psi_2 \) (resp. \( M, s \models \psi_1 \) if and only if \( M, s \models \psi_2 \)).

[4] a) Show that \( \varphi_1 \) implies \( \varphi_2 \), but \( \varphi_1 \) and \( \varphi_2 \) are not equivalent.

Show that \( \varphi_3 \) implies \( \varphi_1 \), but \( \varphi_1 \) and \( \varphi_3 \) are not equivalent.

[4] b) Prove that \( \varphi_1 \) can be expressed in CTL, i.e., give a CTL formula \( \varphi_4 \) and show that \( \varphi_1 \) and \( \varphi_4 \) are equivalent.
2 LTL and Büchi transducers

The flow of time is \((\mathbb{N}, <)\), \(\text{AP} \neq \emptyset\) is the set of atomic propositions and \(\Sigma = 2^{\text{AP}}\).

In addition to the usual LTL future modalities \(X\) and \(U\), we define two new binary modalities, \(U_2\) and \(U'_2\).

The semantics is defined as follows. Let \(w = a_0a_1a_2\cdots \in \Sigma^\omega\) be an infinite word and \(i \in \mathbb{N}\).

\[
\begin{align*}
  w, i \models \varphi & \quad \text{if } \exists k \geq 0 \text{ with } w, i + 2k \models \psi \text{ and } w, i + 2j \models \varphi \text{ for all } 0 \leq j < k, \\
  w, i \models \varphi U'_2 \psi & \quad \text{if } \exists k \geq 0 \text{ with } w, i + 2k \models \psi \text{ and } w, i + j \models \varphi \text{ for all } 0 \leq j < 2k.
\end{align*}
\]

As usual, we denote by \(L(\varphi) = \{w \in \Sigma^\omega | w, 0 \models \varphi\}\). Also, we let \(F_2 \varphi = \top U_2 \varphi\).

Remark: \(F_2 q\) cannot be expressed in LTL\((X, U)\).

[2] a) Show that \(\varphi U'_2 \psi\) can be expressed in LTL\((X, U, U_2)\).

Show that \(\varphi U \psi\) can be expressed in LTL\((X, U_2)\).

[1] b) Let \(p, q \in \text{AP}\). Give a deterministic Büchi automaton which recognizes \(L(p U_2 q)\).

[2] c) Let \(p, q \in \text{AP}\). Give an MSO\((\text{AP}, <)\) formula \(\Phi(x)\) with one (first-order) free variable which is equivalent to \(p U_2 q\), i.e., for all \(w \in \Sigma^\omega\) and all \(i \in \mathbb{N}\), we have

\[
  w, i \models p U_2 q \text{ iff } w, [x \mapsto i] \models \Phi.
\]

[2] d) Let \(q \in \text{AP}\). Give a Büchi automaton \(A_1\) which recognizes \(L_1 = L(G F_2 q)\).

Hint: Give a non-deterministic Büchi automaton with 3 states.

[3] e) Let \(q \in \text{AP}\) and consider the language \(L_2 = \Sigma^* (\Sigma_q \Sigma_{\neg q} (\Sigma_{\neg q} \Sigma_{\neg q})^*)^\omega\).

Give a deterministic Büchi automaton \(A_2\) which recognizes \(L_2\).

Give a formula \(\varphi_2 \in \text{LTL}(X, U_2)\) which defines \(L_2\).

[2] f) Let \(q \in \text{AP}\) and consider \(\varphi = G(q \rightarrow XX F_2 q)\).

Show that \(L_1 \cap L_2 = \emptyset\) and \(L_1 \cup L_2 \subseteq L = L(\varphi)\).

Give a Büchi automaton \(A_3\) which recognizes \(L_3 = L \setminus (L_1 \cup L_2)\).