1 CTL and CTL*  

Fix $AP = \{p, q, r\}$. The goal is to see whether the CTL* formula  

$$\varphi_1 = E((p U q) U r)$$  

can be expressed in CTL. Consider the following CTL formulæ:  

$$\varphi_2 = E((p \lor q) U r)$$  

$$\varphi_3 = E((p \lor q) U (r \land E(p U q)))$$  

Let $\psi_1, \psi_2$ be two CTL* state formulæ. Recall that $\psi_1$ implies $\psi_2$ (resp. $\psi_1$ and $\psi_2$ are equivalent) if for all models $M$ and all states $s$ of $M$, we have $M, s \models \psi$ implies $M, s \models \psi_2$ (resp. $M, s \models \psi_1$ if and only if $M, s \models \psi_2$).

[4] a) Show that $\varphi_1$ implies $\varphi_2$, but $\varphi_1$ and $\varphi_2$ are not equivalent.  

Show that $\varphi_3$ implies $\varphi_1$, but $\varphi_1$ and $\varphi_3$ are not equivalent.

**Answer:** We have $p U q$ implies $p \lor q$, hence also $(p U q) U r$ implies $(p \lor q) U r$. It follows that $\varphi_1$ implies $\varphi_2$.

The converse is false. Consider the model $M_1 = \begin{array}{c} 1 \\ p \end{array}$ 

We have $M_1, 1 \models \varphi_2$ but $M_1, 1 \not\models \varphi_1$.

We show now that $\varphi_3$ implies $\varphi_1$. Let $M$ be a model and $s$ a state such that $M, s \models \varphi_3$. There is a run $\sigma$ starting from $s$ and $j \geq 0$ such that $M, \sigma, j \models r \land E(p U q)$ and $M, \sigma, i \models p \lor q$ for $0 \leq i < j$.

There is a run $\sigma'$ with $\sigma[j] = \sigma'[j]$ (same prefix up to $j$) such that $M, \sigma', j \models p U q$. Using $\sigma[j] = \sigma'[j]$ and $M, \sigma, i \models p \lor q$ for $i < j$, we deduce that $M, \sigma', i \models p U q$ for $i < j$. Since $M, \sigma', j \models r$ we obtain $M, s \models \varphi_1$.

Once again, the converse is false. Consider the model $M_2 = \begin{array}{c} 1 \end{array}$ 

We have $M_2, 1 \models \varphi_1$ but $M_2, 1 \not\models \varphi_3$.  


b) Prove that \( \varphi_1 \) can be expressed in CTL, i.e., give a CTL formula \( \varphi_4 \) and show that \( \varphi_1 \) and \( \varphi_4 \) are equivalent.

**Answer:** \( \varphi_4 = r \lor \varphi_3 \lor \varphi_5 \) where \( \varphi_5 = E(p \lor q) \cup (q \land EXr) \).

We show now that \( \varphi_1 \) implies \( \varphi_4 \). Let \( M \) be a model and \( s \) a state such that \( M, s \models \varphi_1 \).

There is a run \( \sigma \) starting from \( s \) and \( j \geq 0 \) such that \( M, \sigma, j \models r \) and \( M, \sigma, i \models p \lor q \) for \( 0 \leq i < j \).

If \( j = 0 \) then \( M, s \models r \), hence \( M, s \models \varphi_4 \). We assume below that \( j > 0 \).

If \( M, \sigma, j - 1 \models q \) then \( M, \sigma, 0 \models (p \lor q) \cup (q \land EXr) \) and \( M, s \models \varphi_5 \).

Otherwise, \( M, \sigma, j - 1 \models p \land \neg q \). Since \( M, \sigma, j - 1 \models p \lor q \) we deduce \( M, \sigma, j \models p \lor q \).

Therefore, \( M, s \models \varphi_3 \).

Conversely, \( r \) clearly implies \( \varphi_1 \) and we have seen above that \( \varphi_3 \) implies \( \varphi_1 \). It remains to show that \( \varphi_5 \) implies \( \varphi_1 \). If \( M, s \models \varphi_5 \), there is a run \( \sigma \) starting from \( s \) and some \( j \geq 0 \) with \( M, \sigma, j + 1 \models r \), \( M, \sigma, j \models q \) and \( M, \sigma, i \models p \lor q \) for \( i < j \). We deduce that \( M, \sigma, i \models p \lor q \) for all \( i < j + 1 \). Hence, \( M, s \models \varphi_1 \).

2 **LTL and Büchi transducers**

The flow of time is \((\mathbb{N},<)\), \(\text{AP} \neq \emptyset\) is the set of atomic propositions and \(\Sigma = 2^\text{AP}\).

In addition to the usual LTL future modalities \(X\) and \(U\), we define two new binary modalities, \(U_2\) and \(U_2'\).

The semantics is defined as follows. Let \( w = a_0a_1a_2\cdots \in \Sigma^\omega\) be an infinite word and \( i \in \mathbb{N}\).

\[
\begin{align*}
  w, i \models \varphi U_2 \psi & \iff \exists k \geq 0 \text{ with } w, i + 2k \models \psi \text{ and } w, i + 2j \models \varphi \text{ for all } 0 \leq j < k \\
  w, i \models \varphi U_2' \psi & \iff \exists k \geq 0 \text{ with } w, i + 2k \models \psi \text{ and } w, i + j \models \varphi \text{ for all } 0 \leq j < 2k
\end{align*}
\]

As usual, we denote by \(L(\varphi) = \{w \in \Sigma^\omega \mid w, 0 \models \varphi\}\). Also, we let \(F_2 \varphi = \top U_2 \varphi\).

**Remark:** \(F_2 q\) cannot be expressed in \(LTL(X, U)\).

[2] a) Show that \(\varphi U_2' \psi\) can be expressed in \(LTL(X, U, U_2)\).

Show that \(\varphi U \psi\) can be expressed in \(LTL(X, U, U_2)\).

**Answer:** \(\varphi U_2' \psi \equiv (\varphi \land X \varphi) U_2 \psi\) and \(\varphi U \psi \equiv \varphi U_2' (\psi \lor (\varphi \land X \psi))\).

b) Let \(p, q \in \text{AP}\). Give a deterministic Büchi automaton which recognizes \(L(p U_2 q)\).

**Answer:**

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3 \( \quad \rightarrow \quad \) 1 \( \quad \rightarrow \quad \) 2
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\( p \land \neg q \quad q \quad \top \)

[2] c) Let \(p, q \in \text{AP}\). Give an MSO(\(\text{AP},<\)) formula \(\Phi(x)\) with one (first-order) free variable which is equivalent to \(p U_2 q\), i.e., for all \(w \in \Sigma^\omega\) and all \(i \in \mathbb{N}\), we have

\[
 w, i \models p U_2 q \text{ iff } w, [x \mapsto i] \models \Phi.
\]

**Answer:**

\[
\Phi(x) = \exists z \exists Y, q(z) \land z \in Y \land \forall y \in Y \setminus \{x\}, \exists y_1, y_2 \ (p(y_1) \land y_1 < y_2 < y) \]

where \(u < v = u < v \land \neg \exists w (u < w < v)\).
Let $q \in \text{AP}$. Give a Büchi automaton $A_1$ which recognizes $L_1 = \mathcal{L}(GF_2 q)$.

**Hint:** Give a non-deterministic Büchi automaton with 3 states.

**Answer:** Notice that $w \in L_1$ iff $w$ contains infinitely many odd positions satisfying $q$ and infinitely many even positions satisfying $q$. Hence $L_1 = (\Sigma^* \Sigma_q)^\omega$. Hence,

\[ A_1 = \begin{array}{c}
1 \quad 3 \quad 2 \\
\quad q \\
\end{array} \]

Let $q \in \text{AP}$ and consider the language $L_2 = \Sigma^*_q (\Sigma_q \Sigma^-q (\Sigma^-q \Sigma^-q)^*)^\omega$.

Give a deterministic Büchi automaton $A_2$ which recognizes $L_2$.

Give a formula $\varphi_2 \in \text{LTL}(X, U_2)$ which defines $L_2$.

**Answer:** Notice that $w \in L_2$ iff $w$ contains infinitely many odd positions satisfying $q \text{ or infinitely many even positions satisfying } q$. Hence,

\[ \varphi_2 = (\neg q \cup (q \land \neg XF_2 q)) \land G(q \rightarrow XF_2 q) \]

\[ \equiv F q \land G(q \rightarrow XF_2 q) \land \neg GF_2 q \]

Let $q \in \text{AP}$ and consider $\varphi = G(q \rightarrow XF_2 q)$.

Show that $L_1 \cap L_2 = \emptyset$ and $L_1 \cup L_2 \subseteq L = \mathcal{L}(\varphi)$.

Give a Büchi automaton $A_3$ which recognizes $L_3 = L \setminus (L_1 \cup L_2)$.

**Answer:** From the discussion above, we know that $L_1 \cap L_2 = \emptyset$.

Now, $\mathcal{L}(\varphi)$ is the set of words $w$ such that if $w$ satisfies $q$ in some position $i$ (odd or even) then it contains infinitely many positions satisfying $q$ with the same parity as $i$. We deduce that $L_1 \cup L_2 \subseteq L$ and that $L_3$ is the set of words which never satisfy $q$: $L_3 = \mathcal{L}(\neg q) = (\Sigma^-q)^\omega$. Therefore, $A_3 = \begin{array}{c}
1 \\
\neg q \\
\end{array}$.