Efficient computations with pebbles

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joint work with Benjamin Monmege

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Slides at `http://www.lsv.ens-cachan.fr/~gastin/Talks/`

See also CIAA'2012

Outline

1. Introduction: 2-way moves and pebbles
   - Counting Patterns
   - Weighted temporal logic

Weighted expressions with pebbles

Weighted automata with pebbles

From automata to expressions

From expressions to automata

Evaluation of pebble weighted automata

Concluding remarks

Counting Patterns

$$E = \rightarrow^{+} a? \leftarrow^{+} b? \rightarrow^{+} c? \leftarrow^{+} d? \rightarrow^{+}$$

$$w \quad = \quad c \quad a \quad b \quad c \quad d \quad b \quad a \quad d \quad c \quad b \quad a \quad b$$

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$$[\![E]\!](w) = 8$$

An equivalent 1-way expression is more complex and less intuitive.

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$$[\![E]\!](w) = 8$$

An equivalent 1-way expression is more complex and less intuitive.

$$F = \rightarrow^+ a?\, x!\Big((\neg x?\rightarrow)^*\, b?\, (\neg x?\rightarrow)^+\, c?\, \leftarrow^+\, d?\, \rightarrow^+\Big) \rightarrow^*$$

Counting Patterns

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$$F = \rightarrow^{+} a?\, x!\Big((\neg x? \rightarrow)^{*}\, b?\, (\neg x? \rightarrow)^{+}\, c?\, \leftarrow^{+}\, d?\, \rightarrow^{+} \Big) \rightarrow^{*}$$

$$[\![F]\!](w) = 4$$

Future and Past modalities

$$\varphi_1 = \mathsf{F}(a \wedge \mathsf{P}(b \wedge \mathsf{F}(c \wedge \mathsf{P}\, d)))$$

$$\varphi_2 = \mathsf{G}(\mathtt{grant} \rightarrow \mathsf{P}\, \mathtt{request})$$

$$\varphi_3 = \mathsf{G}(\mathtt{grant} \rightarrow \mathsf{Y}((\neg \mathtt{grant})\, \mathsf{S}\, \mathtt{request})$$

Probabilistic LTL

Each LTL formula φ has an implicit free variable x denoting the position where the formula is evaluated. We use a pebble to mark this position.

Let $\mathbb{P}(\varphi, u, i)$ denote the probability that φ holds on word u at position i.

$$\mathbb{P}(\mathsf{G}\,\varphi, u, i) \quad = \quad \prod_{j \geq i} \mathbb{P}(\varphi, u, j)$$

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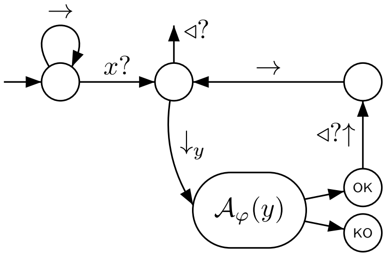

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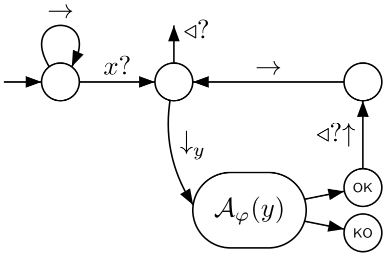

$$E_{\mathsf{G}\,\varphi}(x) \quad = \quad \rhd?{\rightarrow}^* x?\big((y!E_\varphi(y)){\rightarrow}\big)^* \lhd?\,.$$

Probabilistic LTL

$$\mathbb{P}(\mathsf{F}\,\varphi, u, i) \quad = \quad \mathbb{P}(\varphi, u, i) + (1 - \mathbb{P}(\varphi, u, i)) \times \mathbb{P}(\mathsf{F}\,\varphi, u, i+1)$$

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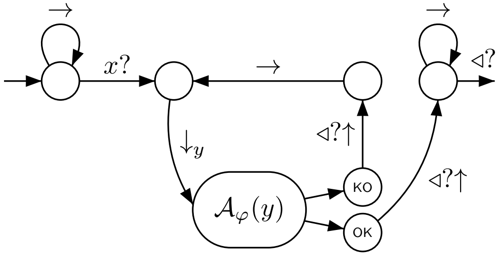

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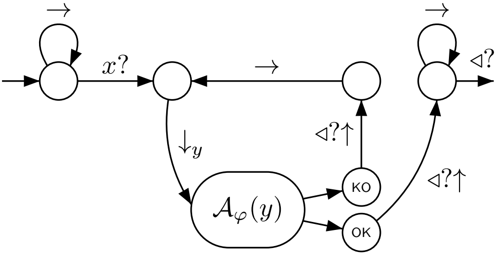

$$E_{\mathsf{F}\,\varphi}(x) = \triangleright?{\rightarrow}^{*}x?\big((y!E_{\neg\varphi}(y)){\rightarrow}\big)^{*}(y!E_{\varphi}(y)){\rightarrow}^{*}\triangleleft?$$

Probabilistic LTL

$$\mathbb{P}(\varphi \,\mathsf{U}\, \psi, u, i) \quad = \quad \mathbb{P}(\psi, u, i) + \mathbb{P}(\neg\psi, u, i) \times \mathbb{P}(\varphi, u, i) \times \mathbb{P}(\varphi \,\mathsf{U}\, \psi, u, i+1)$$

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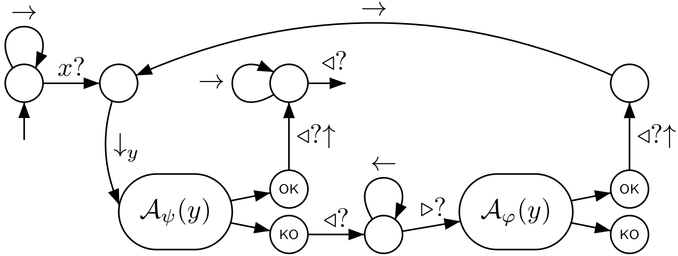

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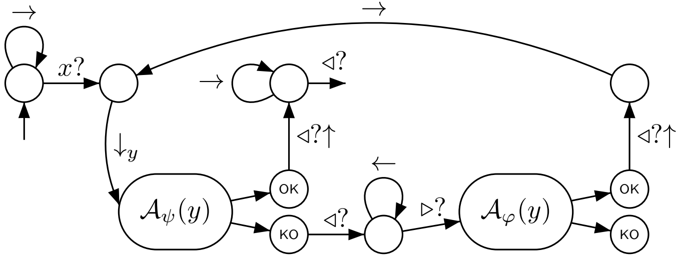

$$E_{\varphi \mathsf{U} \psi}(x) = \triangleright? \rightarrow^* x? \big((y!(E_{\neg\psi}(y) \leftarrow^* E_{\varphi}(y))) \rightarrow \big)^* (y! E_{\psi}(y)) \rightarrow^* \triangleleft?$$

Outline

Continuous semirings

A semiring $\mathbb{S}$ is complete if every family $(s_i)_{i \in I} \subseteq \mathbb{S}$ is summable
and the following conditions are satisfied:

- $\sum_{i \in \emptyset} s_i = 0 \qquad \sum_{i \in \{1\}} s_i = s_1 \qquad \sum_{i \in \{1,2\}} s_i = s_1 + s_2$ Compatibility
- if $I = \bigcup_{j \in J} I_j$ is a partition, $\sum_{j \in J} \left(\sum_{i \in I_j} s_i \right) = \sum_{i \in I} s_i$ Associativity
- $\left(\sum_{i \in I} s_i \right) \times \left(\sum_{j \in J} t_j \right) = \sum_{(i,j) \in I \times J} (s_i \times t_j)$ Distributivity

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A semiring $\mathbb{S}$ is continuous if it is complete and

- The relation $a \leq b$ if $b = a + c$ for some c is an order relation Order
- $\sum_{i \in I} s_i$ is the least upper bound of the finite sums Approximability

$$\sum_{i \in I} s_i = \bigsqcup_{J \subseteq I, J \text{ finite}} \sum_{i \in J} s_i$$

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Star operation: for $s \in \mathbb{S}$, we let $s^* = \sum_{i\in\mathbb{N}} s^i$ (with $s^0 = 1$).

Continuous semirings

Examples:

- The Boolean semiring $(\{0,1\}, \vee, \wedge, 0, 1)$ with $\sum$ defined as an infinite disjunction.
- $(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, \times, 0, 1)$ with $\sum$ defined as usual for positive series: in particular, $s^* = \infty$ if $s \geq 1$ and $s^* = 1/(1-s)$ if $0 \leq s < 1$.
- $(\mathbb{N} \cup \{\infty\}, +, \times, 0, 1)$ as a complete subsemiring of the previous one.
- $(\mathbb{R} \cup \{-\infty\}, \min, +, -\infty, 0)$ with $\sum = \inf$.
- $(\mathbb{R} \cup \{\infty\}, \max, +, \infty, 0)$ with $\sum = \sup$.
- Complete lattices such as $([0,1], \min, \max, 0, 1)$.
- The semiring of languages over an alphabet A: $(2^{A^*}, \cup, +, \emptyset, \{\varepsilon\})$ with $\sum$ defined as (infinite) union.

Marked words

- Let $u = u_0 \cdots u_{n-1} \in A^+$ be a non-empty word.

  The set of positions of u is $\mathrm{pos}(u) = \{0, 1, \ldots, n\}$.

- Let Peb be the (finite) set of pebbles.

- A (statically) *marked* word is a tuple (u, σ, i, j) where $u \in A^+$ is a word, $\sigma : \mathrm{Peb} \to \mathrm{pos}(u)$ is a valuation and $i, j \in \mathrm{pos}(u)$ are positions.

  We denote by $\mathrm{Mk}(A^+)$ the set of marked words.

  We will see below that $\mathrm{Mk}(A^+)$ forms a partial monoid.

Weighted expressions with pebbles

Syntax of pebWE:

$$E ::= s \mid \varphi \mid \rightarrow \mid \leftarrow \mid x!E \mid E + E \mid E \cdot E \mid E^{+}$$
$$\varphi ::= a? \mid \triangleright? \mid \triangleleft? \mid x? \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

with $s \in \mathbb{S}$, $a \in A$, $x \in \mathrm{Peb}$.

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Semantics of Test formulas φ:

- $\triangleright?$ holds on position 0,
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Semantics over marked words: $[\![E]\!] \in \mathbb{S}\langle\langle \mathrm{Mk}(A^+) \rangle\rangle$.

$$\begin{array}{rcl} [\![E]\!] : \mathrm{Mk}(A^+) & \rightarrow & \mathbb{S} \\ (u, \sigma, i, j) & \mapsto & [\![E]\!](u, \sigma, i, j) \end{array}$$

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with $s \in \mathbb{S}$, $a \in A$, $x \in \mathrm{Peb}$.

Semantics:

$$[\![s]\!](u,\sigma,i,j) = \begin{cases} s & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

$$[\![\varphi]\!](u,\sigma,i,j) = \begin{cases} 1 & \text{if } j = i \text{ and } u,\sigma,i \models \varphi \\ 0 & \text{otherwise} \end{cases}$$

$$[\![\rightarrow]\!](u,\sigma,i,j) = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$[\![\leftarrow]\!](u,\sigma,i,j) = \begin{cases} 1 & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

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Semantics:

$$\begin{aligned} [\![E + F]\!](u, \sigma, i, j) &= [\![E]\!](u, \sigma, i, j) + [\![F]\!](u, \sigma, i, j) \\ [\![E \cdot F]\!](u, \sigma, i, j) &= \textstyle\sum_{k \in \mathrm{pos}(u)} [\![E]\!](u, \sigma, i, k) \times [\![F]\!](u, \sigma, k, j) \end{aligned}$$

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[\![E^+]\!](u, \sigma, i, j) &= \textstyle\sum_{n>0} [\![E^n]\!](u, \sigma, i, j) \\
[\![x!E]\!](u, \sigma, i, j) &= \begin{cases} [\![E]\!](u, \sigma[x \mapsto i], 0, |u|) & \text{if } j = i < |u| \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Examples of pebWE

Abbreviations:

- $[\![E]\!](u, \sigma) = [\![E]\!](u, \sigma, 0, |u|)$.
- If E has no free variable:
  $[\![E]\!](u, i, j) = [\![E]\!](u, \sigma, i, j)$ and $[\![E]\!](u) = [\![E]\!](u, 0, |u|)$.

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Examples in the natural semiring

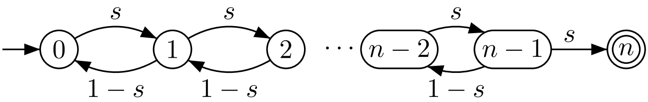
- $[\![\rightarrow^* a? \rightarrow^*]\!](baaba) = 3$
- $[\![(2\rightarrow)^+]\!](u) = 2^{|u|}$
- $[\![E_1 \lhd? \leftarrow^* \rhd? \, E_2]\!](u) = [\![E_1]\!](u) \times [\![E_2]\!](u)$
- $[\![(x!((2\rightarrow)^+)\rightarrow)^+]\!](u) = 2^{|u|^2}$

Examples of pebWE

Consider the *continuous* semiring $(\mathbb{R}^{\infty}_{\geq 0}, +, \times, 0, 1)$, and let $0 < s < 1$:

$$E = (\neg\triangleleft?(s \rightarrow + (1-s)\neg\triangleright?\leftarrow))^* \triangleleft?$$

Random walk on a word u of length n (Markov chain):



With $\alpha = \frac{1-s}{s}$, one can show that

$$[\![E]\!](u) = \frac{1}{1 + \alpha + \ldots + \alpha^{|u|}}$$

Expressions for probabilistic LTL

$A = 2^{\mathrm{AP}}$ with AP the set of atomic propositions.

$$
\begin{aligned}
E_p(x) &= \rhd ? \rightarrow^* x ? p ? \rightarrow^* \lhd ? \\
E_{\varphi \wedge \psi}(x) &= E_\varphi(x) \leftarrow^* E_\psi(x) \\
E_{\varphi \vee \psi}(x) &= E_\varphi(x) + E_{\neg\varphi \wedge \psi}(x)
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E_{\mathsf{X}\,\varphi}(x) &= \triangleright ? \rightarrow^* x? \rightarrow (x! E_\varphi(x)) \rightarrow^* \triangleleft ? \\
E_{\mathsf{Y}\,\varphi}(x) &= \triangleright ? \rightarrow^* x? \leftarrow (x! E_\varphi(x)) \rightarrow^* \triangleleft ?
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Reusable pebbles!

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E_{\mathsf{X}\,\varphi}(x) & = & \triangleright ? \rightarrow^* x ? \rightarrow (x ! E_\varphi(x)) \rightarrow^* \triangleleft ? \\
E_{\mathsf{Y}\,\varphi}(x) & = & \triangleright ? \rightarrow^* x ? \leftarrow (x ! E_\varphi(x)) \rightarrow^* \triangleleft ? \\
E_{\varphi \mathsf{U} \psi}(x) & = & \triangleright ? \rightarrow^* x ? \big((x ! E_{\neg\psi \wedge \varphi}(x)) \rightarrow \big)^* (x ! E_\psi(x)) \rightarrow^* \triangleleft ? \\
E_{\varphi\,\mathsf{S}\,\psi}(x) & = & \triangleright ? \rightarrow^* x ? \big((x ! E_{\neg\psi \wedge \varphi}(x)) \leftarrow \big)^* (x ! E_\psi(x)) \rightarrow^* \triangleleft ?
\end{array}
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Reusable pebbles!

Semantics at a higher level

$\mathbb{S}\langle\langle A^*\rangle\rangle$ is a semiring with pointwise sum and Cauchy product:

$$(f+g)(w) = f(w) + g(w)$$
$$(f \times g)(w) = \sum_{w=uv} f(u) \times g(v)$$

If $\mathbb{S}$ is continuous, so is $\mathbb{S}\langle\langle A^*\rangle\rangle$.

Hence we have a star operation on $\mathbb{S}\langle\langle A^*\rangle\rangle$: $f^* = \sum_{n\geq 0} f^n$.

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If $\mathbb{S}$ is continuous, so is $\mathbb{S}\langle\langle A^*\rangle\rangle$.
Hence we have a star operation on $\mathbb{S}\langle\langle A^*\rangle\rangle$: $f^* = \sum_{n\geq 0} f^n$.

Compositional semantics of rational expressions over $\mathbb{S}\langle\langle A^*\rangle\rangle$:

$$[\![E+F]\!] = [\![E]\!] + [\![F]\!] \qquad [\![E\cdot F]\!] = [\![E]\!] \times [\![F]\!] \qquad [\![E^*]\!] = [\![E]\!]^*$$

If M is an arbitrary monoid, the same holds in $\mathbb{S}\langle\langle M\rangle\rangle$.

Semantics at a higher level

$\mathbb{S}\langle\langle A^*\rangle\rangle$ is a semiring with pointwise sum and Cauchy product:

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What about 2-way moves and pebbles?

Partial monoid of marked words

$$\mathrm{Mk}(A^+) = \{(u, \sigma, i, j) \mid u \in A^+,\ \sigma\colon \mathrm{Peb} \to \mathrm{pos}(u),\ i, j \in \mathrm{pos}(u)\}$$

Partial composition over $\mathrm{Mk}(A^+)$ (associative):

$$(u, \sigma, i, k) \cdot (u, \sigma, k, j) = (u, \sigma, i, j)$$

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Cauchy product over $\mathbb{S}\langle\!\langle \mathrm{Mk}(A^+) \rangle\!\rangle$ (associative):

$$\begin{aligned}(f \times g)(u, \sigma, i, j) &= \sum_{(u,\sigma,i,j)=xy} f(x) \times g(y) \\ &= \sum_{k \in \mathrm{pos}(w)} f(u, \sigma, i, k) \times g(u, \sigma, k, j)\end{aligned}$$

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Unit of $\mathbb{S}\langle\langle \mathrm{Mk}(A^+) \rangle\rangle$ for the Cauchy product:

$1_{\mathrm{Unit}(A^+)}$ the characteristic function of $\mathrm{Unit}(A^+)$.

$$(f \times 1_{\mathrm{Unit}(A^+)})(u, \sigma, i, j) = \sum_{k \in \mathrm{pos}(w)} f(u, \sigma, i, k) \times 1_{\mathrm{Unit}(A^+)}(u, \sigma, k, j) = f(u, \sigma, i, j)$$

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$(\mathbb{S}\langle\langle \mathrm{Mk}(A^+)\rangle\rangle, +, \times, 0, 1_{\mathrm{Unit}(A^+)})$ is a semiring.

$\mathbb{S}\langle\langle \mathrm{Mk}(A^+)\rangle\rangle$ is continuous if $\mathbb{S}$ is continuous. Star operation: $f^* = \sum_{n \geq 0} f^n$

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Partial monoids

A partial monoid is a triple $(Z, \cdot, Y)$ where

- Z is the set of elements,
- $\cdot\colon Z^2 \to Z$ is a partially defined associative concatenation,
- $Y \subseteq Z$ is a set of partial units satisfying:

$$\begin{array}{rll}
\forall z \in Z & \exists! y \in Y & y \cdot z = z \\
\forall z \in Z & \exists! y \in Y & z \cdot y = z \\
\forall x, z \in Z & \forall y \in Y & x \cdot y = z \implies x = z \\
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Proposition: Series over partial monoids

If $\mathbb{S}$ is a continuous semiring and $(Z, \cdot, Y)$ is a partial monoid, then

- the series $\mathbb{S}\langle\langle Z \rangle\rangle$ forms a continuous semiring $(\mathbb{S}\langle\langle Z \rangle\rangle, +, \times, 0, 1_Y)$,
- the star operation is defined on $\mathbb{S}\langle\langle Z \rangle\rangle$.

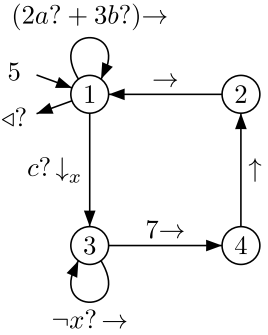
Outline

Weighted automata with pebbles

$\mathrm{Move} = \{\leftarrow, \rightarrow, \uparrow\} \cup \{\downarrow_x \mid x \in \mathrm{Peb}\}$ is the set of possible moves of an automaton.

A *pebble weighted automaton* (pebWA) is a tuple $\mathcal{A} = (Q, A, I, M, T)$ with

- Q a finite set of states,
- $I \in \mathbb{S}^Q$ a row vector of initial weights,
- $T \in \mathbb{S}\langle \mathrm{Test} \rangle^Q$ a column vector of terminal weighted tests,
- $M \in (\mathbb{S}\langle \mathrm{Test} \rangle \langle \mathrm{Move} \rangle)^{Q \times Q}$ the transition matrix.

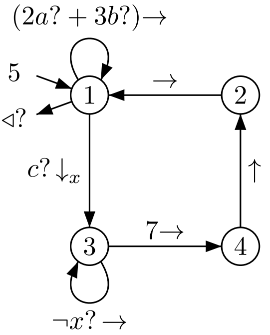


$$I = \begin{pmatrix} 5 & 0 & 0 & 0 \end{pmatrix}$$

$$M = \left(\begin{array}{cc|cc} (2a? + 3b?)\rightarrow & 0 & c?\downarrow_x & 0 \\ \rightarrow & 0 & 0 & 0 \\ \hline 0 & 0 & \neg x? \rightarrow & 7\rightarrow \\ 0 & \uparrow & 0 & 0 \end{array}\right)$$

$$T = \begin{pmatrix} \lhd ? \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Weighted automata with pebbles



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Each accepting run of $\mathcal{A}$ over u has weight $5 \times 2^{|u|_a} \times 3^{|u|_b} \times 7^{|u|_c}$

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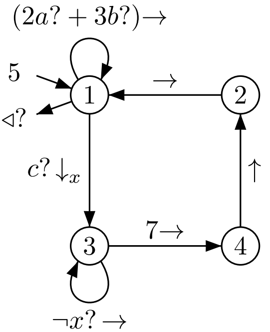

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Non-deterministic choice in state 3 yields $i+1$ runs if x is dropped on position i

$$[\![\mathcal{A}]\!](u) = 5 \times 2^{|u|_a} \times 3^{|u|_b} \times 7^{|u|_c} \times \prod_{i \mid u_i = c} (i+1)$$

Formal semantics

A configuration of $\mathcal{A}$ is a tuple (u, σ, q, i, π) with

- $u \in A^+$ a word,
- $\sigma\colon \mathrm{Peb} \to \mathrm{pos}(u)$ a valuation,
- $q \in Q$ the current state,
- $i \in \mathrm{pos}(u)$ the current position,
- $\pi \in (\mathrm{Peb} \times \mathrm{pos}(u))^*$ the stack of pebbles currently dropped.

Reusable pebbles: σ_π defined inductively by $\sigma_\varepsilon = \sigma$ and $\sigma_{\pi(x,i)} = \sigma_\pi[x \mapsto i]$.

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Configurations are locations of the weighted transition system $\mathrm{TS}(\mathcal{A})$.

The weight of transition $(u, \sigma, p, i, \pi) \rightsquigarrow (u, \sigma, q, j, \pi')$ is

$$\begin{aligned}
&[\![M^{\rightarrow}_{p,q}]\!](u, \sigma_\pi, i, i) && \text{if } j = i + 1 \text{ and } \pi' = \pi && (\rightarrow)\\
&[\![M^{\leftarrow}_{p,q}]\!](u, \sigma_\pi, i, i) && \text{if } j = i - 1 \text{ and } \pi' = \pi && (\leftarrow)\\
&[\![M^{\downarrow_x}_{p,q}]\!](u, \sigma_\pi, i, i) && \text{if } j = 0,\ i < |u| \text{ and } \pi' = \pi(x, i) && (\downarrow_x)\\
&[\![M^{\uparrow}_{p,q}]\!](u, \sigma_\pi, i, i) && \text{if } \pi = \pi'(y, j) \text{ for some } y \in \mathrm{Peb} && (\uparrow)
\end{aligned}$$

where $M^d_{p,q} \in \mathbb{S}\langle \mathrm{Test} \rangle$ is the coefficient of move d in $M_{p,q}$.

Formal semantics

For ρ run of $\mathrm{TS}(\mathcal{A})$, weight(ρ) is the product of the weights of its transitions.
Given $(u, \sigma, i, j) \in \mathrm{Mk}(A^+)$ and $p, q \in Q$, we define

$$[\![\mathcal{A}_{p,q}]\!](u, \sigma, i, j) = \sum_{\rho} \mathsf{weight}(\rho)$$

sum over runs ρ from configuration $(u, \sigma, p, i, \varepsilon)$ to configuration $(u, \sigma, q, j, \varepsilon)$.

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$$[\![\mathcal{A}]\!](u, \sigma, i, j) = \sum_{p,q \in Q} I_p \times [\![\mathcal{A}_{p,q}]\!](u, \sigma, i, j) \times [\![T_q]\!](u, \sigma, j, j)$$

Semantics at a higher level

Let $\mathcal{A} = (Q, A, I, M, T)$ be a 2-way weighted automaton.

We have $M \in (\mathbb{S}\langle \mathrm{Test} \rangle \langle \{\leftarrow, \rightarrow\} \rangle)^{Q \times Q}$.

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The semiring $\mathbb{K} = \mathbb{S}\langle\langle \mathrm{Mk}(A^+) \rangle\rangle$ is continuous.
Let $[\![M]\!] = ([\![M_{p,q}]\!])_{p,q \in Q} \in \mathbb{K}^{Q \times Q}$.
$[\![M]\!]^n$ gives the semantics restricted to paths of length n.

$$[\![\mathcal{A}_{p,q}]\!] = \sum_{n \geq 0} ([\![M]\!]^n)_{p,q} = ([\![M]\!]^*)_{p,q} \qquad [\![\mathcal{A}]\!] = I \times [\![M]\!]^* \times T$$

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The semiring of matrices $\mathbb{K}^{Q \times Q}$ is also continuous.

$[\![M]\!] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then $[\![M]\!]^* = \begin{pmatrix} (A + BD^*C)^* & A^*B(D + CA^*B)^* \\ D^*C(A + BD^*C)^* & (D + CA^*B)^* \end{pmatrix}$

The entries of the matrix $[\![M]\!]^*$ are in the rational closure of the entries of $[\![M]\!]$.

Corollary:

We can construct a pebWE $E(\mathcal{A}) = I \times M^* \times T$ which is equivalent to $\mathcal{A}$.

Semantics at a higher level

What about pebbles?

Partial monoid of dynamically marked words: $(u, \sigma, i, \pi, j, \pi')$.

Partial composition (associative):

$$(u, \sigma, i, \pi, k, \pi') \cdot (u, \sigma, k, \pi', j, \pi'') = (u, \sigma, i, \pi, j, \pi'')$$

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Semantics of drop and lift:

$$[\![\downarrow_x]\!](u, \sigma, i, \pi, j, \pi') = \begin{cases} 1 & \text{if } \pi' = \pi(x, i) \text{ and } j = 0 \text{ and } i < |u| \\ 0 & \text{otherwise} \end{cases}$$

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Then, $[\![\mathcal{A}]\!] = I \times [\![M]\!]^* \times T$.
But this does not give a pebWE equivalent to $\mathcal{A}$.

Outline

Layered automata

Fix $K \geq 0$ and $\ell\colon Q \to \{0, \ldots, K\}$ mapping each state to its layer.

Automaton $\mathcal{A} = (Q, A, I, M, T)$ is K-layered if

- $\ell(q) \neq K$ implies $I_q = 0 = T_q$,
- $\ell(p) \neq \ell(q)$ implies $M^{\leftarrow}_{p,q} = 0 = M^{\rightarrow}_{p,q}$,
- $M^{\downarrow_x}_{p,q} \neq 0$ implies $\ell(q) = \ell(p) - 1$,
- $M^{\uparrow}_{p,q} \neq 0$ implies $\ell(q) = \ell(p) + 1$.

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A 2-layered automaton has the following form:

$$I = \begin{pmatrix} \begin{array}{|c|c|c|} \hline I^{(2)} & 0 & 0 \\ \hline \end{array} \end{pmatrix}, \quad M = \begin{pmatrix} \begin{array}{|c|c|c|} \hline N^{(2)} & D^{(2)} & 0 \\ \hline L^{(1)} & N^{(1)} & D^{(1)} \\ \hline 0 & L^{(0)} & N^{(0)} \\ \hline \end{array} \end{pmatrix}, \quad T = \begin{pmatrix} \begin{array}{|c|} \hline T^{(2)} \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \end{pmatrix}$$

2-way transitions: entries in $N^{(i)}$ are in $\mathbb{S}\langle \mathrm{Test} \rangle \langle \{\leftarrow, \rightarrow\} \rangle$,
Lift transitions: entries in $L^{(i)}$ are in $\mathbb{S}\langle \mathrm{Test} \rangle \langle \{\uparrow\} \rangle$,
Drop transitions: entries in $D^{(i)}$ are in $\mathbb{S}\langle \mathrm{Test} \rangle \langle \{\downarrow_x \mid x \in \mathrm{Peb}\} \rangle$.

From automata to expressions

Theorem:

Let $\mathcal{A} = (Q, A, I, M, T)$ be a K-layered pebWA.

We can construct a matrix $H \in \text{pebWE}^{Q \times Q}$ such that

$$[\![H_{p,q}]\!] = [\![\mathcal{A}^{\leq i}_{p,q}]\!]$$

for all $i \leq K$ and $p, q \in Q^{(i)} = \ell^{-1}(i)$ be the set of states in layer i.

The pebWE $E(\mathcal{A}) = I \times H \times T$ is equivalent to $\mathcal{A}$:

$$[\![E(\mathcal{A})]\!] = [\![\mathcal{A}]\!]$$

Moreover, the pebble-depth of $E(\mathcal{A})$ is at most K.

From automata to expressions

$$M = \begin{pmatrix} N & D \\ L & P \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & D \\ L & P \end{pmatrix}$$

$$H = \begin{pmatrix} N+G & 0 \\ 0 & P \end{pmatrix}$$

$$G_{p,q} = \sum_{p',q'} \sum_{x \in \mathrm{Peb}} d^x_{p,p'} \cdot x!\big((P^*)_{p',q'} \cdot \ell_{q',q} \cdot \to^*\big)$$

Outline

From expressions to automata

Litteral-length of a pebWE:

$$\ell\ell((2a? + b?)\rightarrow(2b? + 3c?)) = 1$$
$$\ell\ell(\rightarrow^{+}a?\leftarrow^{+}b?\rightarrow^{+}c?\leftarrow^{+}d?\rightarrow^{+}) = 5$$
$$\ell\ell(\rightarrow^{+}\, a?\, x!\Big((\neg x?\rightarrow)^{*}\, b?\, (\neg x?\rightarrow)^{+}\, c?\leftarrow^{+}\, d?\rightarrow^{+}\Big)\rightarrow^{*}) = 8$$

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Theorem:

For each pebWE E we can construct a *layered* pebWA $\mathcal{A}(E)$ such that

$$[\![\mathcal{A}(E)]\!] = [\![E]\!]$$

i.e., for all $(u, \sigma, i, j) \in \mathrm{Mk}(A^+)$ we have

$$[\![\mathcal{A}(E)]\!](u, \sigma, i, j) = [\![E]\!](u, \sigma, i, j)\,.$$

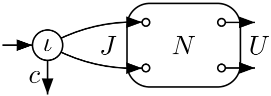
The number of layers in $\mathcal{A}(E)$ is the pebble-depth of E.

The number of states of $\mathcal{A}(E)$ is $1 + \ell\ell(E)$.

The time complexity is cubic.

Standard automata

A pebWA $\mathcal{A} = (Q, A, I, M, T)$ is standard if it has a single initial state ι with no ingoing transition, and the initial weight is 1.

$\mathcal{A} =$ 

$$\mathcal{A} = \begin{pmatrix} 1 & \boxed{\;0\;} \end{pmatrix} \begin{pmatrix} 0 & \boxed{\;J\;} \\ \boxed{0} & \boxed{\;N\;} \end{pmatrix} \begin{pmatrix} c \\ \boxed{U} \end{pmatrix}$$

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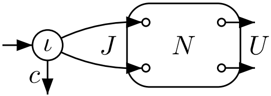
$\mathcal{A} =$ 

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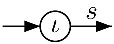
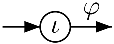
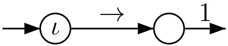
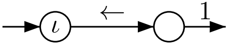
pebWA for the atomic pebWE:

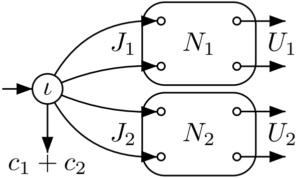
$\mathcal{A}(s) =$ 

$\mathcal{A}(\varphi) =$ 

$\mathcal{A}(\rightarrow) =$ 

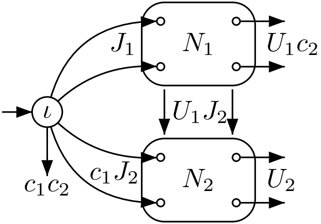
$\mathcal{A}(\leftarrow) =$ 

Sum

$\mathcal{A}_1 + \mathcal{A}_2 =$ 

$$\mathcal{A}_1 + \mathcal{A}_2 = \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \end{pmatrix} \begin{pmatrix} 0 & J_1 & J_2 \\ 0 & N_1 & 0 \\ 0 & 0 & N_2 \end{pmatrix} \begin{pmatrix} c_1 + c_2 \\ U_1 \\ U_2 \end{pmatrix}$$

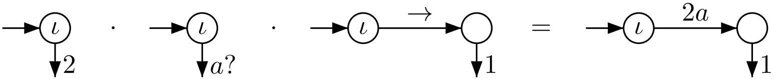
Product

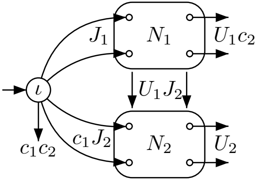
$\mathcal{A}_1 \cdot \mathcal{A}_2 =$ 

$$\mathcal{A}_1 \cdot \mathcal{A}_2 = \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \end{pmatrix} \begin{pmatrix} 0 & J_1 & c_1 J_2 \\ 0 & N_1 & U_1 J_2 \\ 0 & 0 & N_2 \end{pmatrix} \begin{pmatrix} c_1 c_2 \\ U_1 c_2 \\ U_2 \end{pmatrix}$$

Product

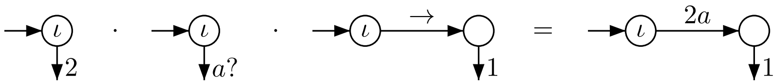
The automaton for $2a = 2 \cdot a? \cdot \rightarrow$ is computed as follows:



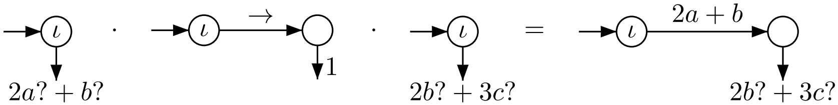


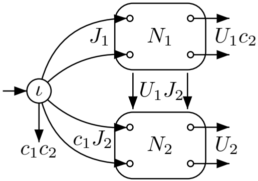
Product

The automaton for $2a = 2 \cdot a? \cdot \rightarrow$ is computed as follows:

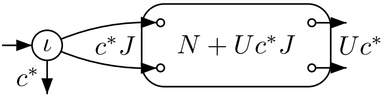


The automaton for $E = (2a? + b?){\rightarrow}(2b? + 3c?)$





Star

$\mathcal{A}^* =$ 

$$\mathcal{A}^* = \begin{pmatrix} 1 & \boxed{0} \end{pmatrix} \begin{pmatrix} 0 & \boxed{c^* J} \\ \boxed{0} & \boxed{N + U c^* J} \end{pmatrix} \begin{pmatrix} c^* \\ \boxed{U c^*} \end{pmatrix}$$

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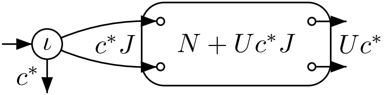
$\mathcal{A}^* =$ 

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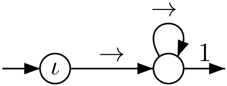
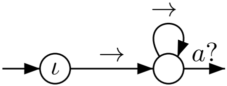
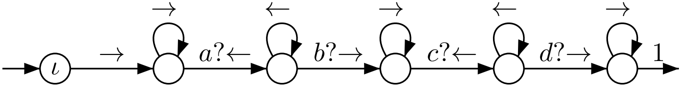
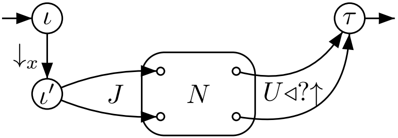
For $E = \to^+ a? \leftarrow^+ b? \to^+ c? \leftarrow^+ d? \to^+$, we compute:

$\mathcal{A}(\to^+) =$ 

$\mathcal{A}(\to^+ \cdot a?) =$ 

$\mathcal{A}(E) =$ 

Pebbles

$x!\mathcal{A} \;=$ 

$$x!\mathcal{A} \;=\; \left(\begin{array}{cc|cc} 1 & 0 & 0 & \boxed{0} \end{array}\right) \left(\begin{array}{cc|cc} 0 & 0 & \downarrow_x & \boxed{0} \\ 0 & 0 & 0 & \boxed{0} \\ \hline 0 & 0 & 0 & \boxed{J} \\ \boxed{0} & \boxed{U\triangleleft?\uparrow} & \boxed{0} & \boxed{N} \end{array}\right) \left(\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ \boxed{0} \end{array}\right)$$

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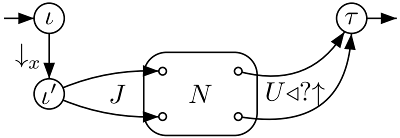

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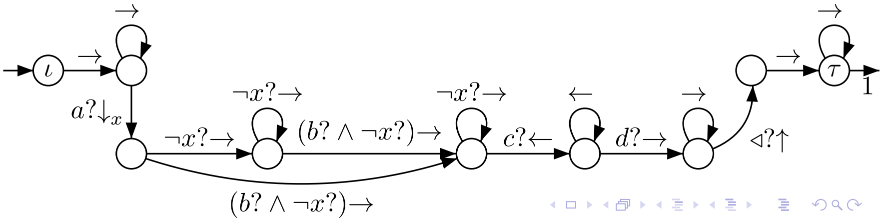
For $E = \rightarrow^+ a?\, x!\Big((\neg x?\rightarrow)^*\, b?\, (\neg x?\rightarrow)^+\, c?\leftarrow^+\, d?\rightarrow^+\Big)\rightarrow^*$, we compute:



Outline

Evaluation of pebble weighted automata

Let $\mathcal{A} = (Q, A, I, M, T)$ be a K-layered pebWA.

Recall that $Q^{(i)} = \ell^{-1}(i)$ is the set of states in layer i.

Theorem:

Given a K-layered pebWA with p pebbles and a word $w \in A^+$,
we can compute with $\mathcal{O}((K+1)|w|^{p+1})$ matrix operations (sum, product, iteration) all values $[\![\mathcal{A}_{p,q}]\!](w, \sigma)$ for all $p, q \in Q^{(K)}$ and valuations $\sigma \colon \mathrm{Peb} \to \mathrm{pos}(w)$.

Note that the number of valuations is $|w|^p$.

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Note that the number of valuations is $|w|^p$.

With reusable pebbles p may be much smaller than K.

Probabilistic LTL can be translated to K-layered pebWA with only 1 pebble.

The number K of layers is the nesting depth of the formula.

Evaluation of pebble weighted automata

A K-layered pebWA $\mathcal{A} = (Q, A, I, M, T)$ is strongly K-layered
if in each layer $i \leq K$, only a fixed pebble x_i may be dropped.

Theorem:

Given a strongly K-layered pebWA with p pebbles and a word $w \in A^+$,
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If a K-layered pebWA uses at most 1 pebble then it is strongly K-layered.

pebWA associated with probabilistic LTL formulas are strongly K-layered.

Corollary:

The evaluation problem for probabilistic LTL is linear in $|w|$.

Outline

Some future directions

Some variations:

- Restrict the syntax of pebWE and pebWA to avoid infinite sums.
  E.g., forward proper or backward proper iterations or loops.
- Restrict the syntax of pebWE and pebWA to fit the probabilistic setting (next talk).
- Weighted extension of regular XPath and tree walking automata.
  Use marked trees instead of marked words.

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Some open problems:

- Try to obtain a quadratic time algorithm for the translation of pebWE to pebWA.
  Generalize the notion of star normal form introduced by Brüggeman-Klein (TCS'93) for word languages
  Generalize the algorithm of Allauzen and Mohri (MFCS'06) for classical weighted expressions and automata.
- Replace the $x!-$ construction of pebWE with a *chop* product $E\,;F$ which evaluates E on the current prefix and F on the current suffix.