WEIGHTED TILING SYSTEMS

EVALUATION COMPLEXITY

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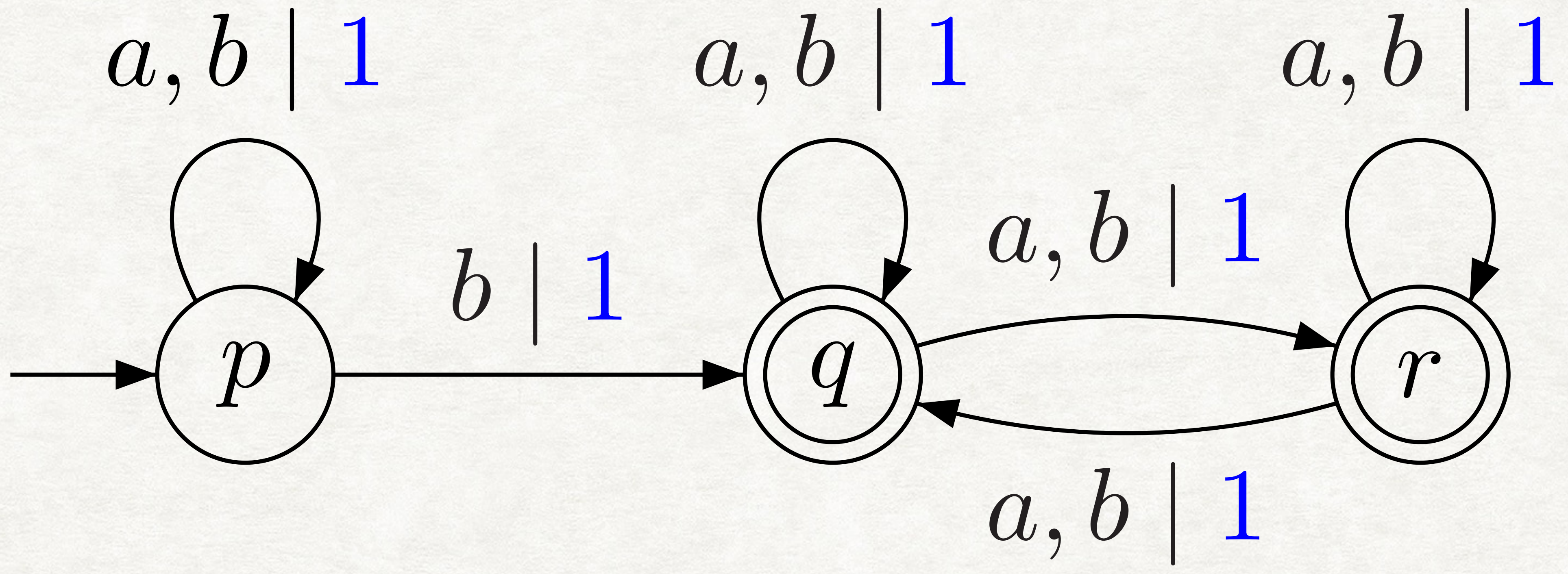
ENS Paris-Saclay

IRL ReLaX

Joint work with C. Aiswarya, Chennai Mathematical Institute

WEIGHTED WORD AUTOMATA



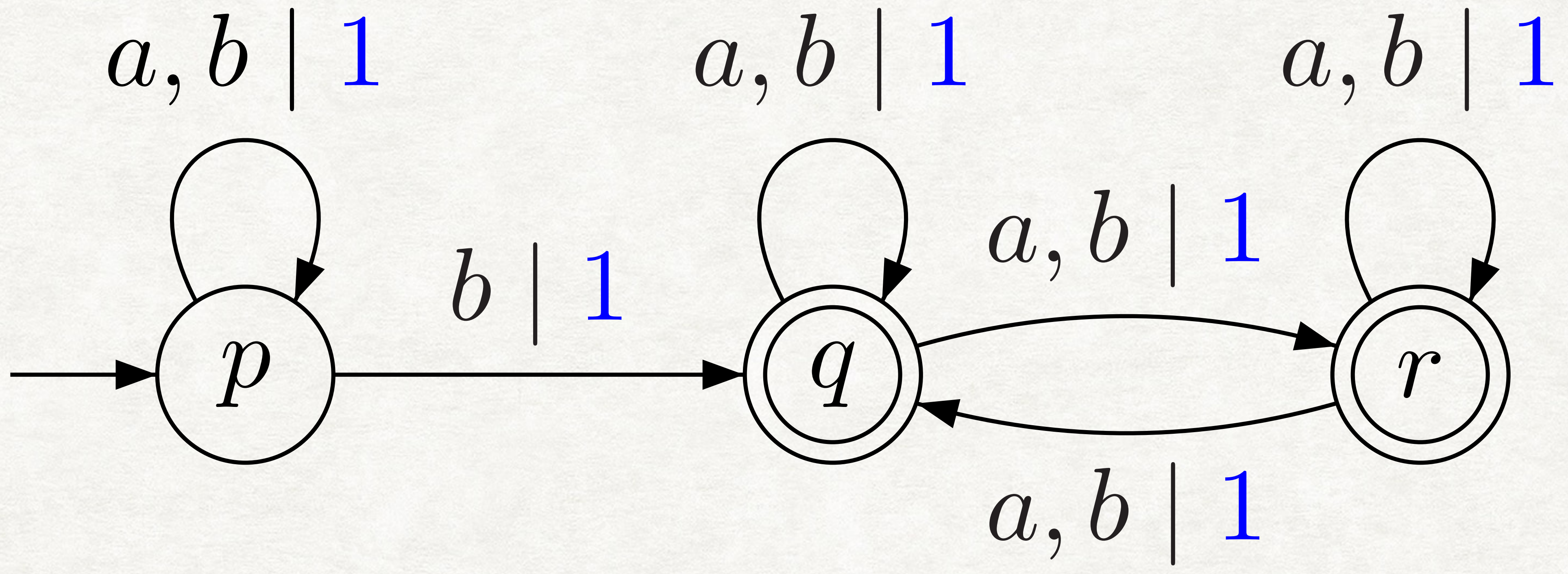
WEIGHTED WORD AUTOMATA

$w = abbab$

$\rho = p \xrightarrow{a} p \xrightarrow{b} p \xrightarrow{b} q \xrightarrow{a} r \xrightarrow{b} r$

$\text{weight}(\rho) = 1 \times 1 \times 1 \times 1 \times 1$

$[\![\mathcal{A}]\!](w) = \sum_{\rho} \text{weight}(\rho)$



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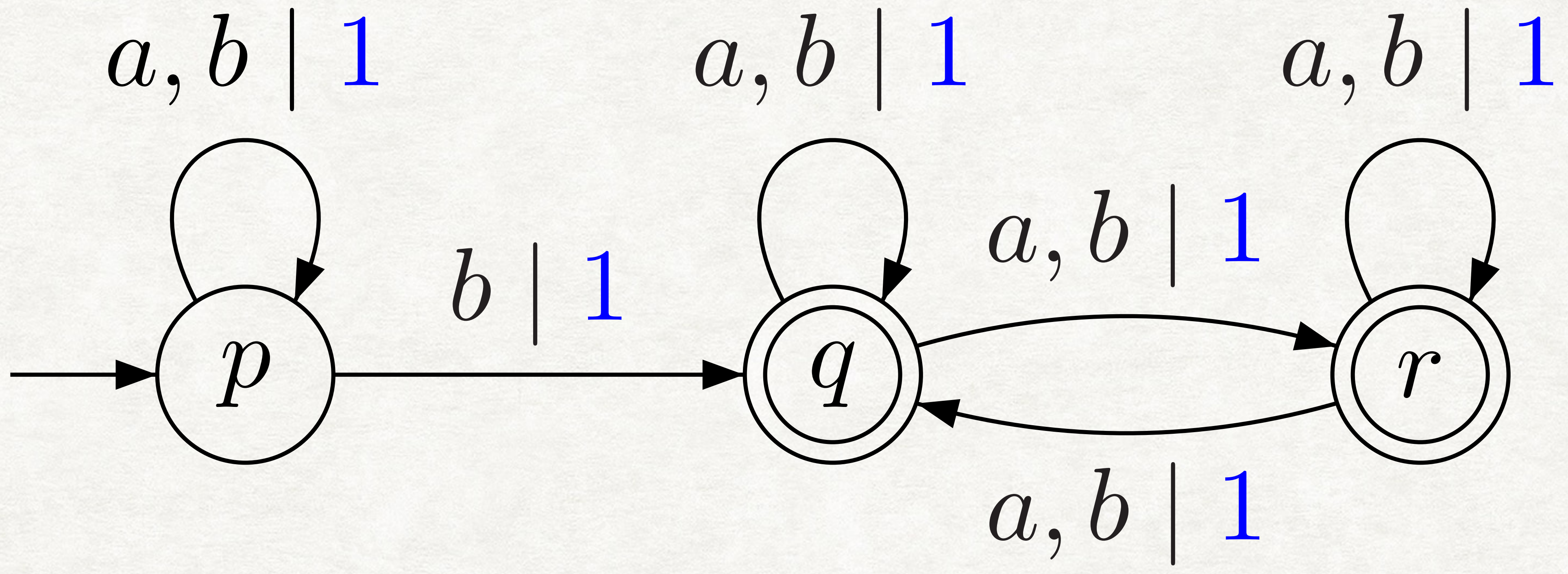

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COMPLEXITY OF THE EVALUATION PROBLEM?

Naive evaluation: $\mathcal{O}(|w| \cdot |Q|^{|w|+1})$

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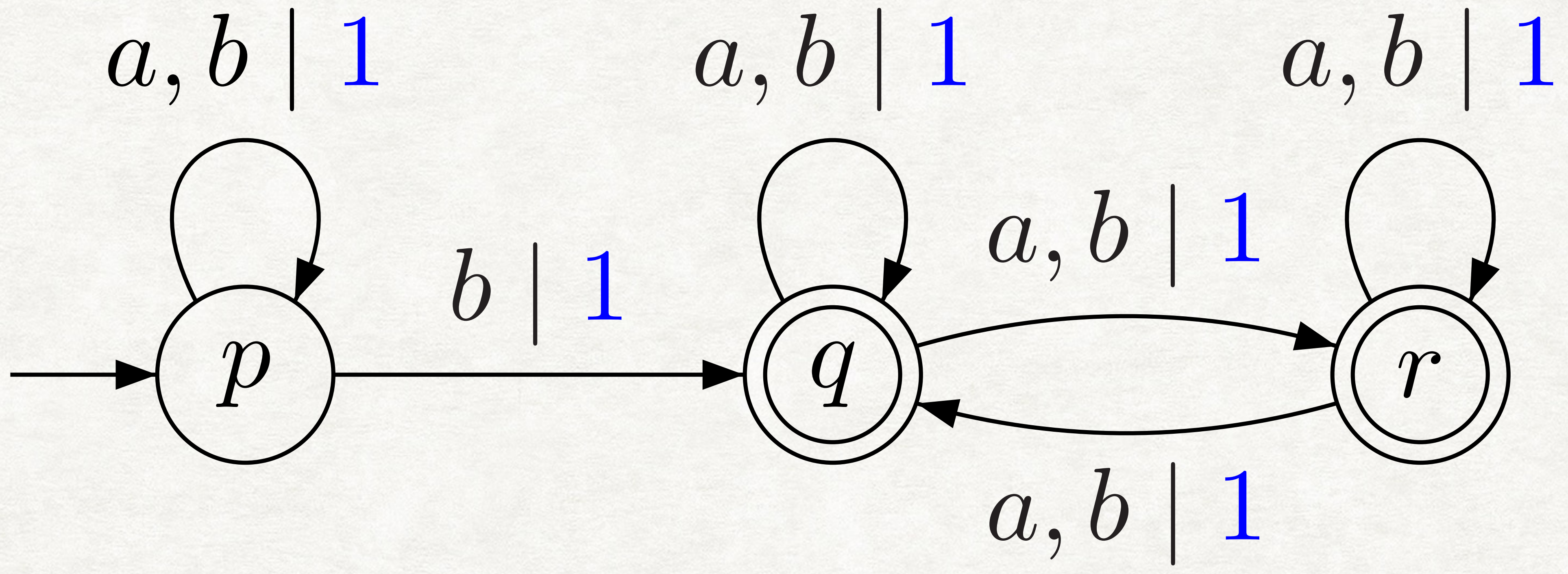

$$M_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad M_b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

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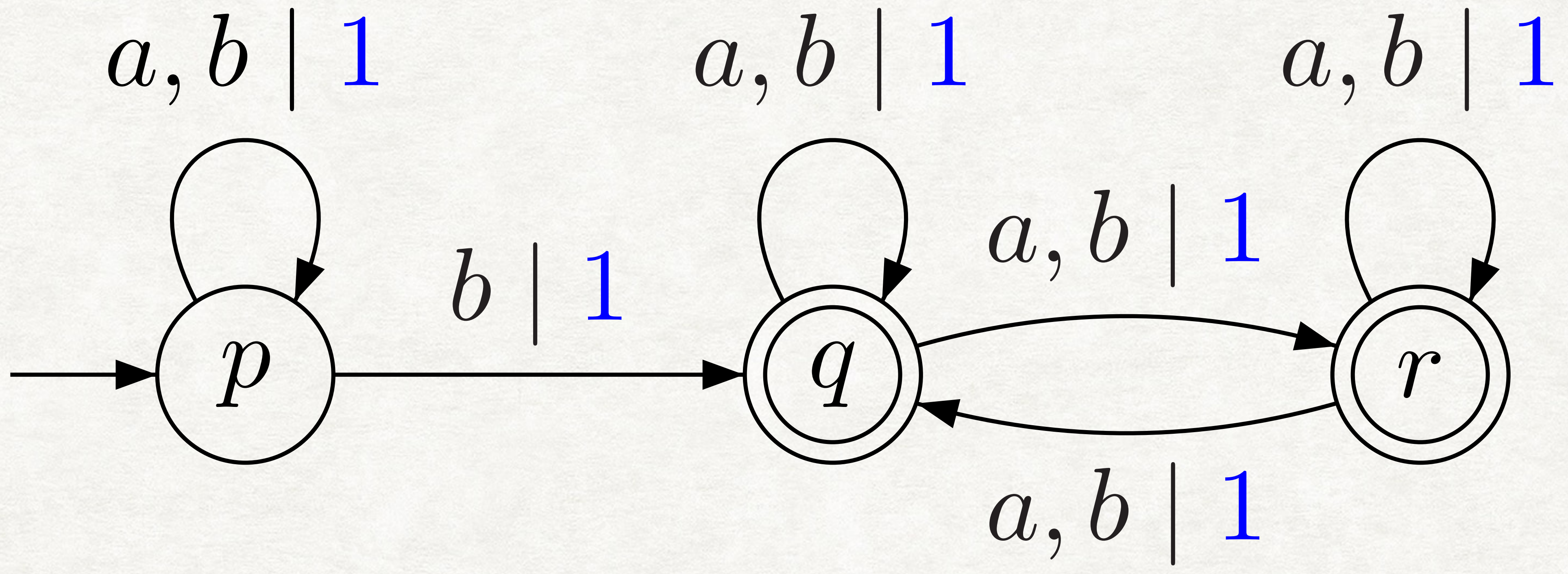

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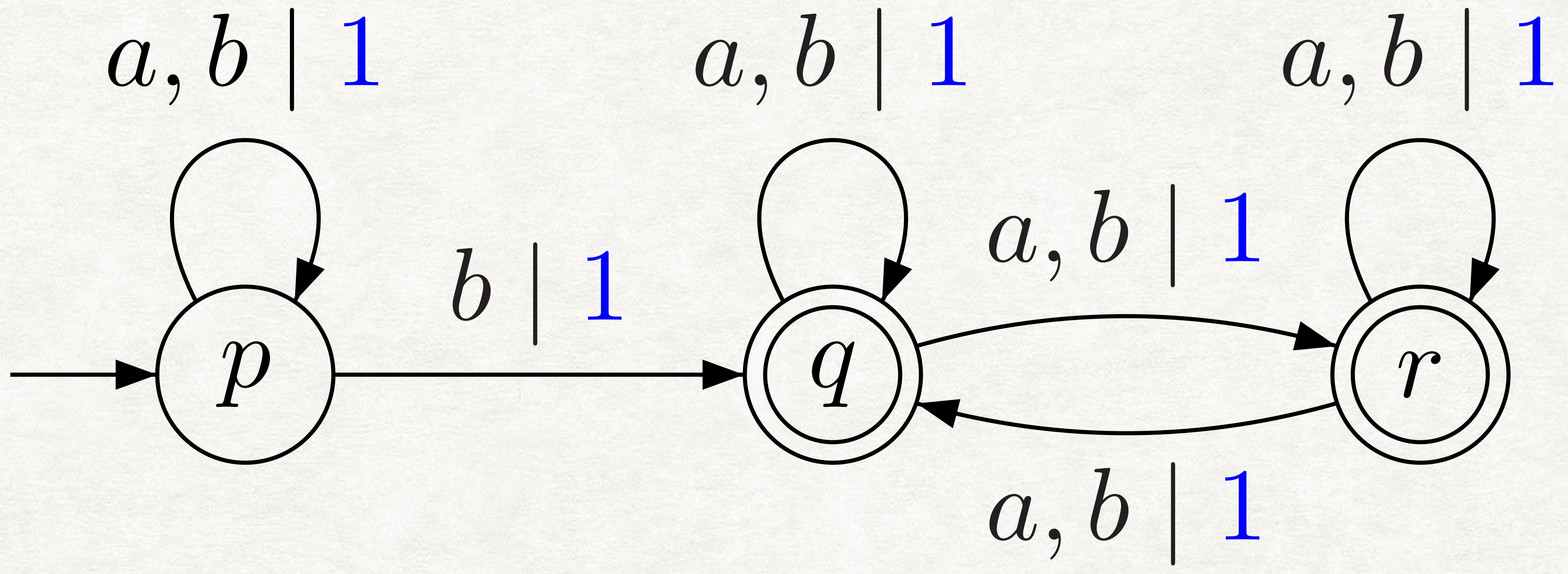
$$[\![\mathcal{A}]\!](abbab) = I \times M_a \times M_b \times M_b \times M_a \times M_b \times F = 13$$

WEIGHTED WORD AUTOMATA



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$$[\![\mathcal{A}]\!](abbab) = I \times M_a \times M_b \times M_b \times M_a \times M_b \times F = 13$$

$$I \xrightarrow{M_a} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \xrightarrow{M_b} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \xrightarrow{M_b} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$
$$\xrightarrow{M_a} \begin{pmatrix} 1 & 3 & 3 \end{pmatrix} \xrightarrow{M_b} \begin{pmatrix} 1 & 7 & 6 \end{pmatrix} \xrightarrow{F} 13$$

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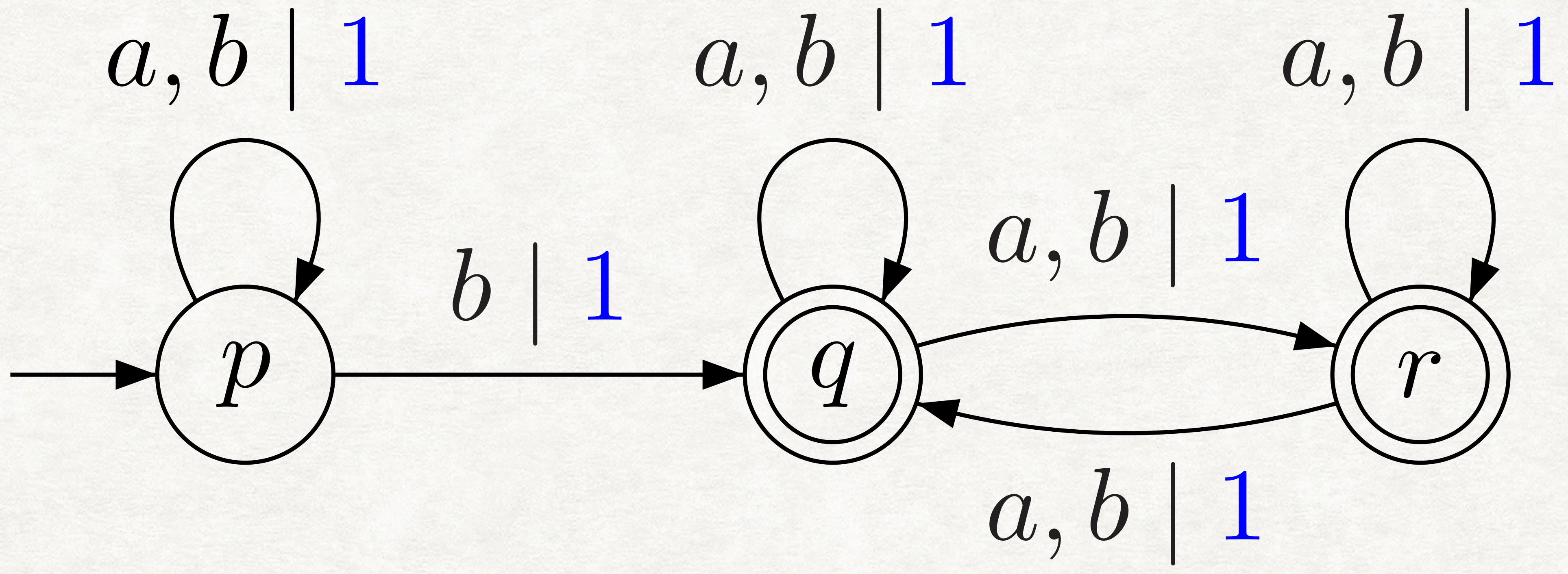

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Efficient evaluation: $\mathcal{O}(|w| \cdot |Q|^2)$

$\mathcal{O}(|w| \cdot |\mathcal{A}|)$

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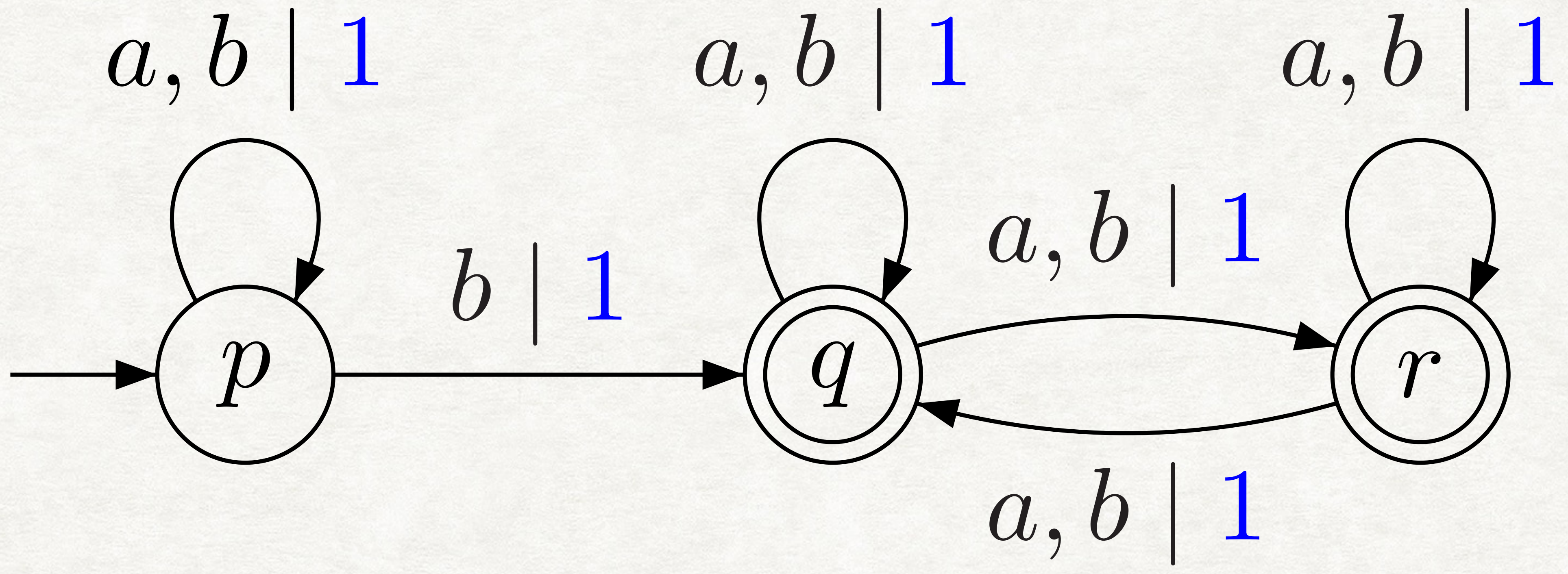

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COMPLEXITY OF THE EVALUATION PROBLEM?

Naive evaluation: $\mathcal{O}(|w| \cdot |Q|^{|w|+1})$

$w = ubv\colon \qquad p \xrightarrow{u} p \xrightarrow{b} q \xrightarrow{v} \{q, r\} \qquad 2^{|v|}$ runs

$[\![\mathcal{A}]\!](abbab) = 2^3 + 2^2 + 2^0 = 13$

$[\![\mathcal{A}]\!](abbab) = I \times M_a \times M_b \times M_b \times M_a \times M_b \times F = 13$

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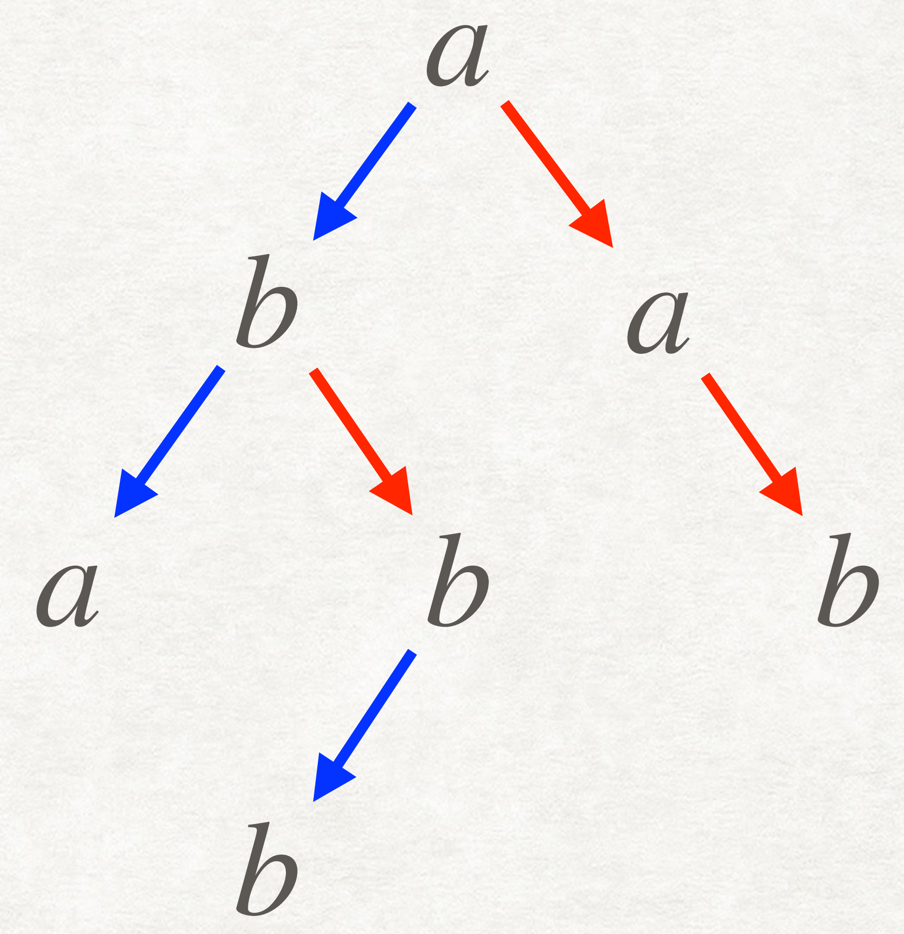
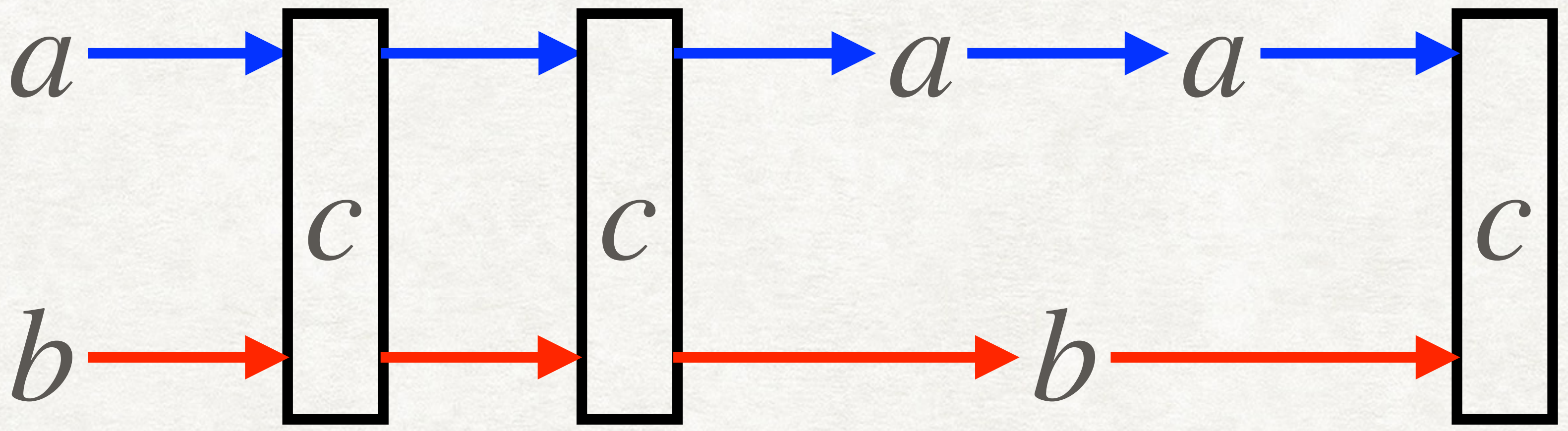
$\mathcal{O}(|w| \cdot |\mathcal{A}|)$

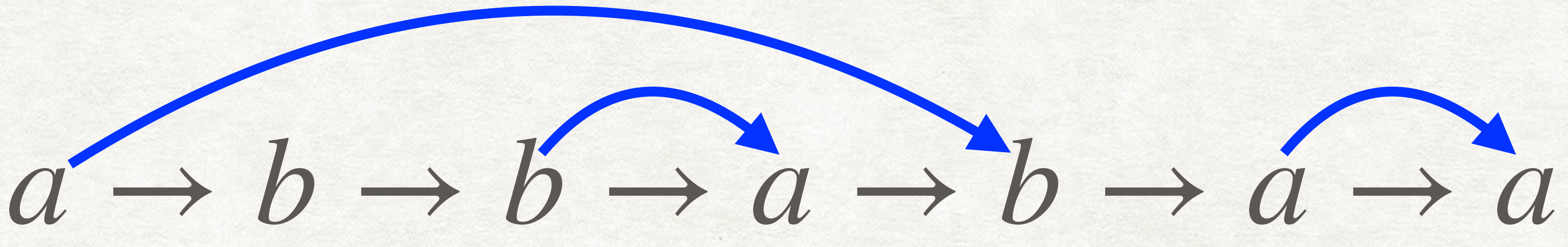
OTHER STRUCTURES

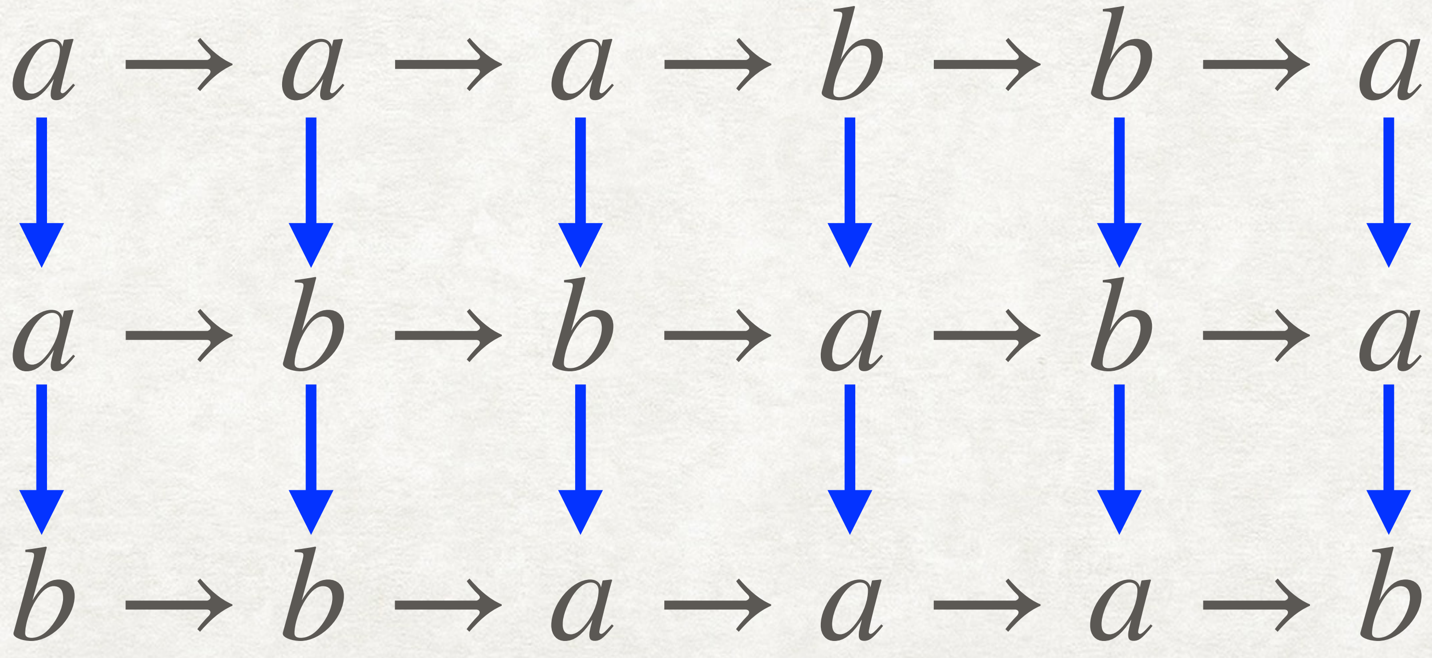
- Words
- Trees
- Mazurkiewicz traces
- Message sequence charts
- Nested words
- Multiply nested words
- Pictures (grids)

$$a \to b \to b \to a \to b \to a \to a$$









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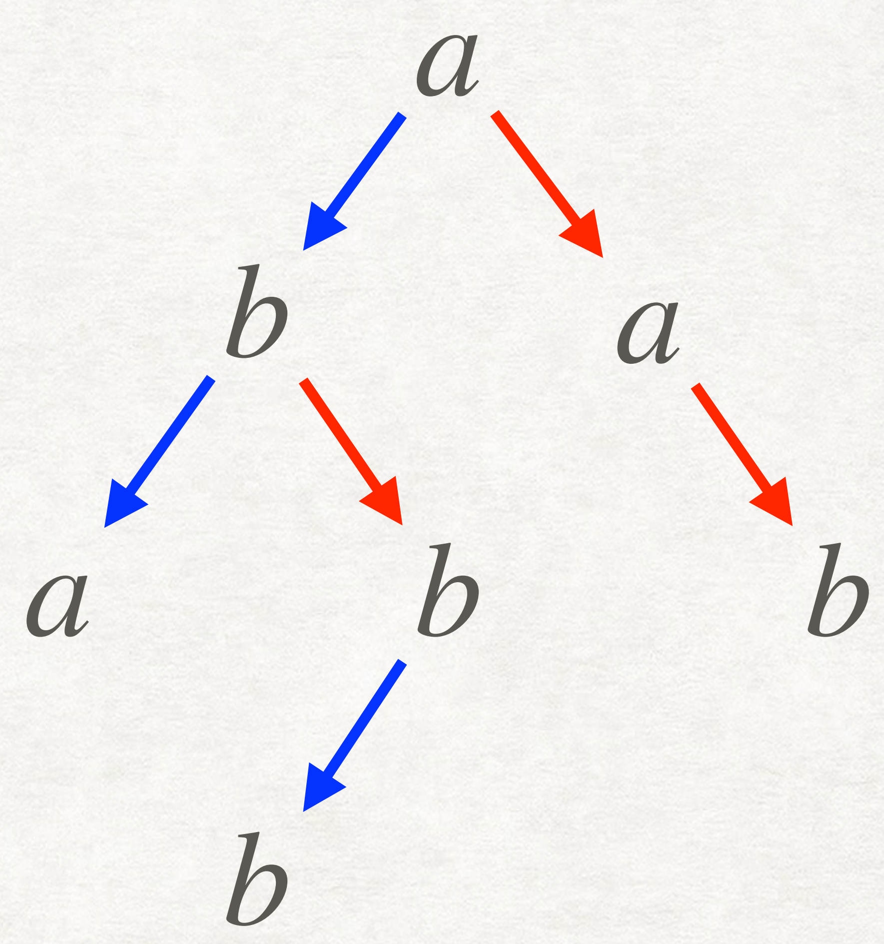
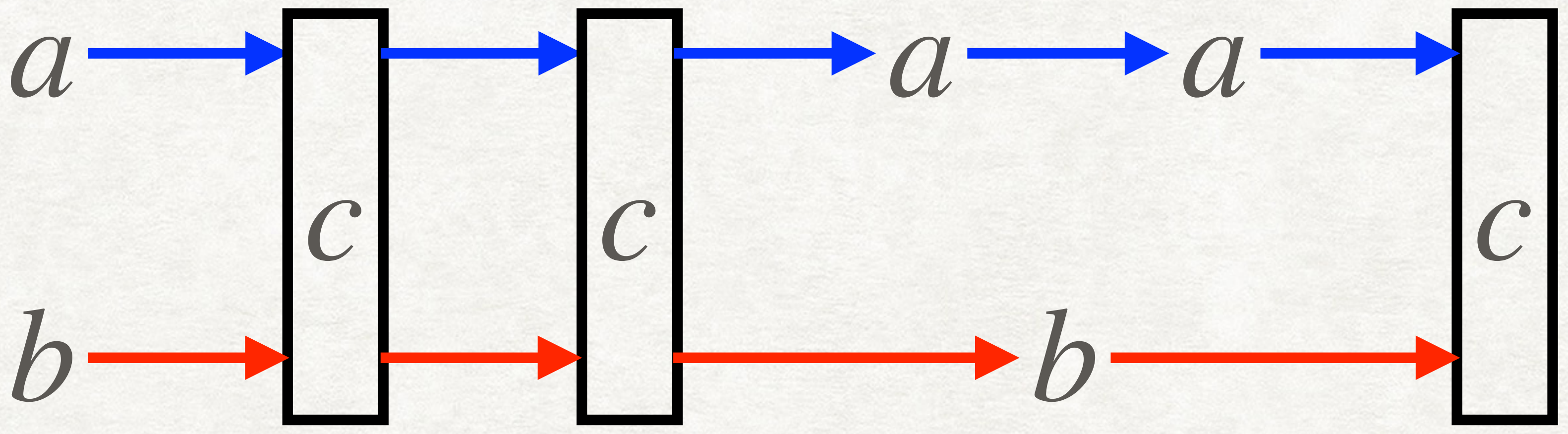
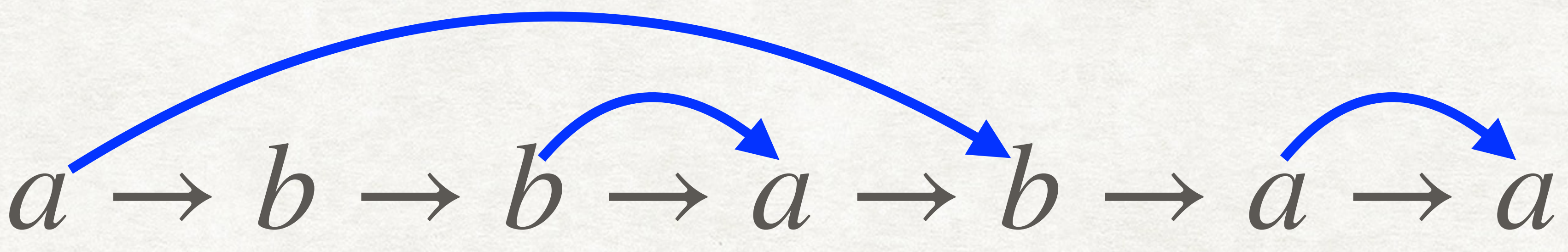
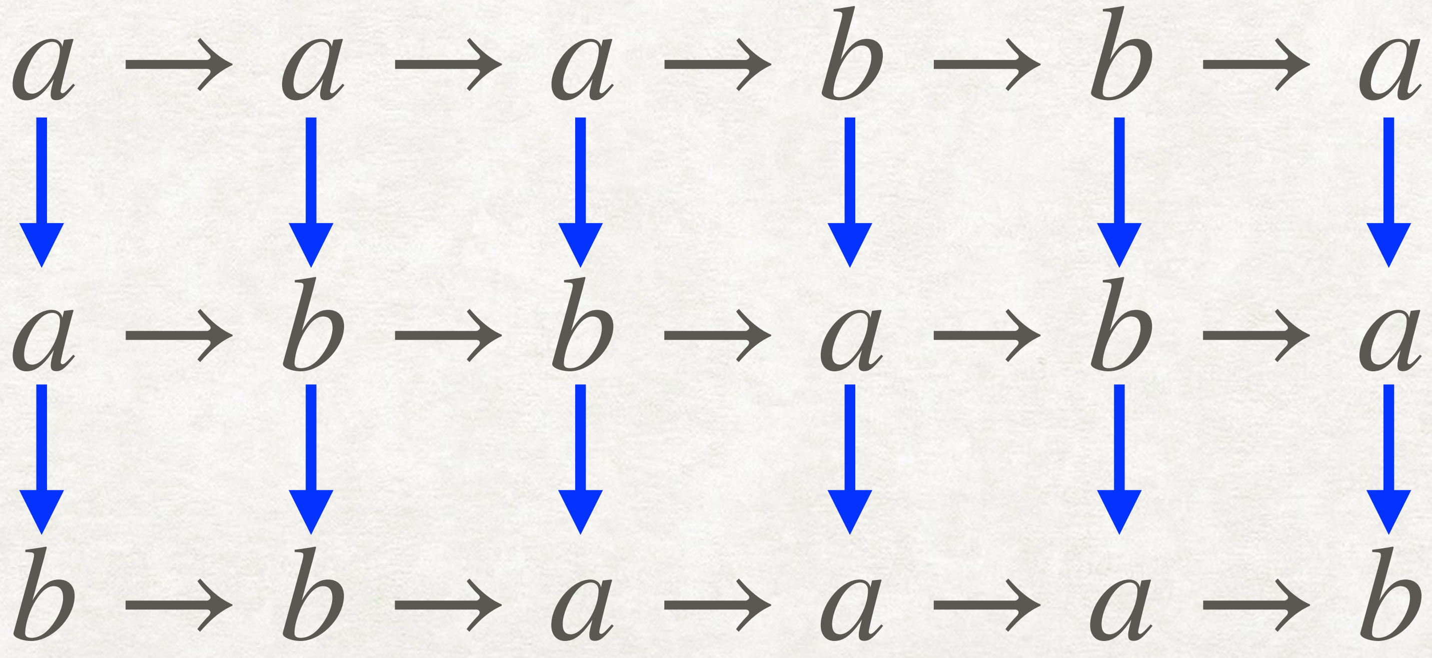








EVALUATION PROBLEM

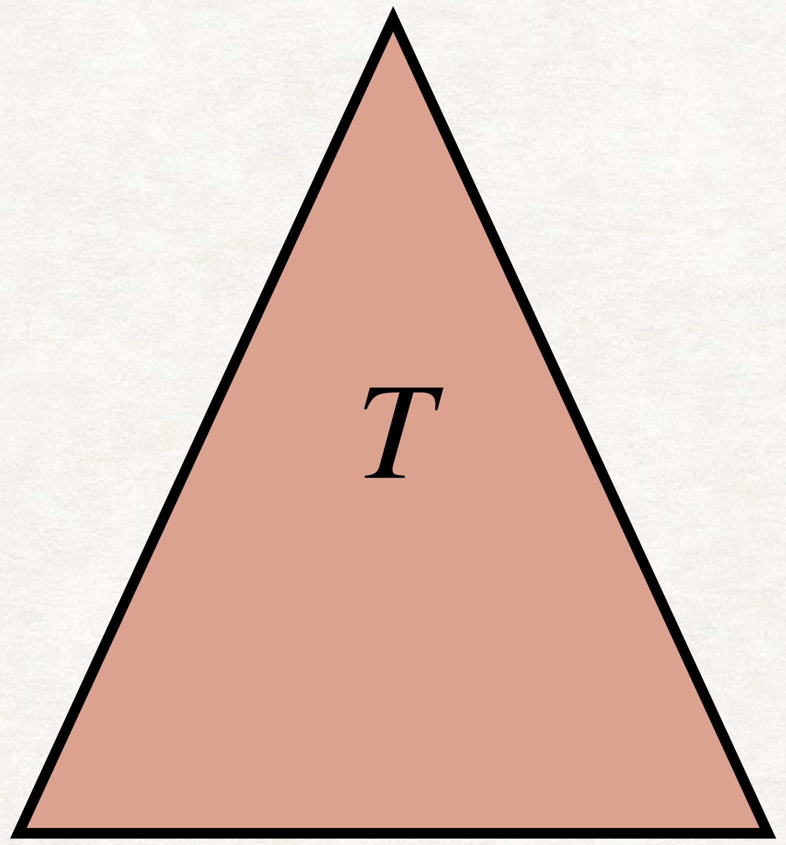
WEIGHTED TREE AUTOMATA

A run on a tree T is a map $\rho\colon$ Nodes $\to$ States

$$\text{weight}(\rho) = \prod \text{weight(transitions)}$$

$$[\![\mathscr{A}]\!](T) = \sum_{\rho} \text{weight}(\rho)$$

Naive evaluation: $\mathcal{O}(|T| \cdot |Q|^{|T|})$



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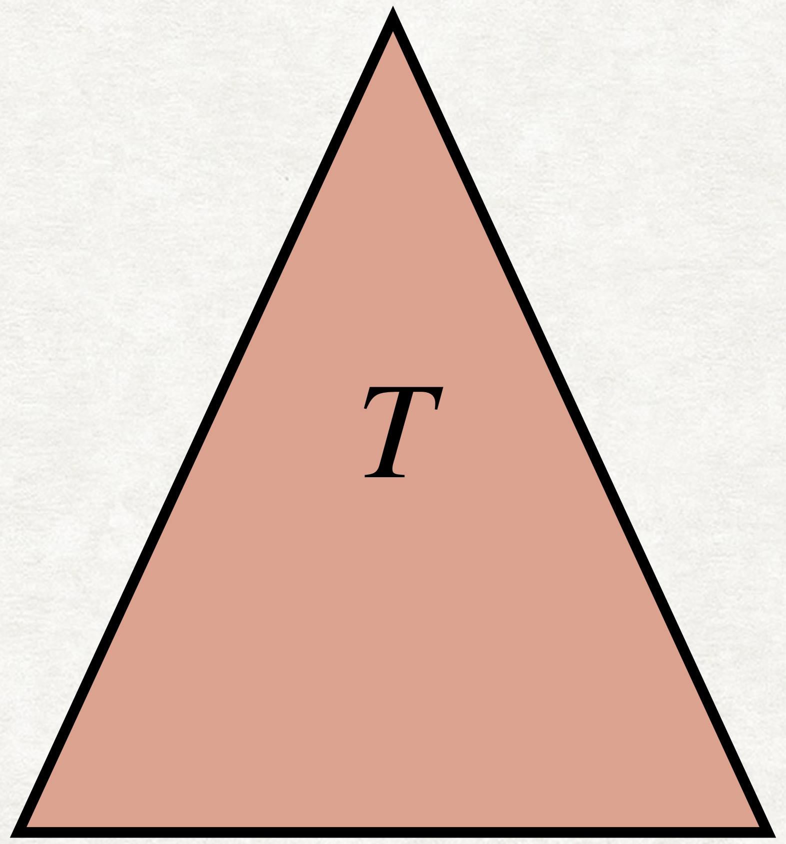
$$x_q = \sum_{\rho} \text{weight}(\rho)$$

ρ run on T with state q at the root

(x_p, x_q, x_r)



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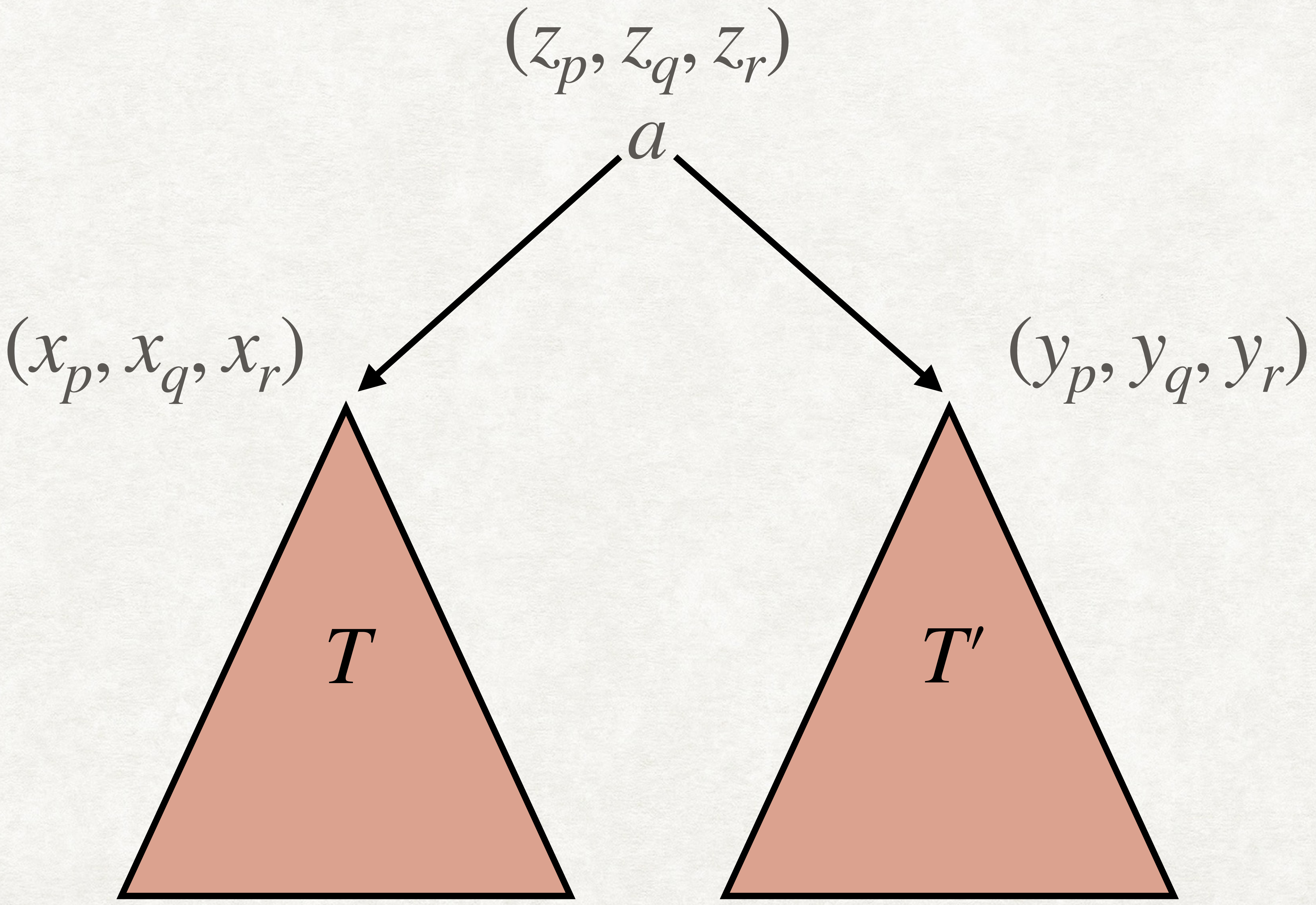

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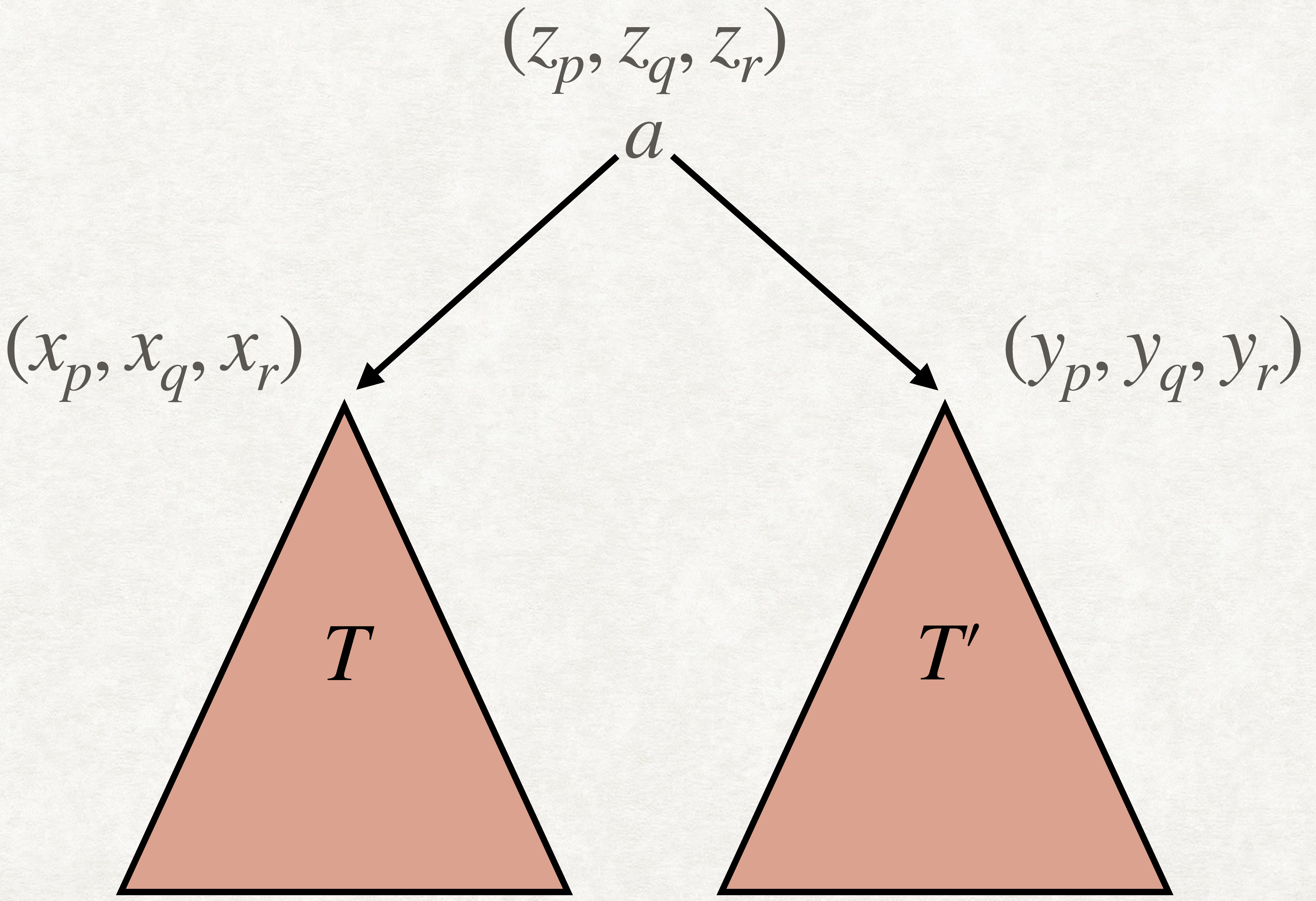

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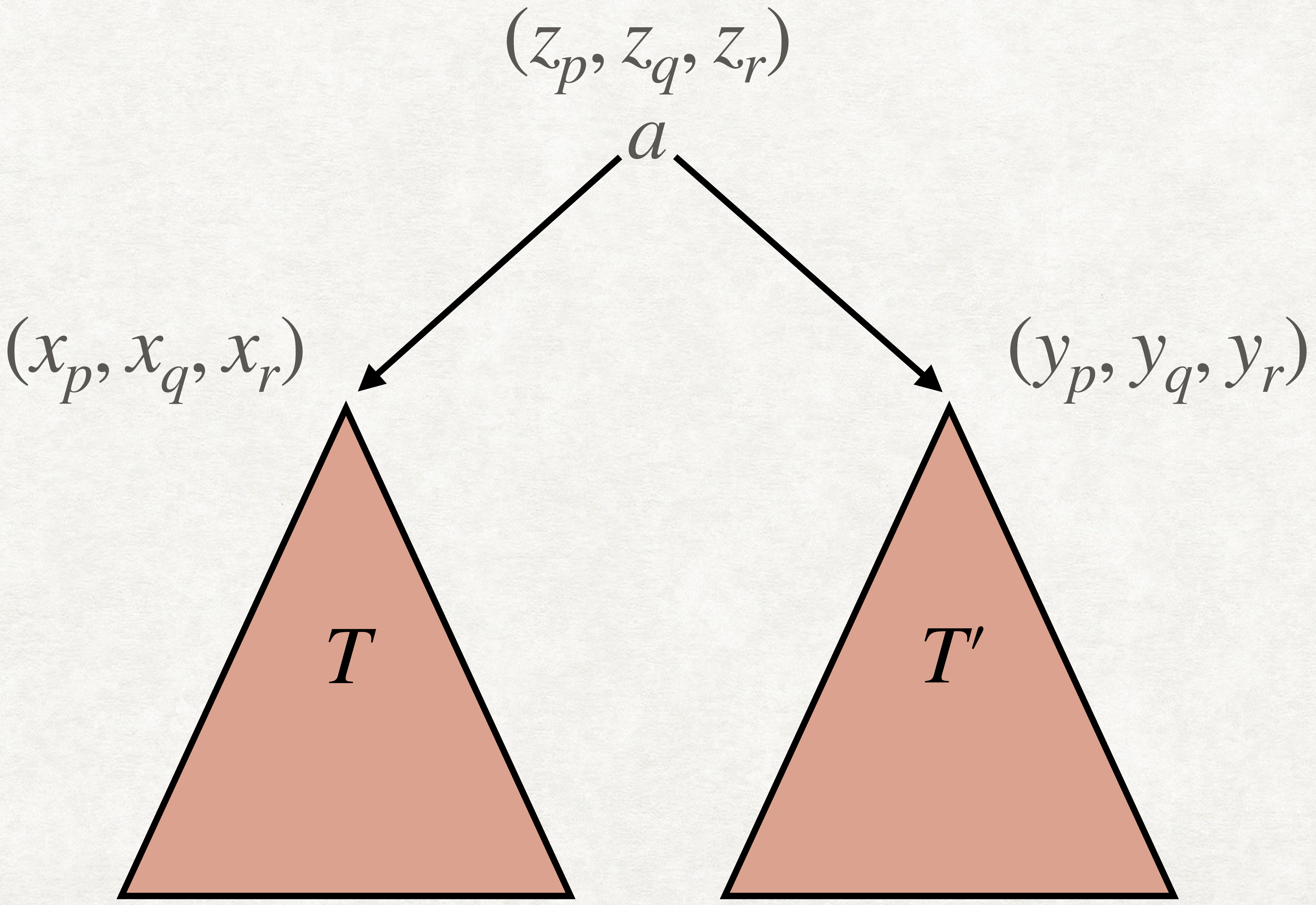

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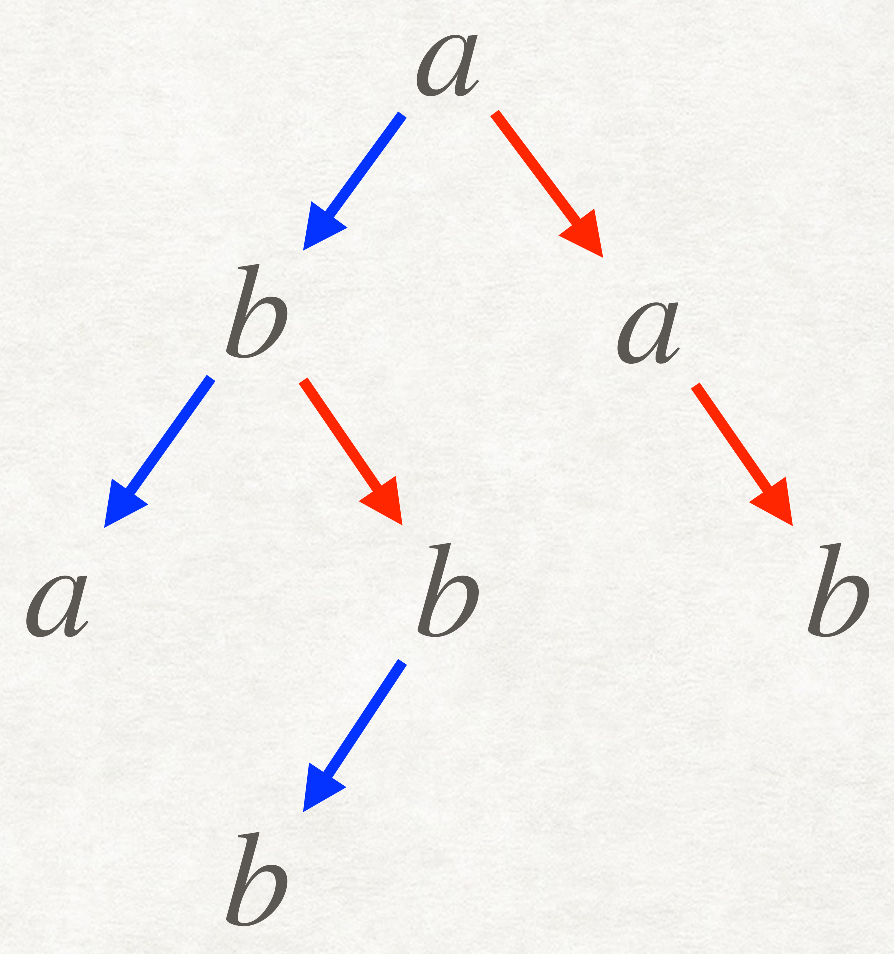
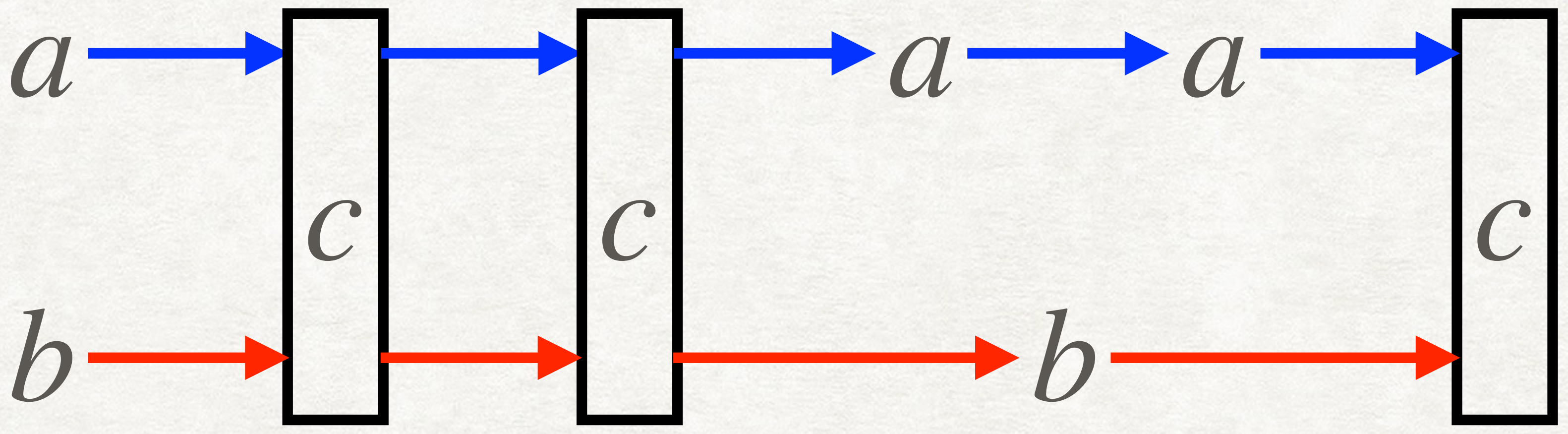
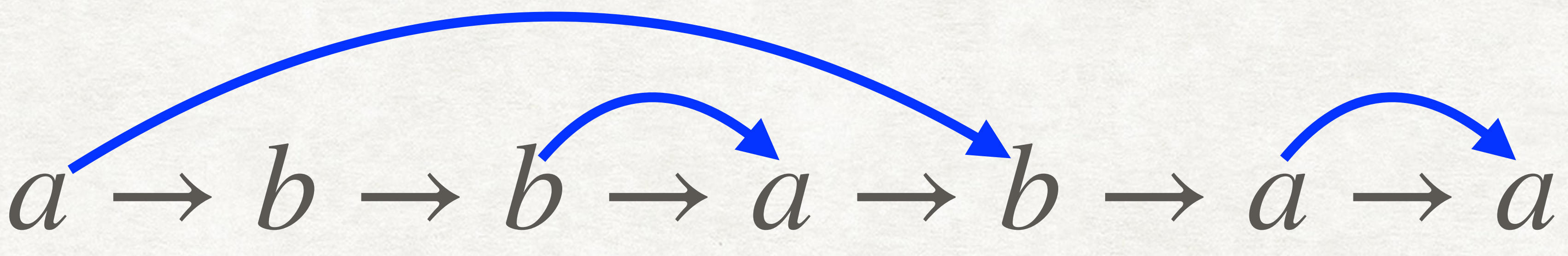
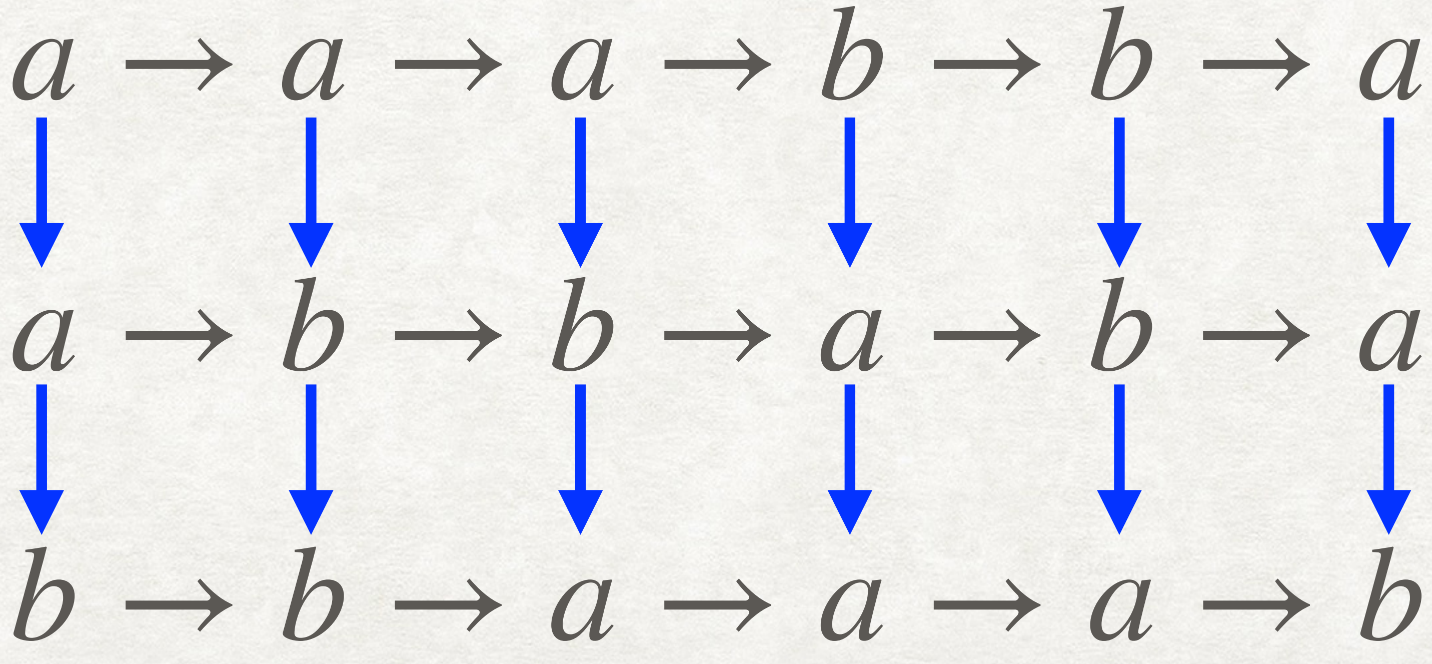


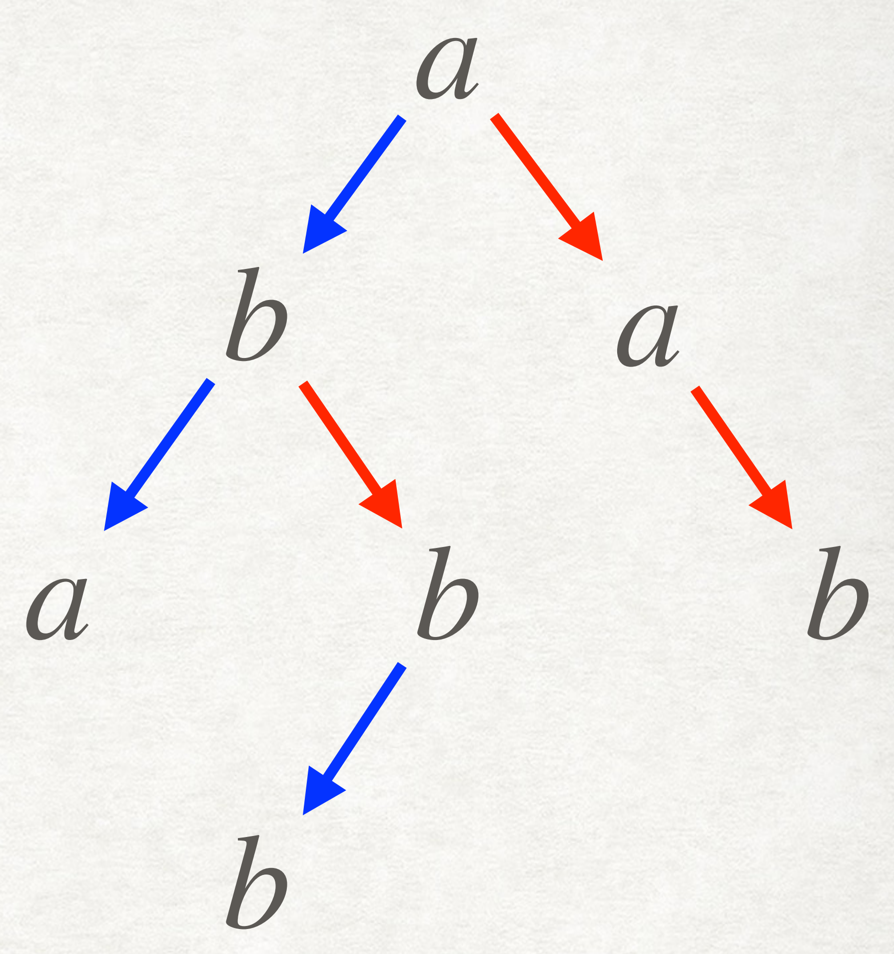






EVALUATION PROBLEM

OTHER STRUCTURES: LABELLED GRAPHS



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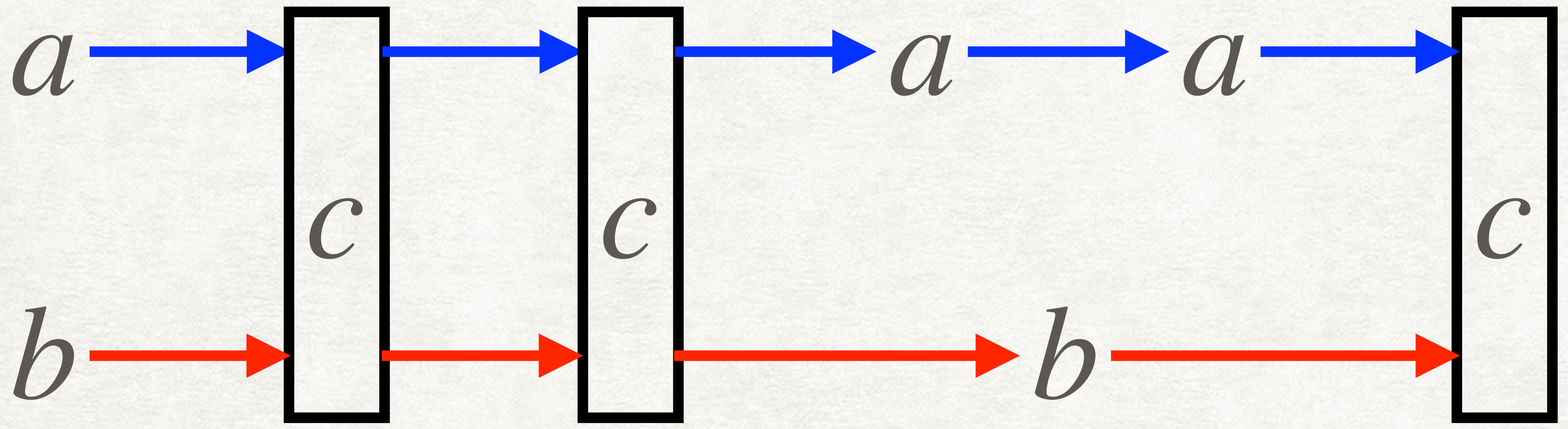

$$\begin{tikzcd}
a \arrow[r] \arrow[rrrr, bend left] & b \arrow[r] & b \arrow[r] \arrow[r, bend left] & a \arrow[r] & b \arrow[r] & a \arrow[r] \arrow[r, bend left] & a
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & a \arrow[d] \\
a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & a \arrow[d] \\
b \arrow[r] & b \arrow[r] & a \arrow[r] & a \arrow[r] & a \arrow[r] & b
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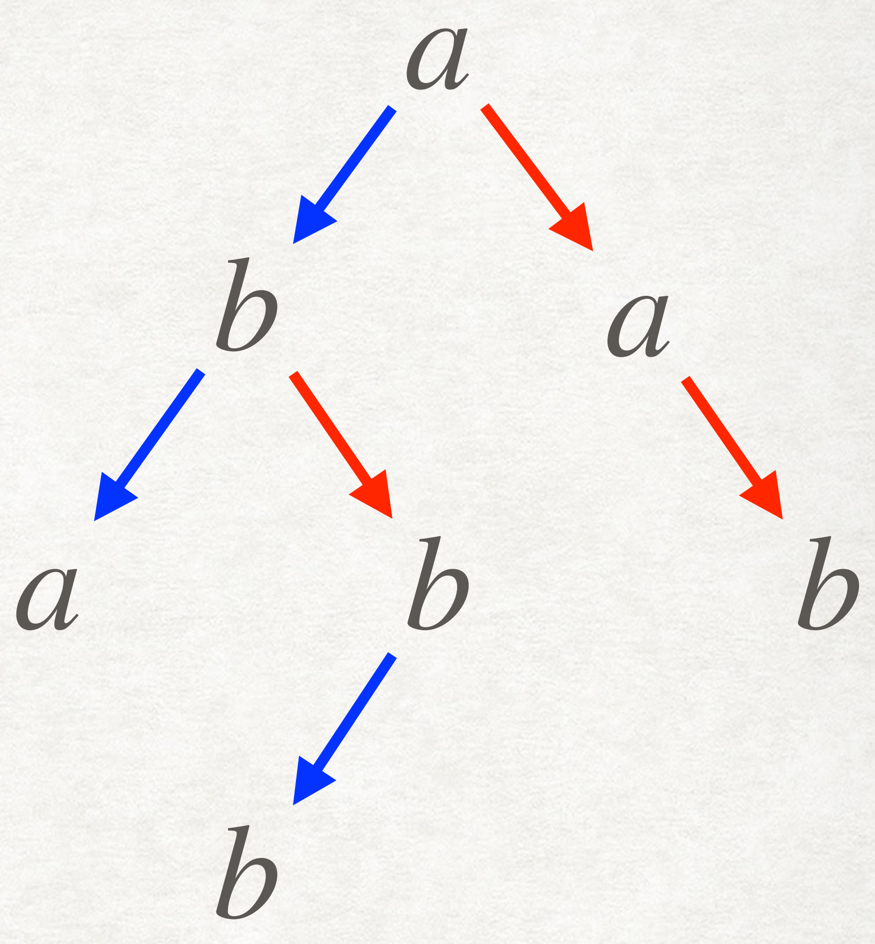

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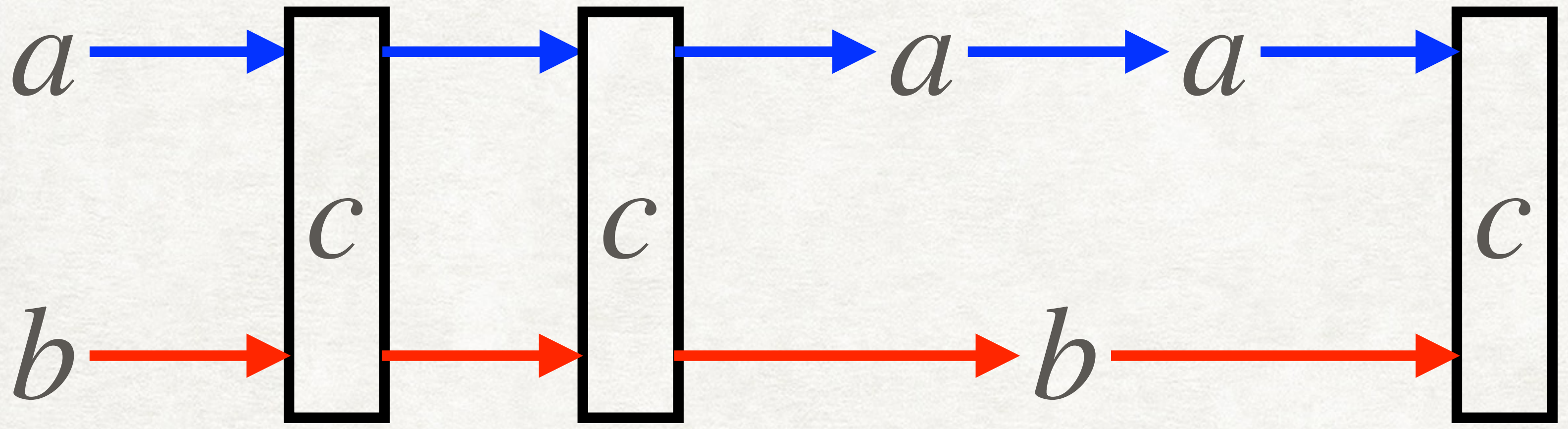

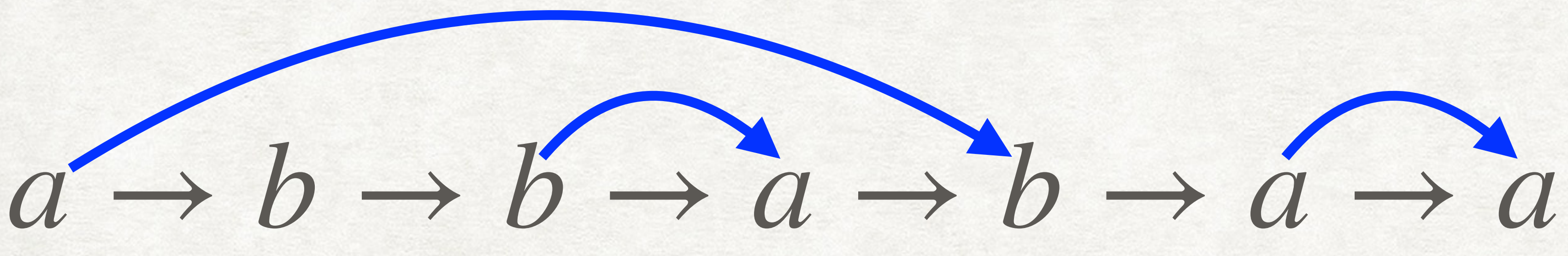

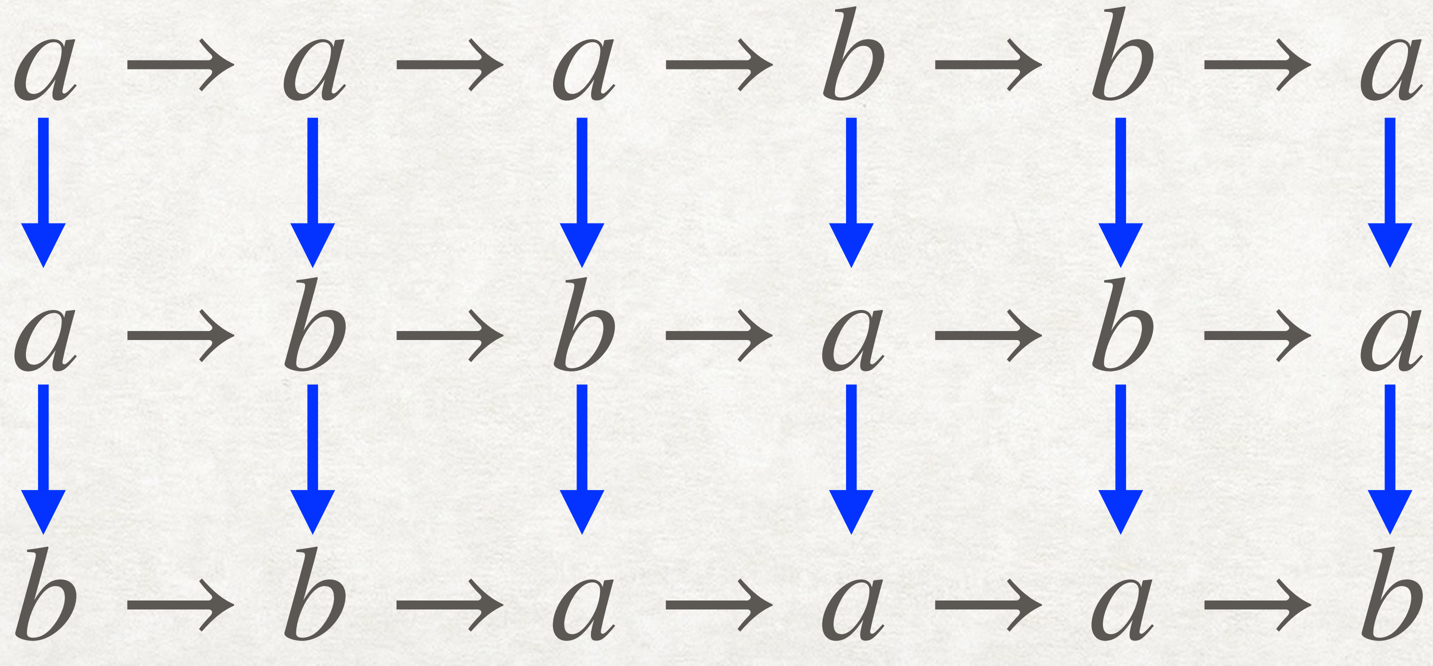

$$G = \left(V, (V_\sigma)_{\sigma \in \Sigma}, (E_\gamma)_{\gamma \in \Gamma}\right)$$

- V: Vertices
- Σ: Vertex labels $\quad V_\sigma \subseteq V$
- Γ: Edge labels $\quad E_\gamma \subseteq V \times V$

WEIGHTED TILING SYSTEMS

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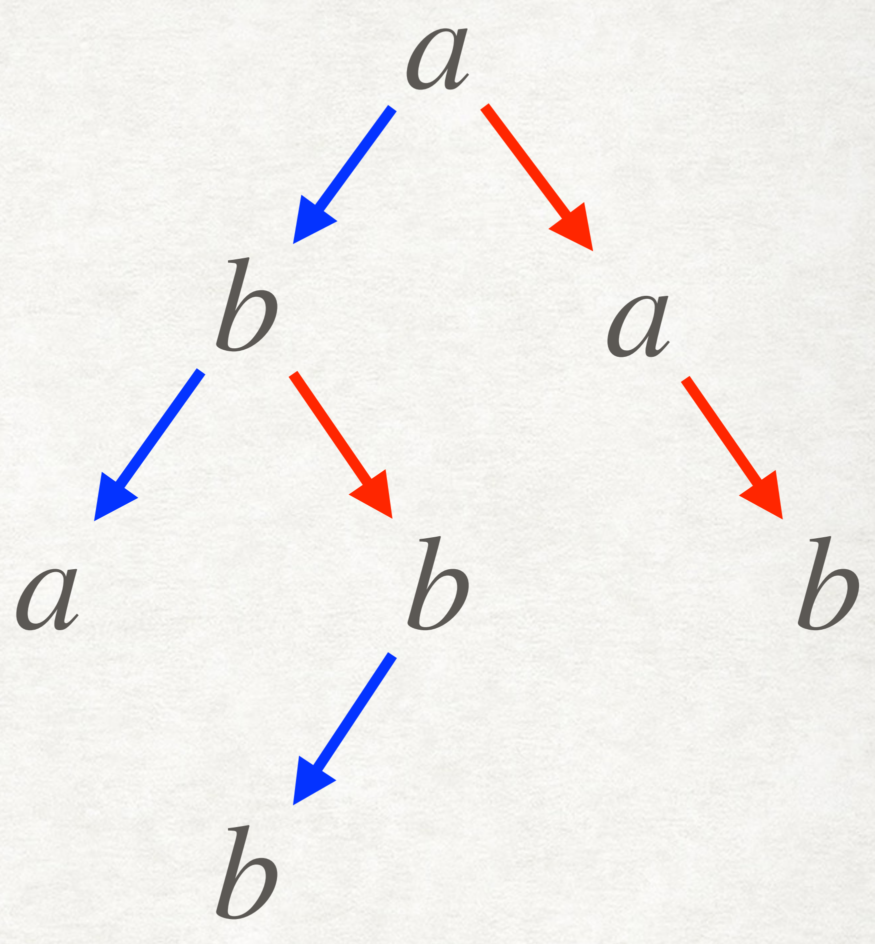

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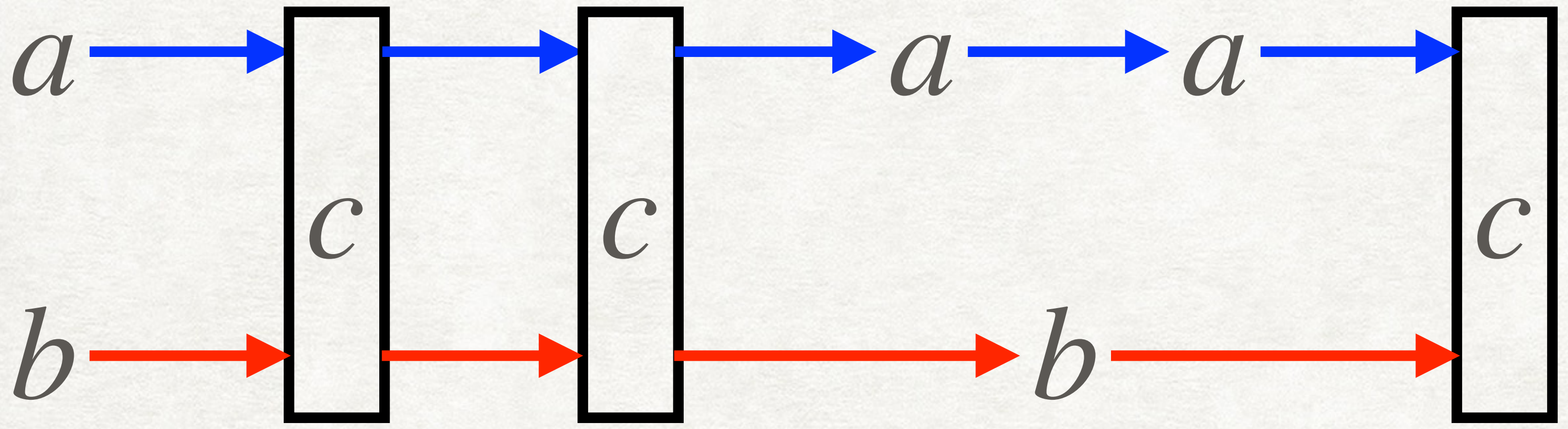

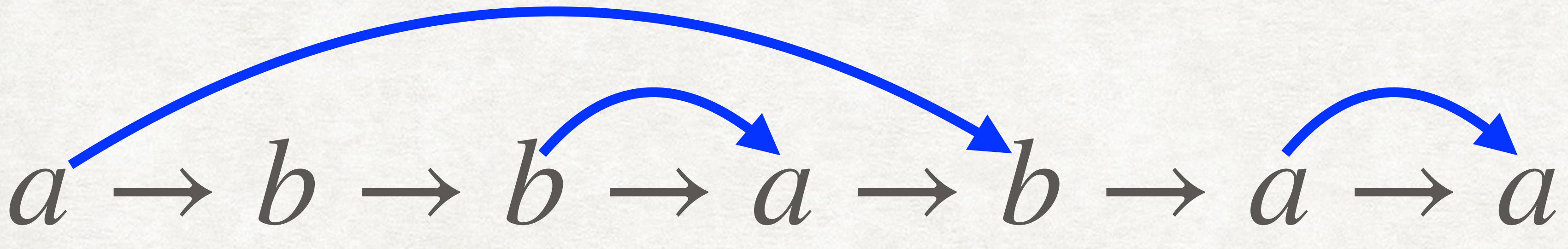

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & a \arrow[d] \\ a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & b \arrow[r] \arrow[d] & a \arrow[d] \\ b \arrow[r] & b \arrow[r] & a \arrow[r] & a \arrow[r] & a \arrow[r] & b \end{tikzcd}$$

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EVALUATION PROBLEM: COMPLEXITY?

- Words
- Trees
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- Message sequence charts
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- Pictures (grids)

$$\text{Boolean} = (\{0,1\}, \vee, \wedge, 0, 1)$$

$$\text{Natural} = (\mathbb{N}, +, \times, 0, 1)$$
$$\text{Integer} = (\mathbb{Z}, +, \times, 0, 1)$$
$$\text{Rational} = (\mathbb{Q}, +, \times, 0, 1)$$

$$\text{max-plus-}\mathbb{N} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$$
$$\text{max-plus-}\mathbb{Z} = (\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$$
$$\text{min-plus-}\mathbb{N} = (\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$$
$$\text{min-plus-}\mathbb{Z} = (\mathbb{Z} \cup \{+\infty\}, \min, +, +\infty, 0)$$

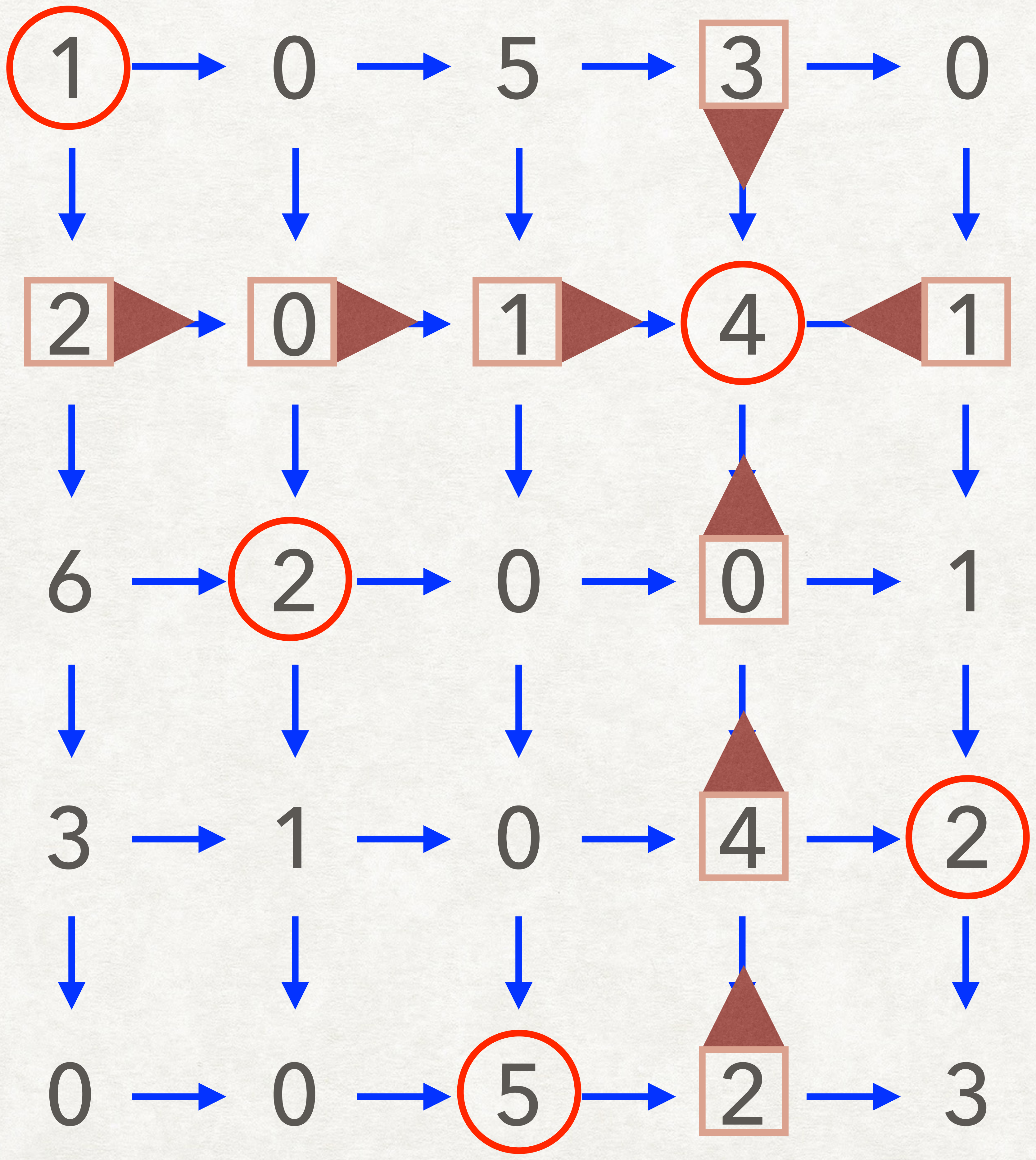
WEIGHTED TILING SYSTEM

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 5 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & 0 \arrow[d] \\
2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 4 \arrow[r] \arrow[d] & 1 \arrow[d] \\
6 \arrow[r] \arrow[d] & 2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\
3 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 4 \arrow[r] \arrow[d] & 2 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 5 \arrow[r] & 2 \arrow[r] & 3
\end{tikzcd}
$$

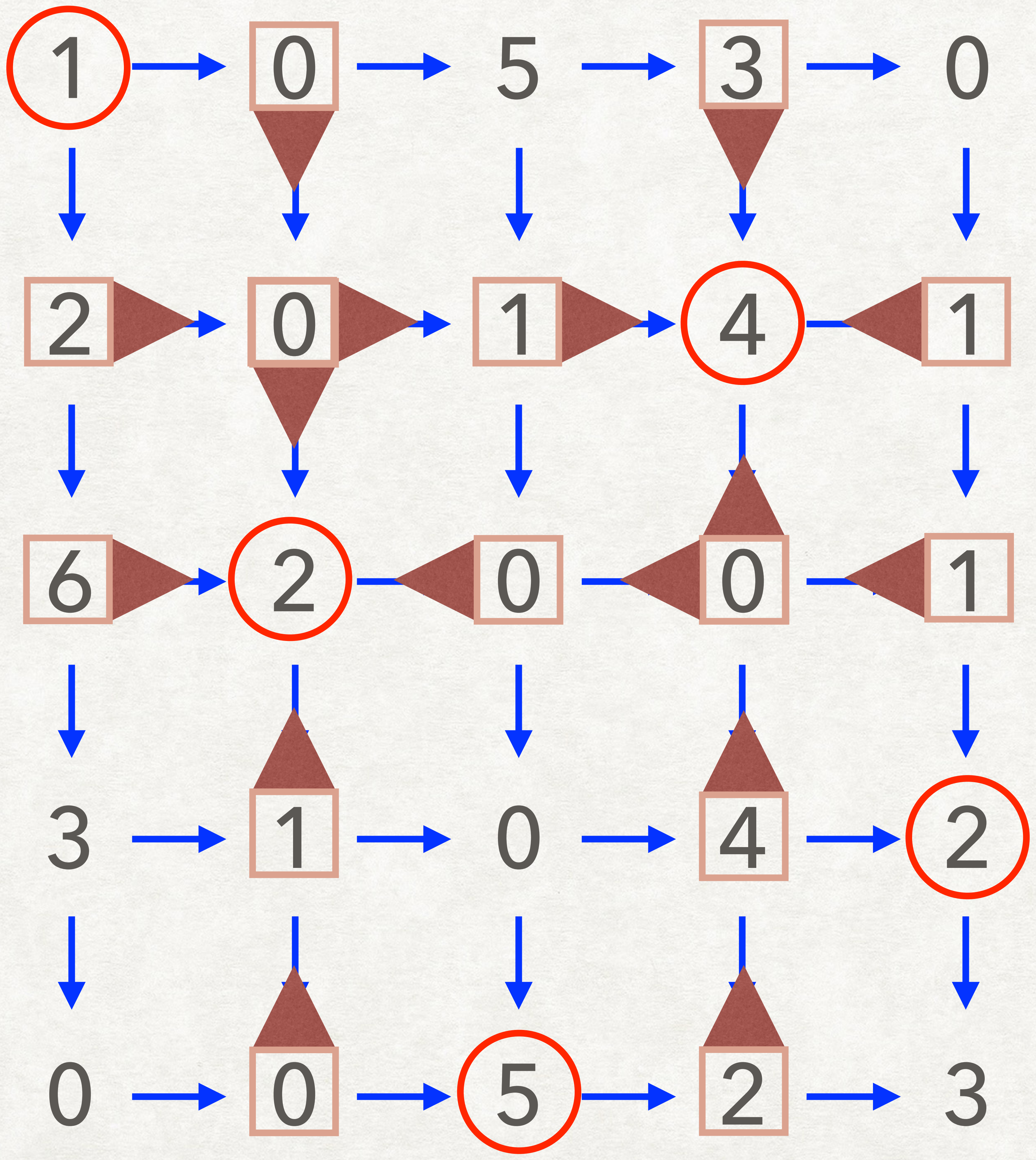
WEIGHTED TILING SYSTEM

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 5 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & 0 \arrow[d] \\
2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 4 \arrow[r] \arrow[d] & 1 \arrow[d] \\
6 \arrow[r] \arrow[d] & 2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\
3 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 4 \arrow[r] \arrow[d] & 2 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 5 \arrow[r] & 2 \arrow[r] & 3
\end{tikzcd}
$$

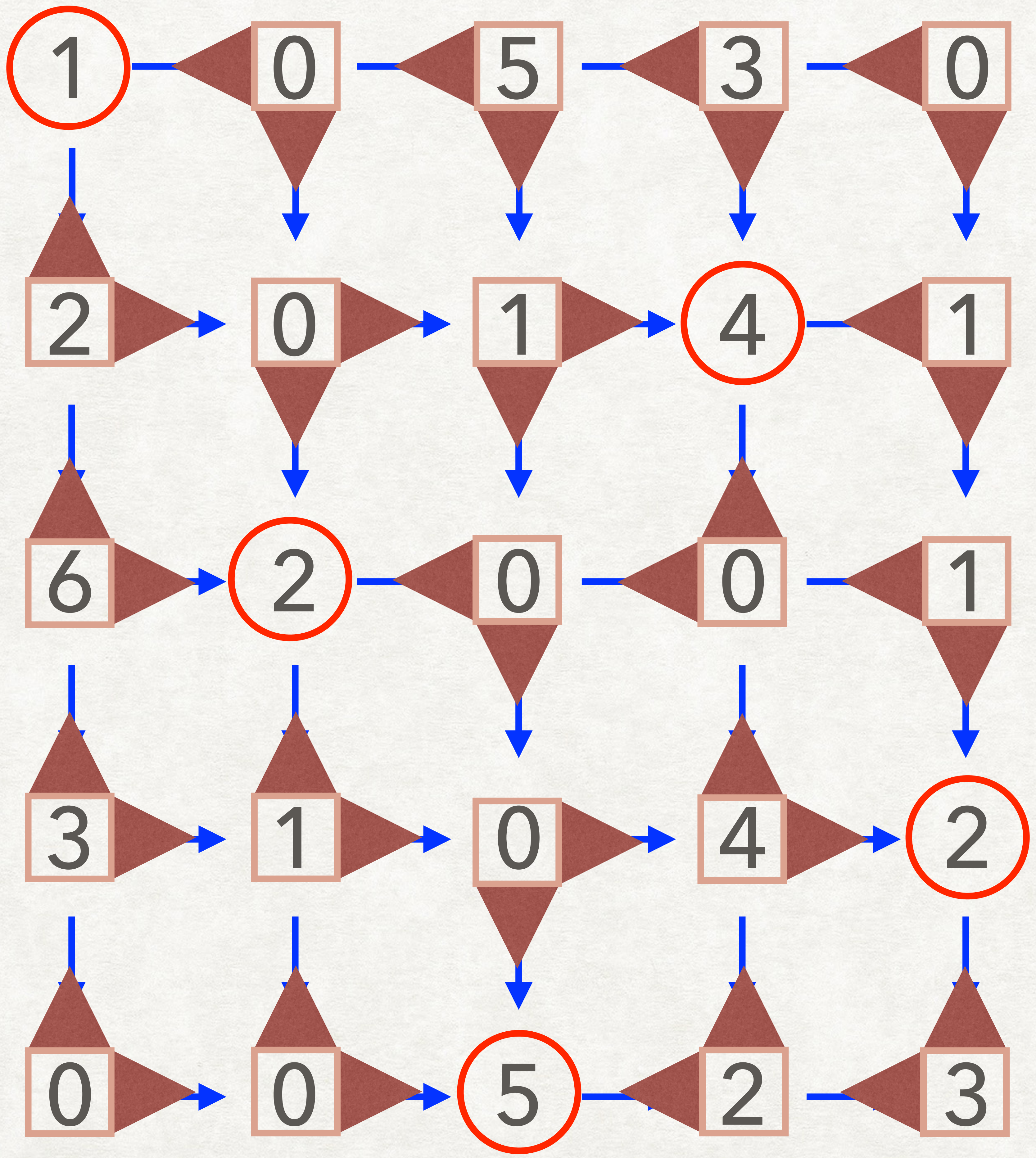
WEIGHTED TILING SYSTEM



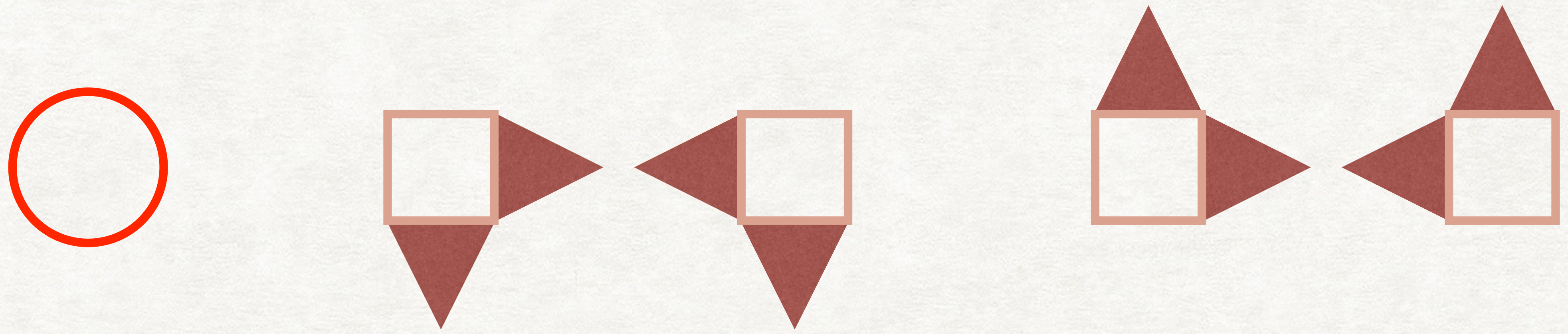
WEIGHTED TILING SYSTEM



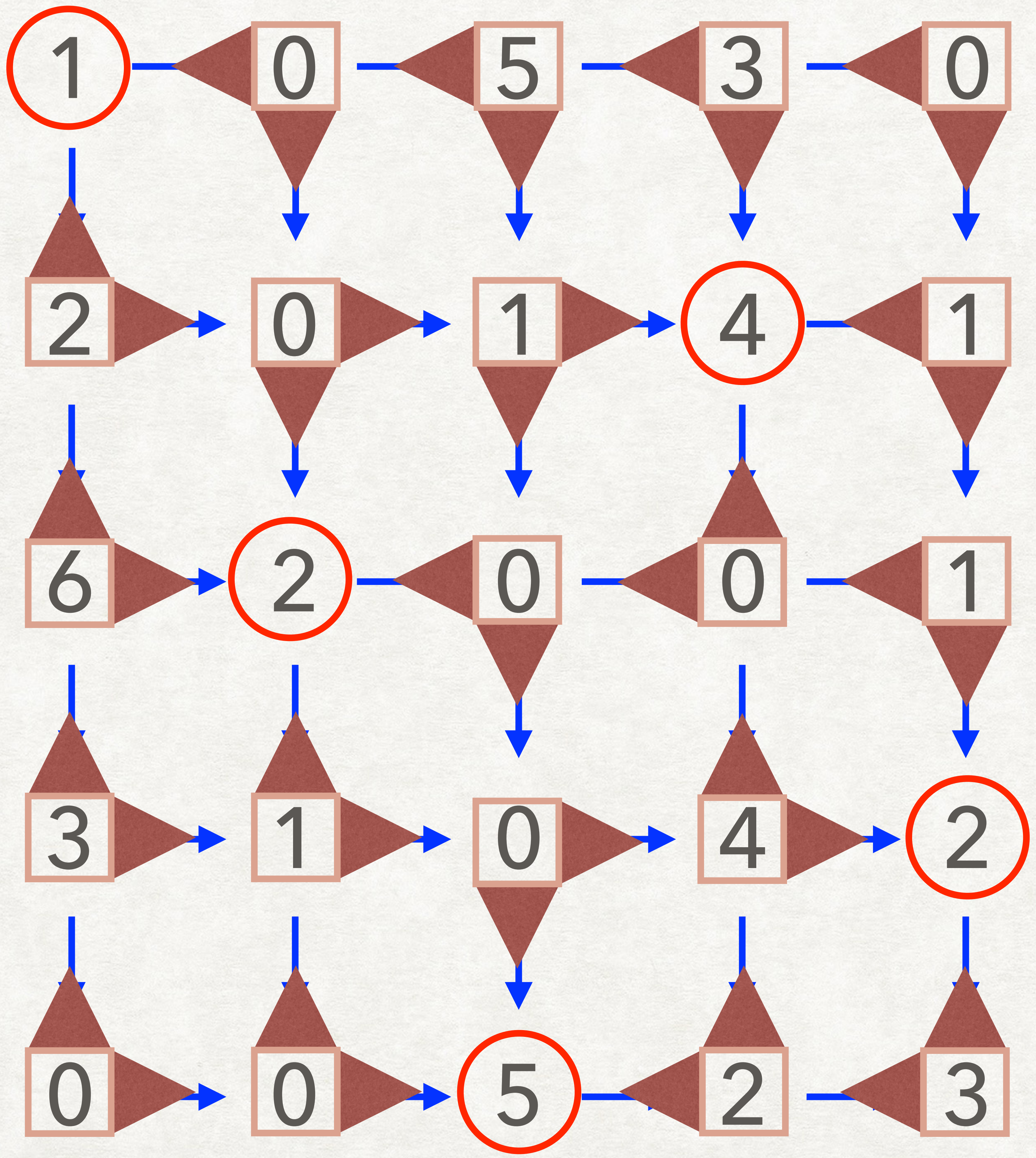
WEIGHTED TILING SYSTEM



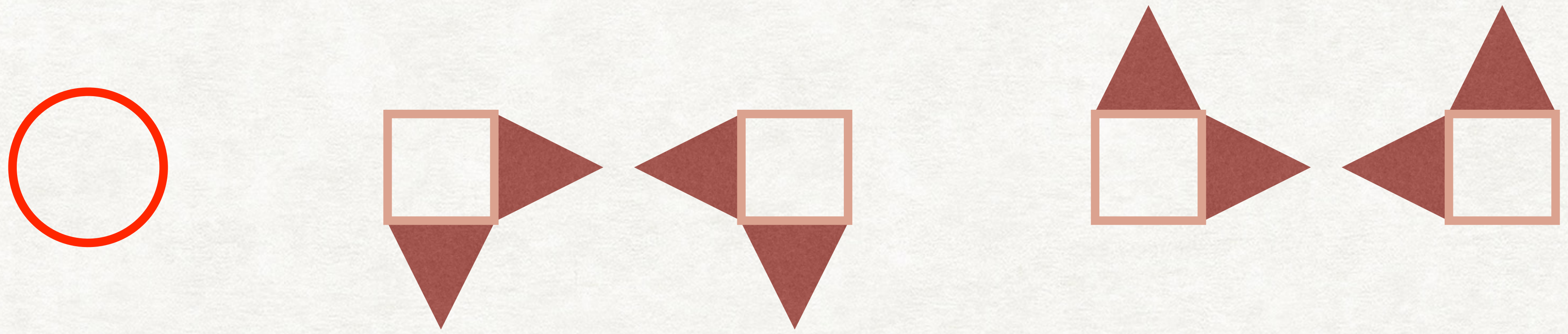
States



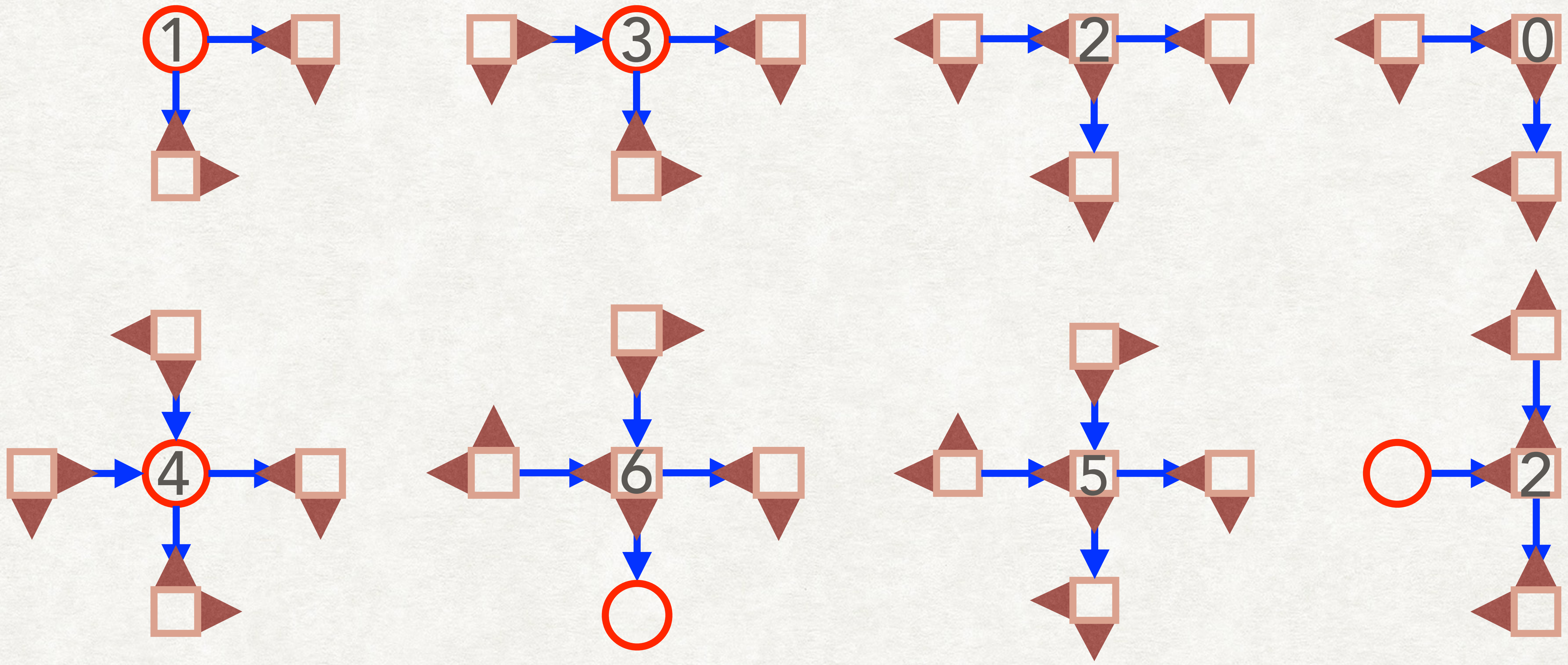
WEIGHTED TILING SYSTEM



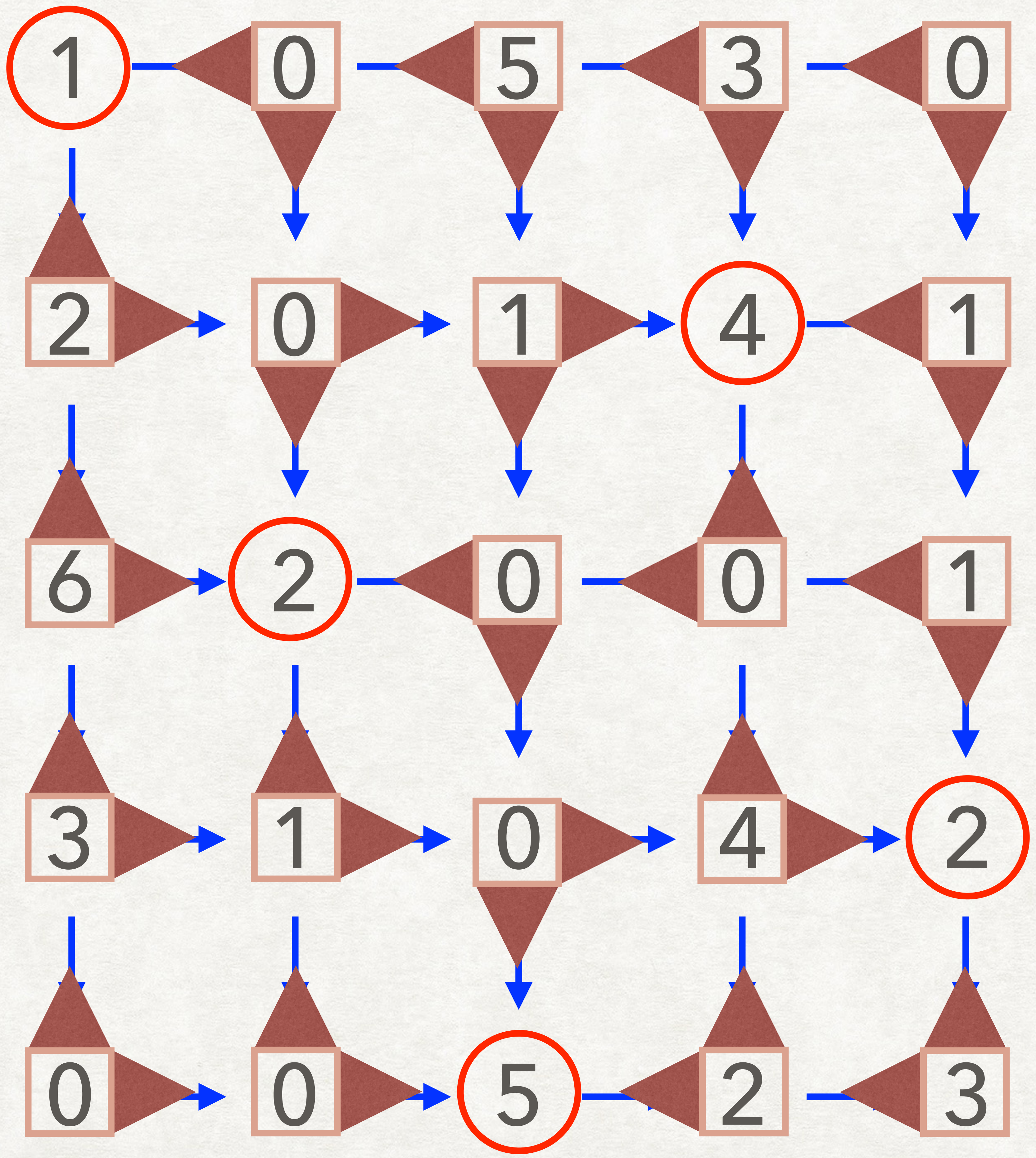
States



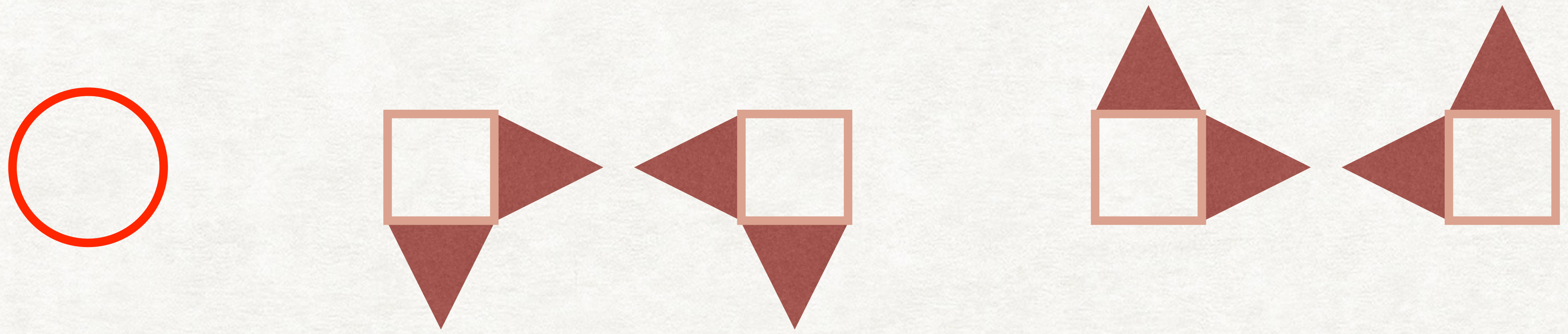
Tiles



WEIGHTED TILING SYSTEM

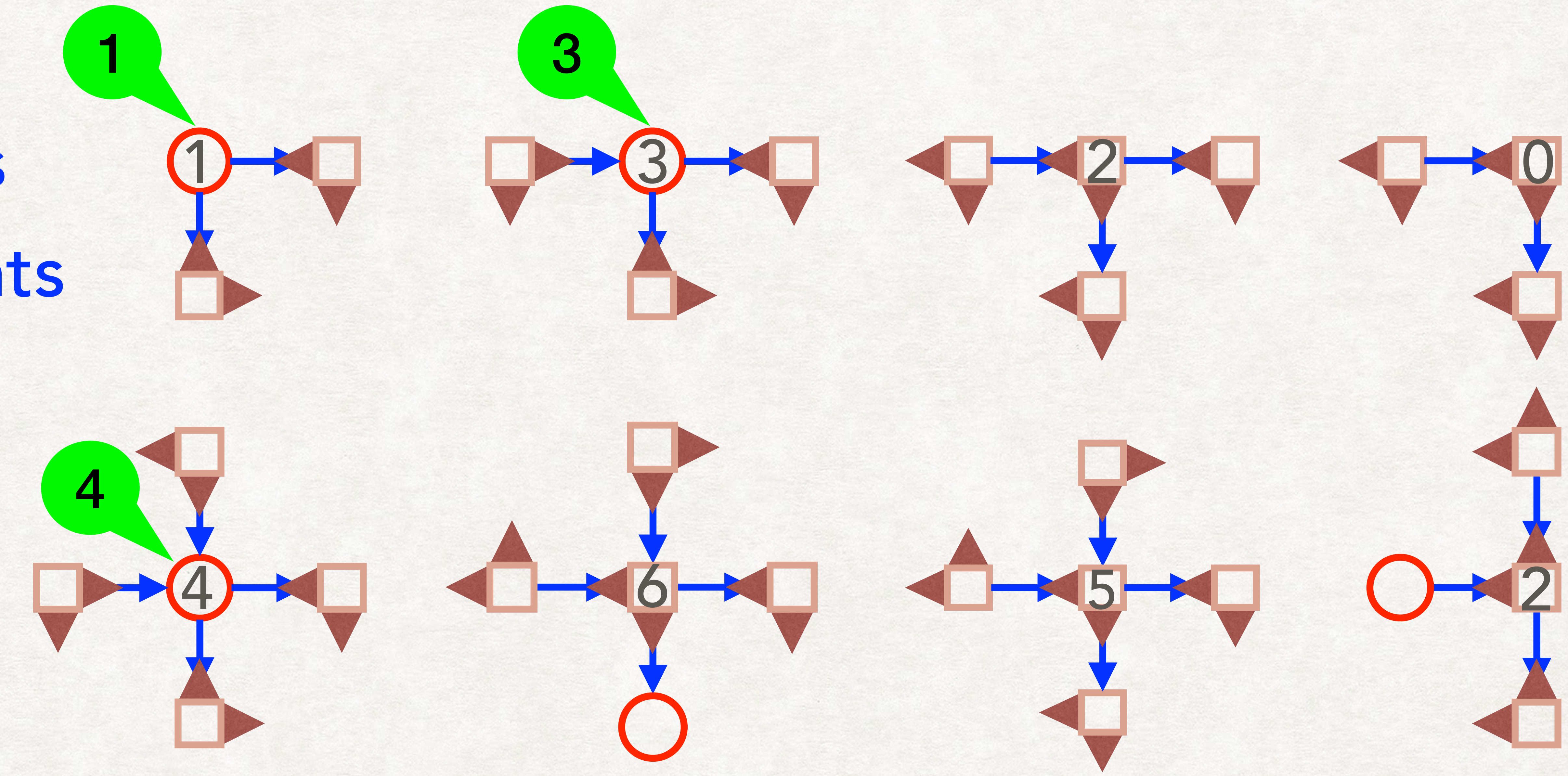


States



Tiles

Weights



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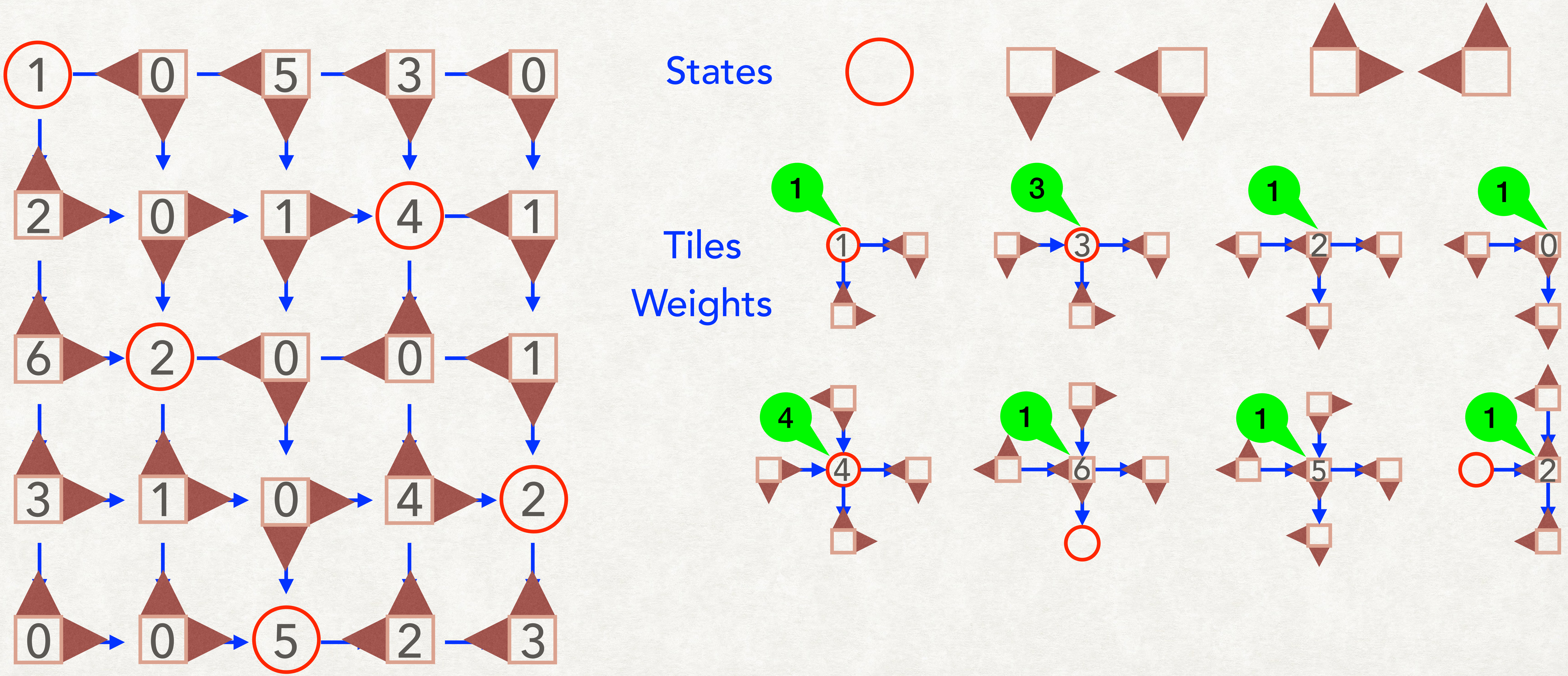

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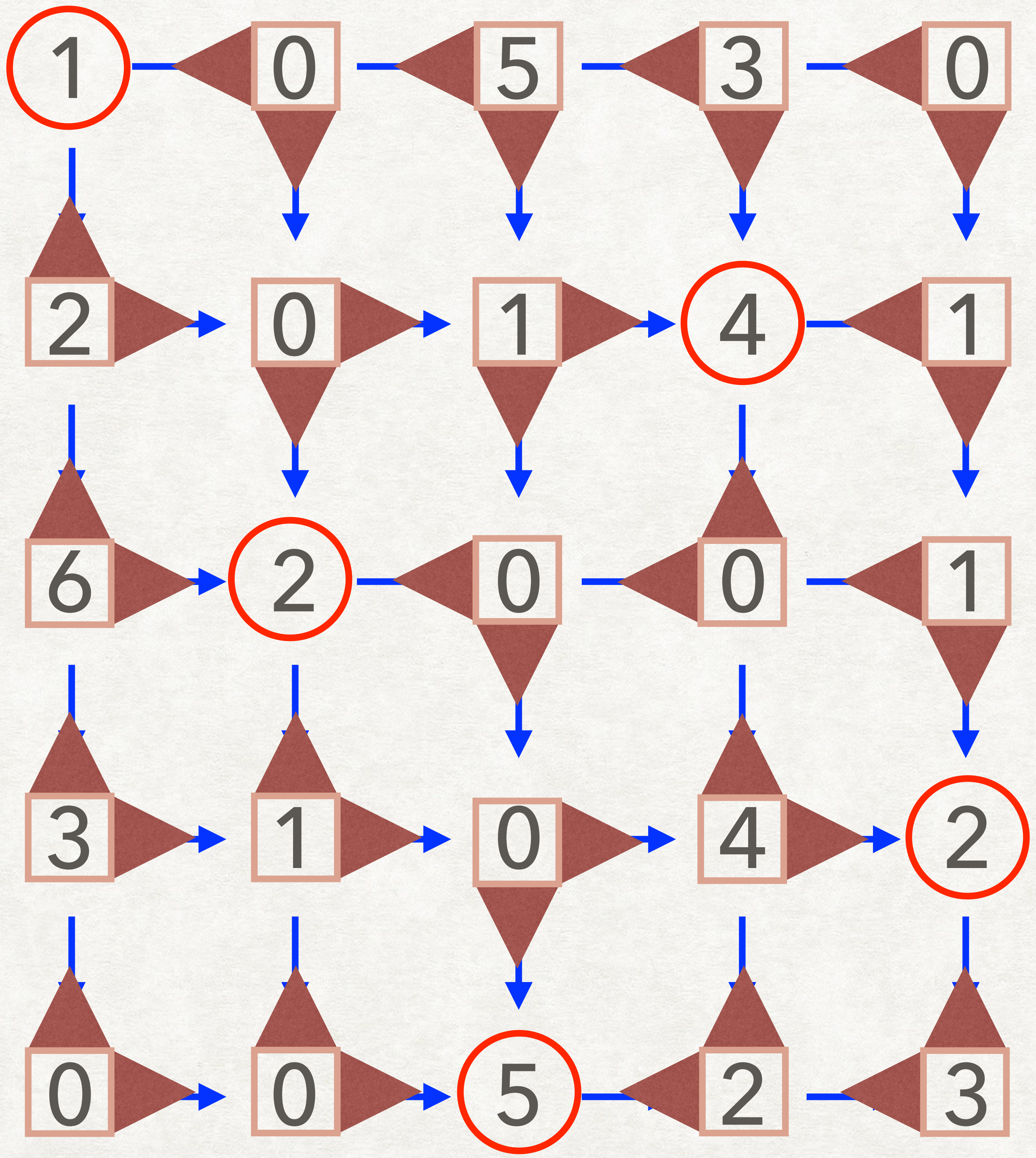
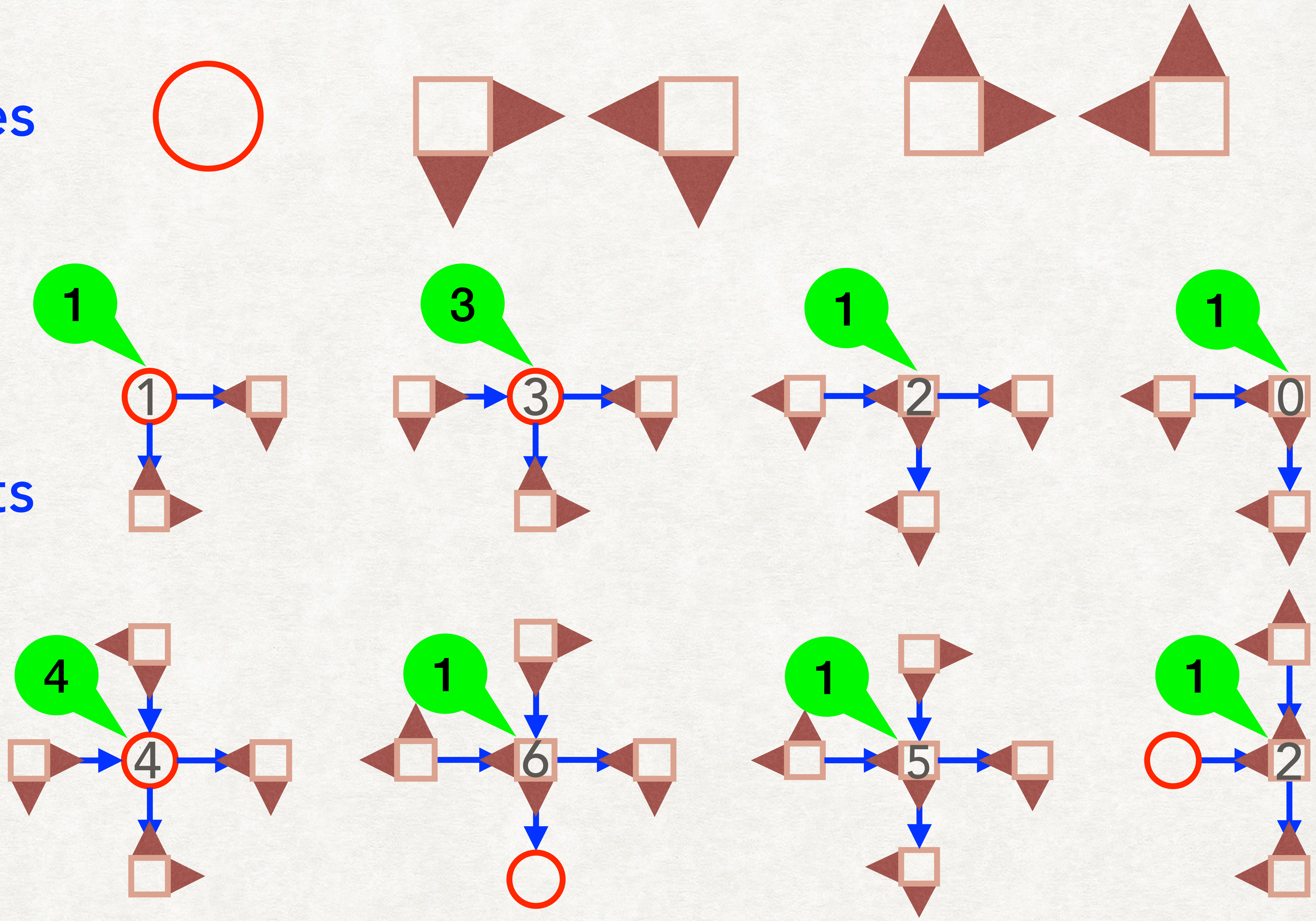


States

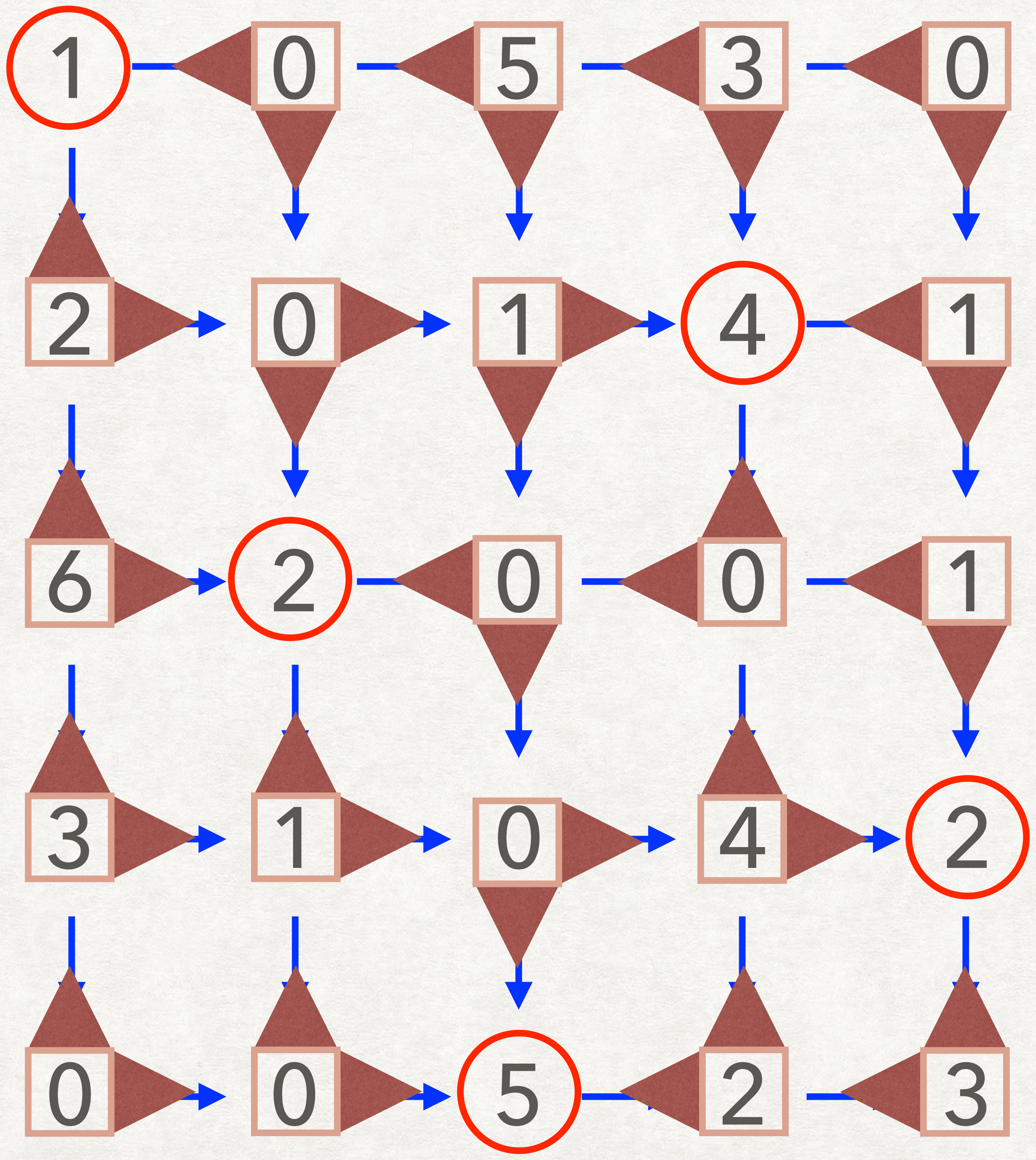
Tiles

Weights

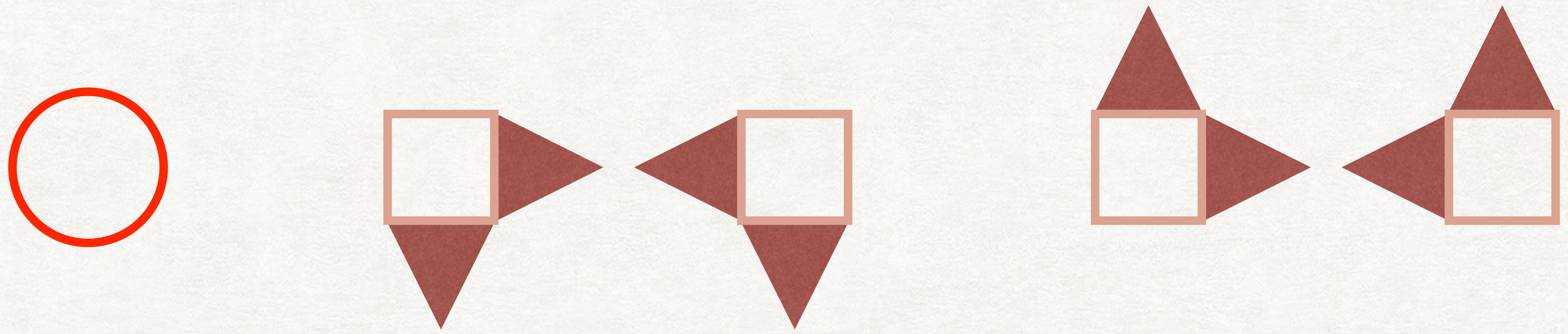


ρ : Vertices $\rightarrow$ States

WEIGHTED TILING SYSTEM

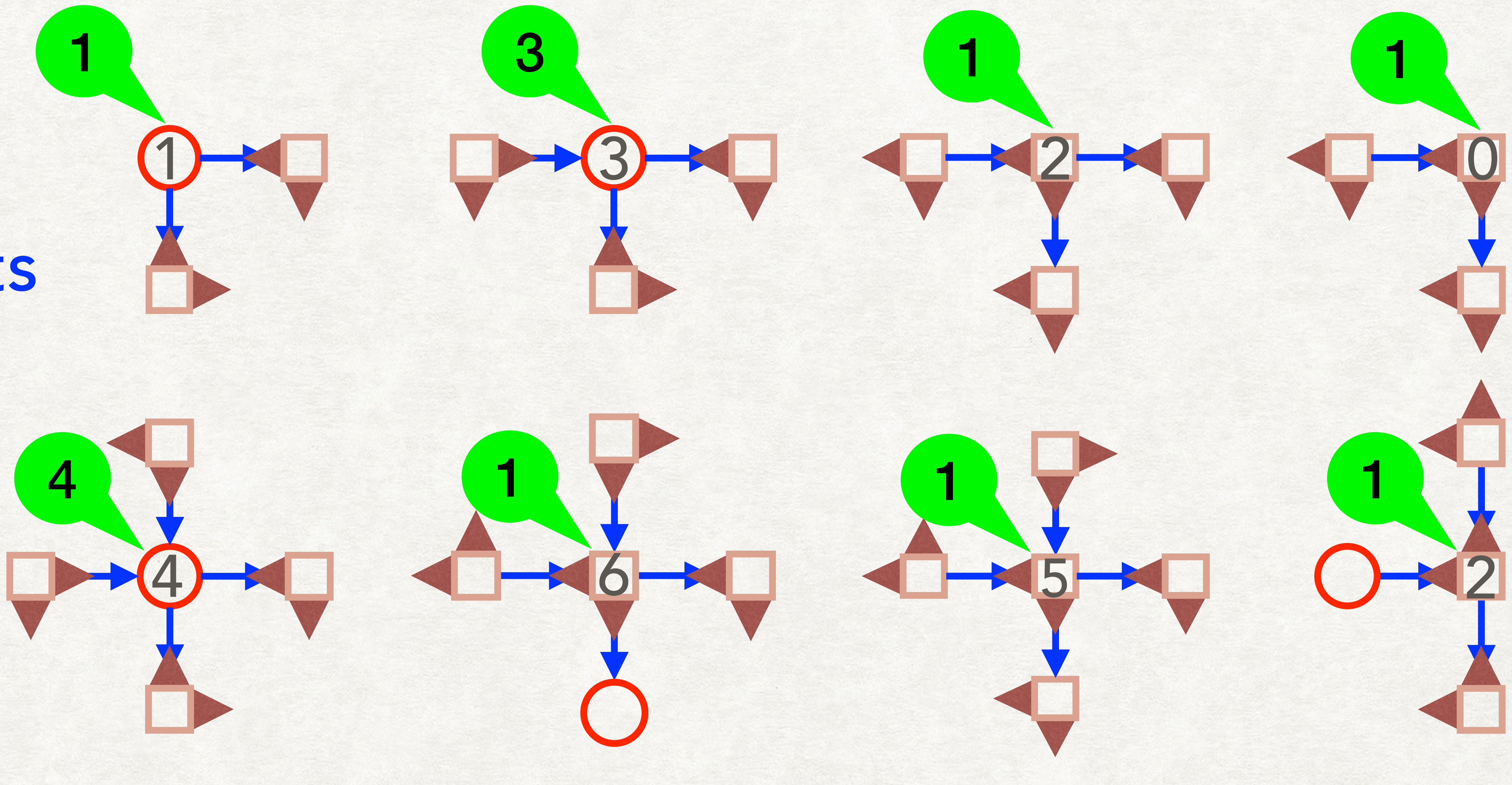


States



Tiles

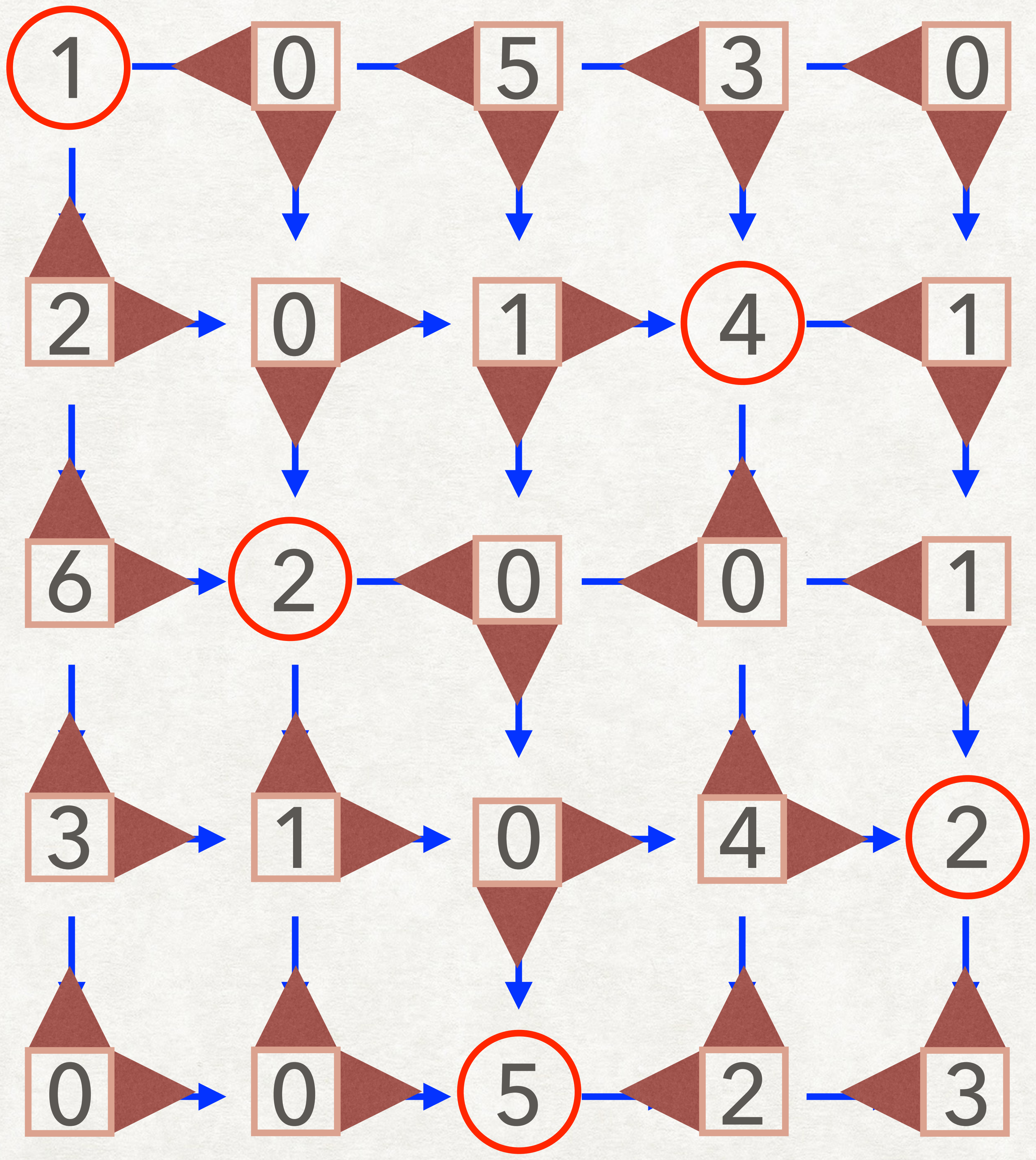
Weights

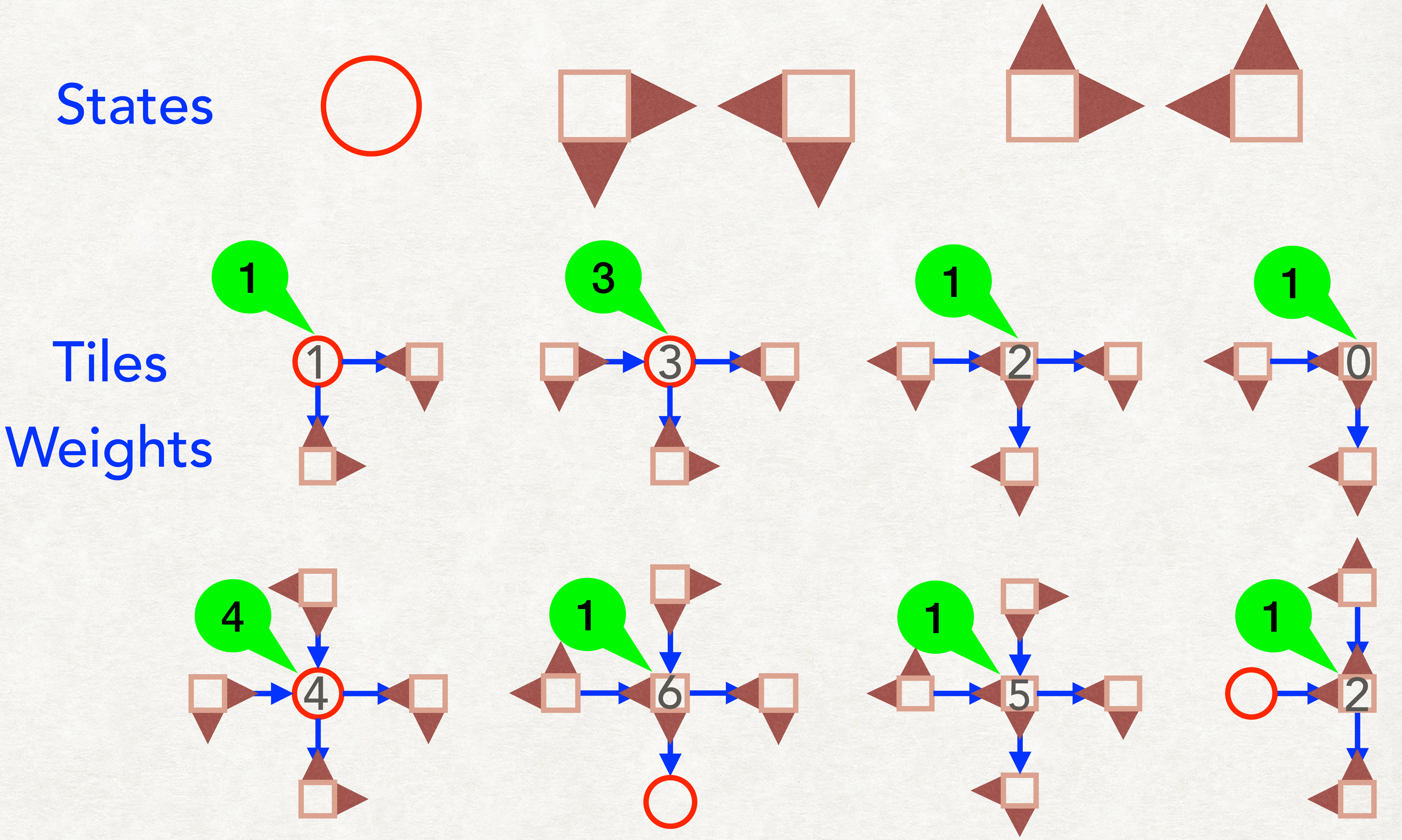


ρ: Vertices $\to$ States

$\text{weight}(\rho) = 1 \times 2 \times 5 \times 4 \times 2 \times 1^{20}$

WEIGHTED TILING SYSTEM





ρ: Vertices $\rightarrow$ States

$[\![\mathscr{T}]\!](G) = \sum_{\rho} \text{weight}(\rho)$

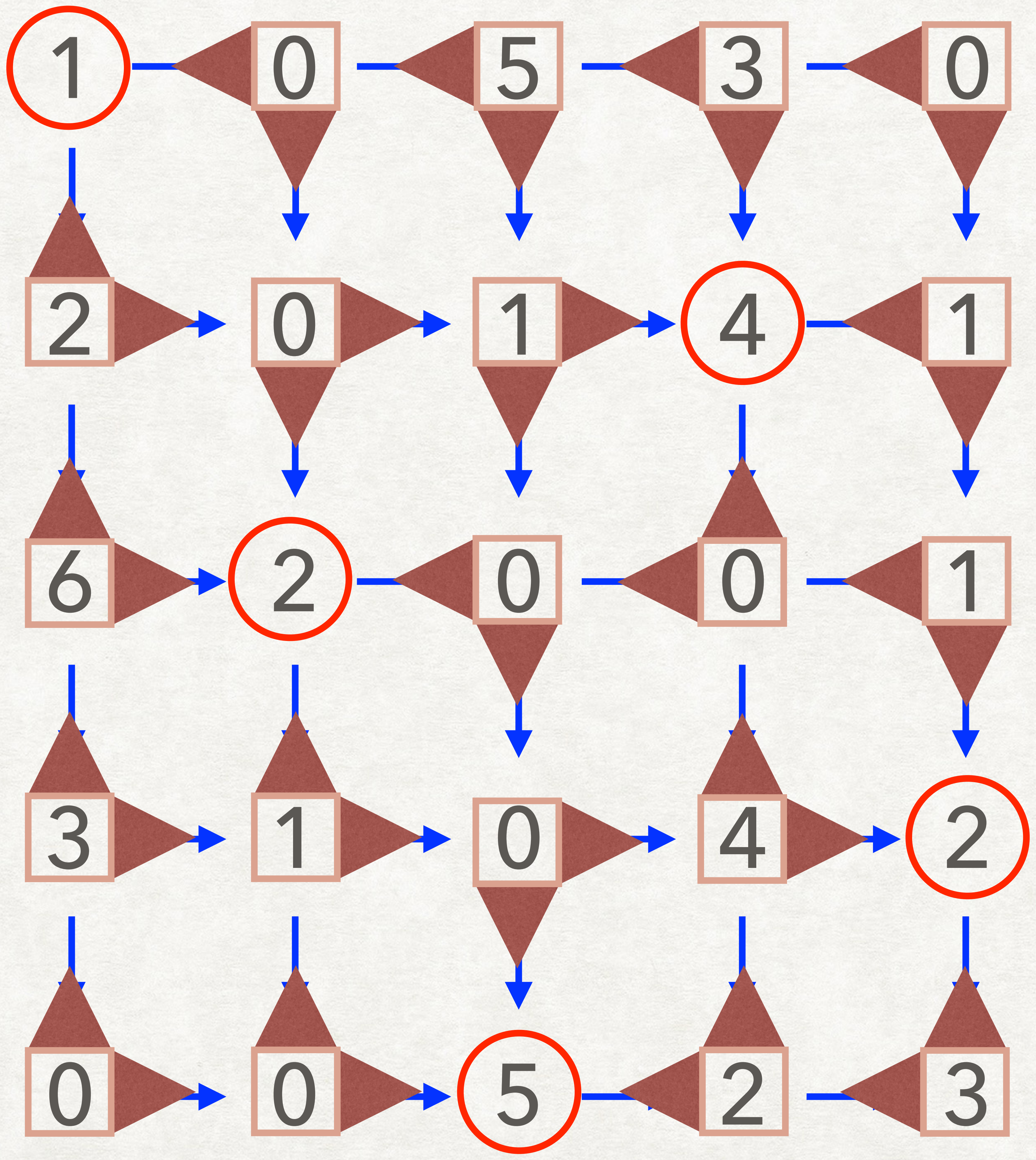
$\text{weight}(\rho) = 1 \times 2 \times 5 \times 4 \times 2 \times 1^{20}$

EVALUATION PROBLEM

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

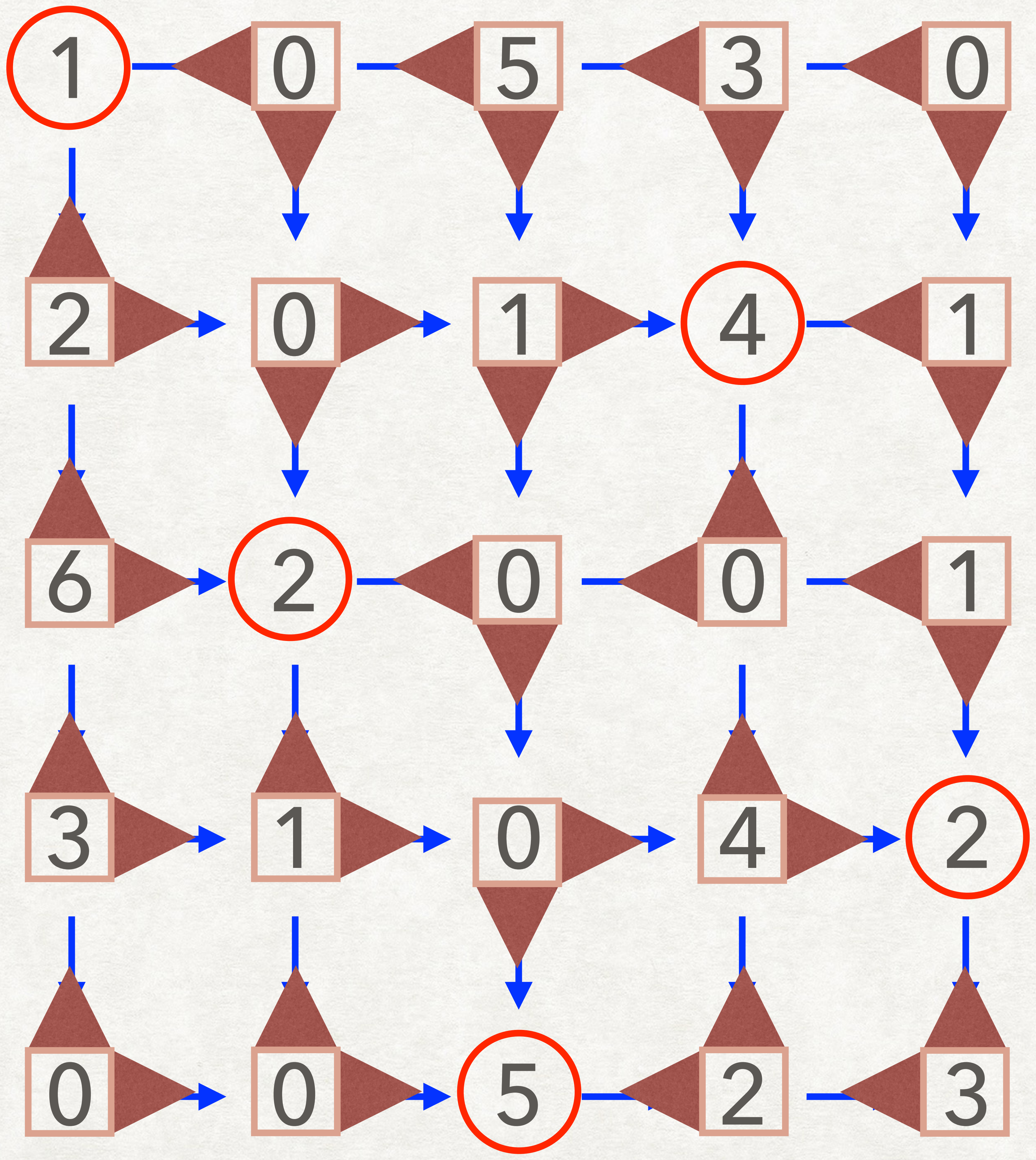
Naive evaluation: $\mathcal{O}(|G| \cdot |\mathscr{T}|^{|G|})$ FPSpace

WEIGHTED TILING SYSTEM



$$M = \begin{pmatrix} 1 & 0 & 5 & 3 & 0 \\ 2 & 0 & 1 & 4 & 1 \\ 6 & 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 4 & 2 \\ 0 & 0 & 5 & 2 & 3 \end{pmatrix}$$

WEIGHTED TILING SYSTEM



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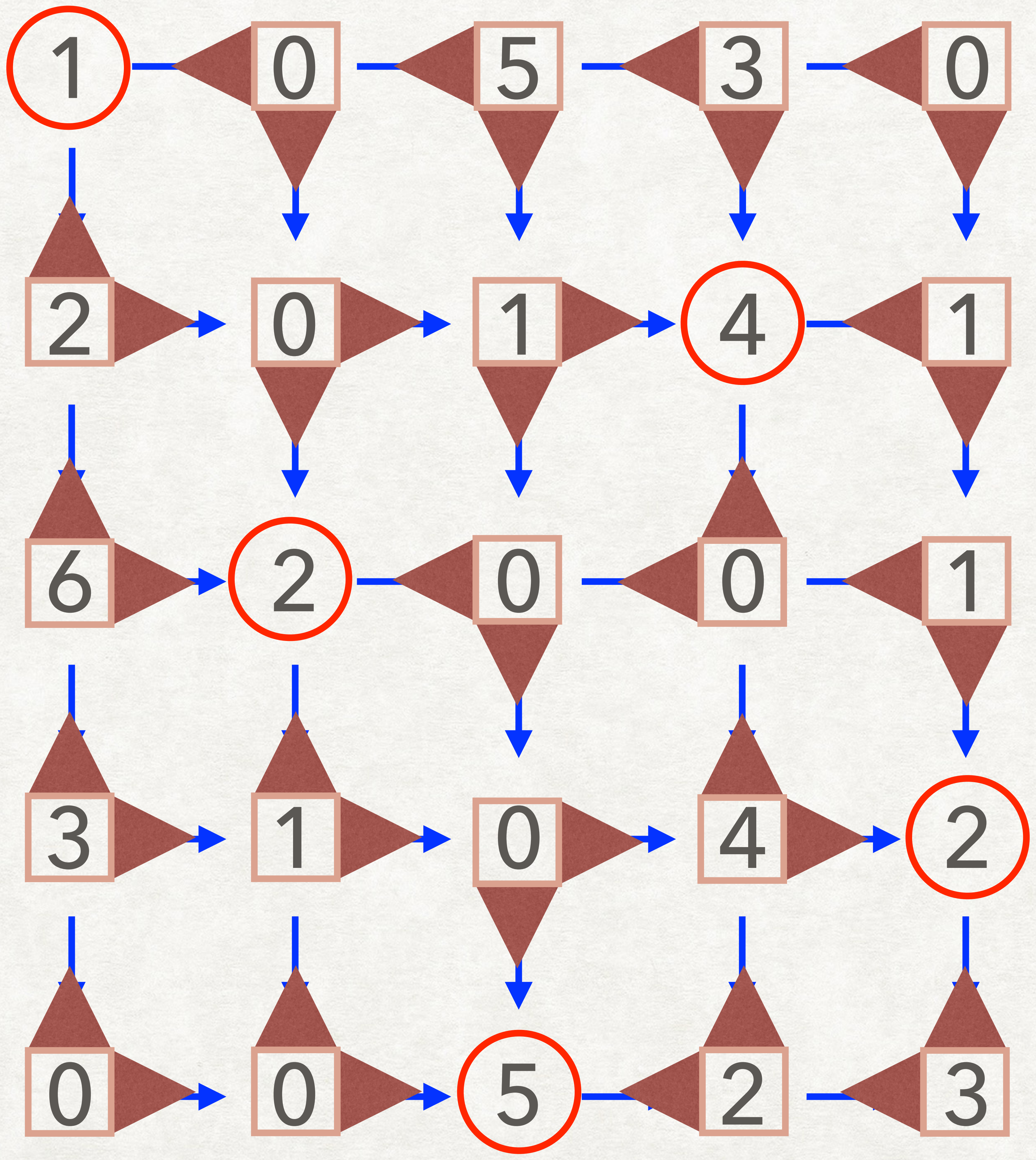
$$[\![\mathcal{T}]\!](M) = \sum_{\rho \text{ valid tiling}} \mathsf{weight}(\rho)$$

WEIGHTED TILING SYSTEM



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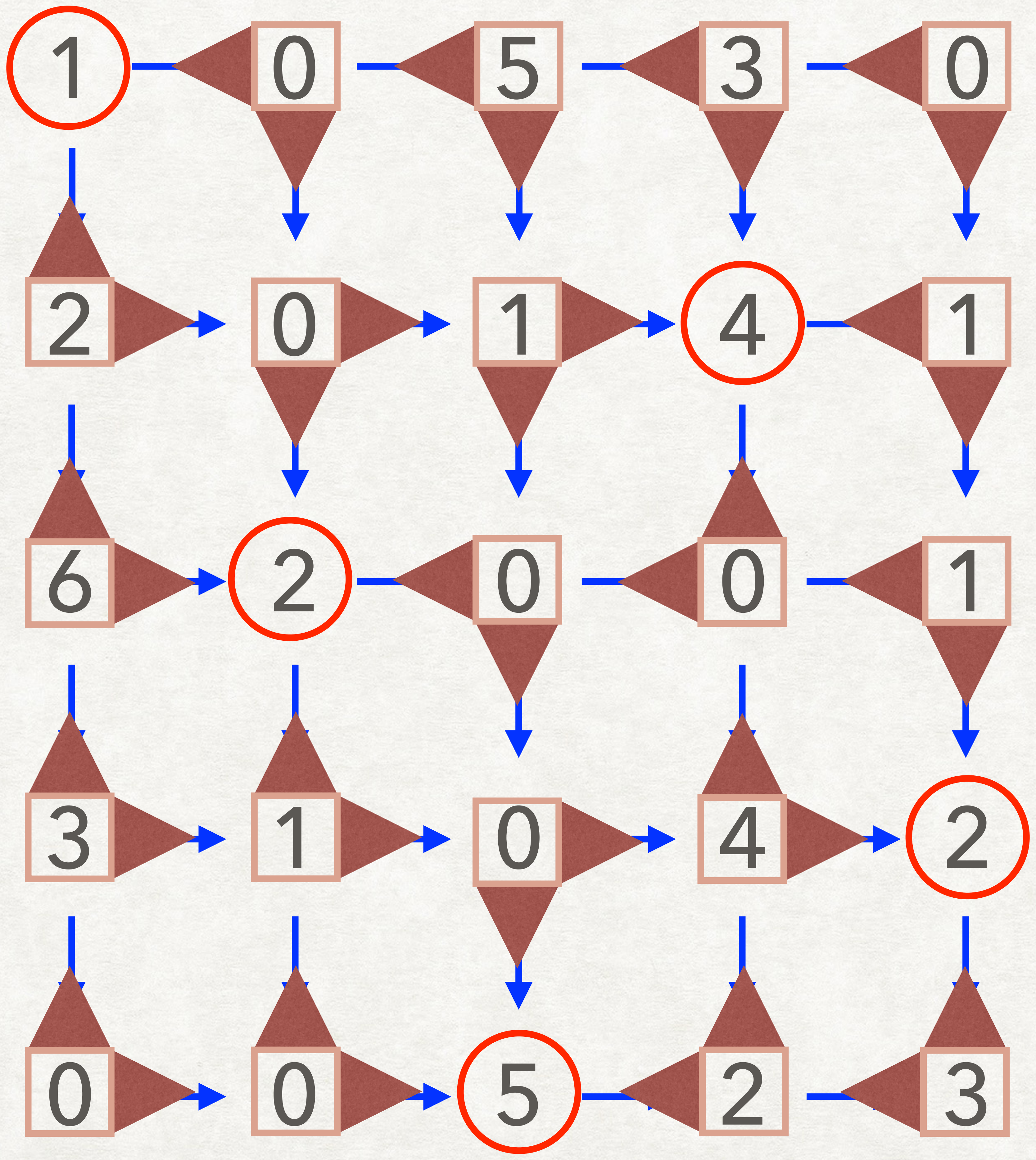
$$[\![\mathcal{T}]\!](M) = \sum_{\rho \text{ valid tiling}} \mathsf{weight}(\rho)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} M_{i,\sigma(i)}$$

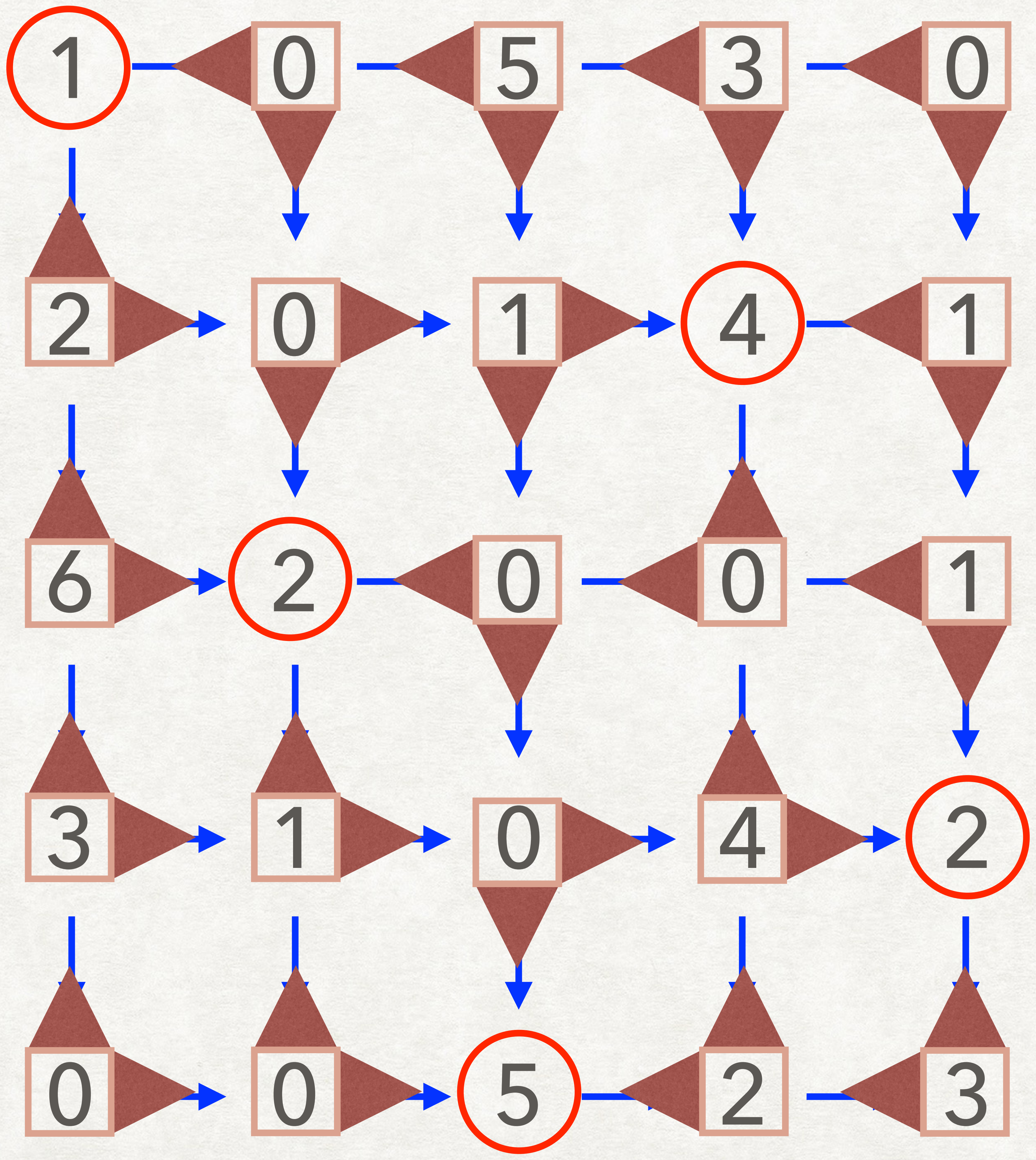
WEIGHTED TILING SYSTEM



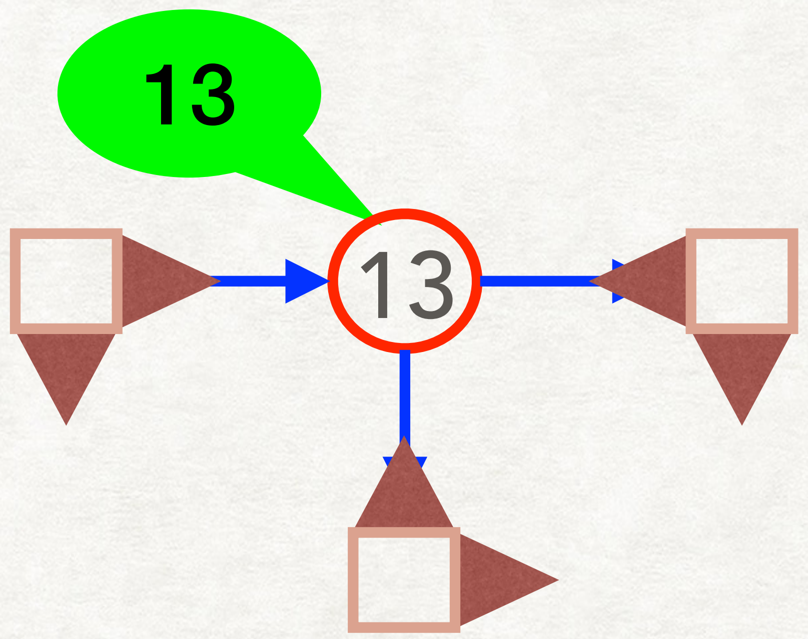
$$M = \begin{pmatrix} 1 & 0 & 5 & 3 & 0 \\ 2 & 0 & 1 & 4 & 1 \\ 6 & 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 4 & 2 \\ 0 & 0 & 5 & 2 & 3 \end{pmatrix}$$

$$\begin{aligned} [\![\mathcal{T}]\!](M) &= \sum_{\rho \text{ valid tiling}} \mathsf{weight}(\rho) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} M_{i,\sigma(i)} \\ &= \mathsf{permanent}(M) \end{aligned}$$

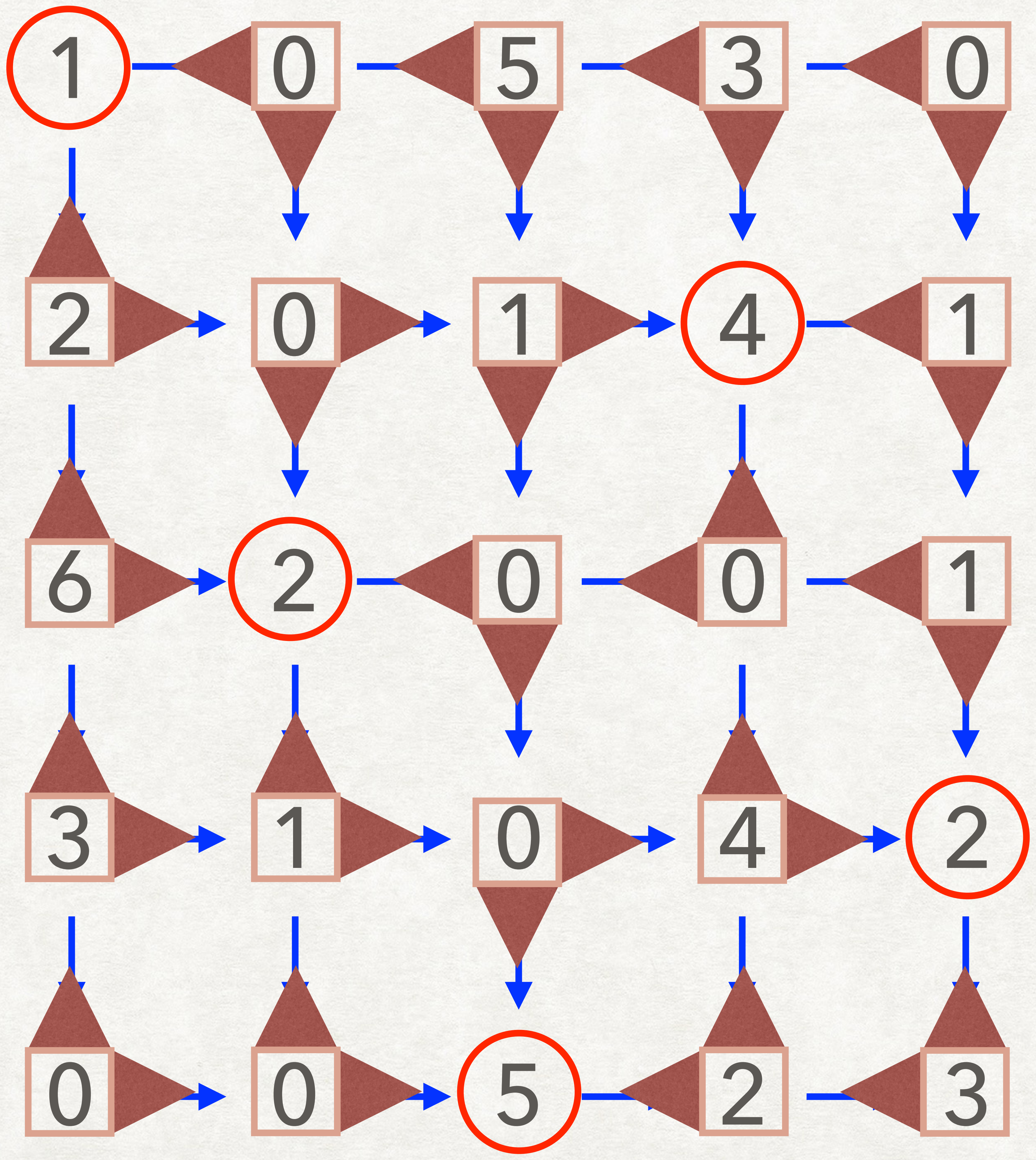
WEIGHTED TILING SYSTEM



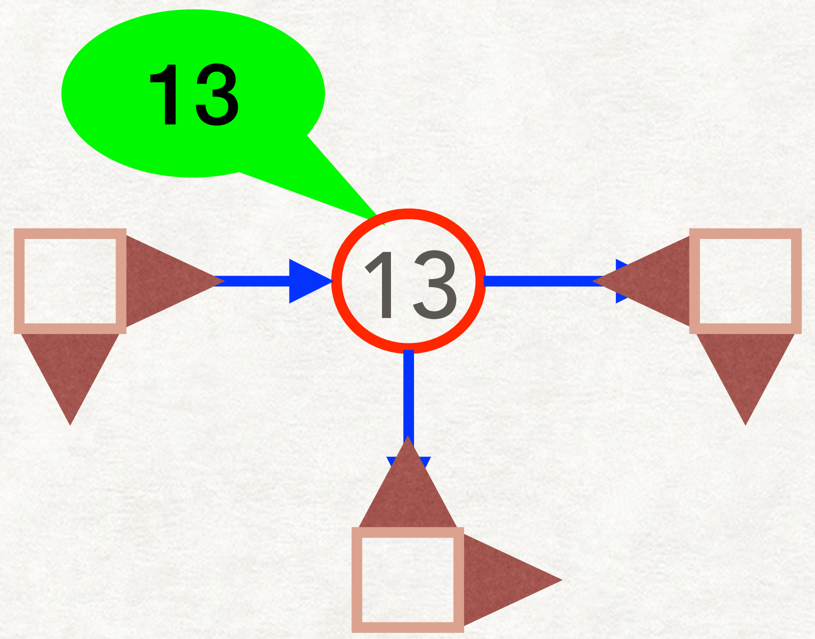
Tiles



WEIGHTED TILING SYSTEM

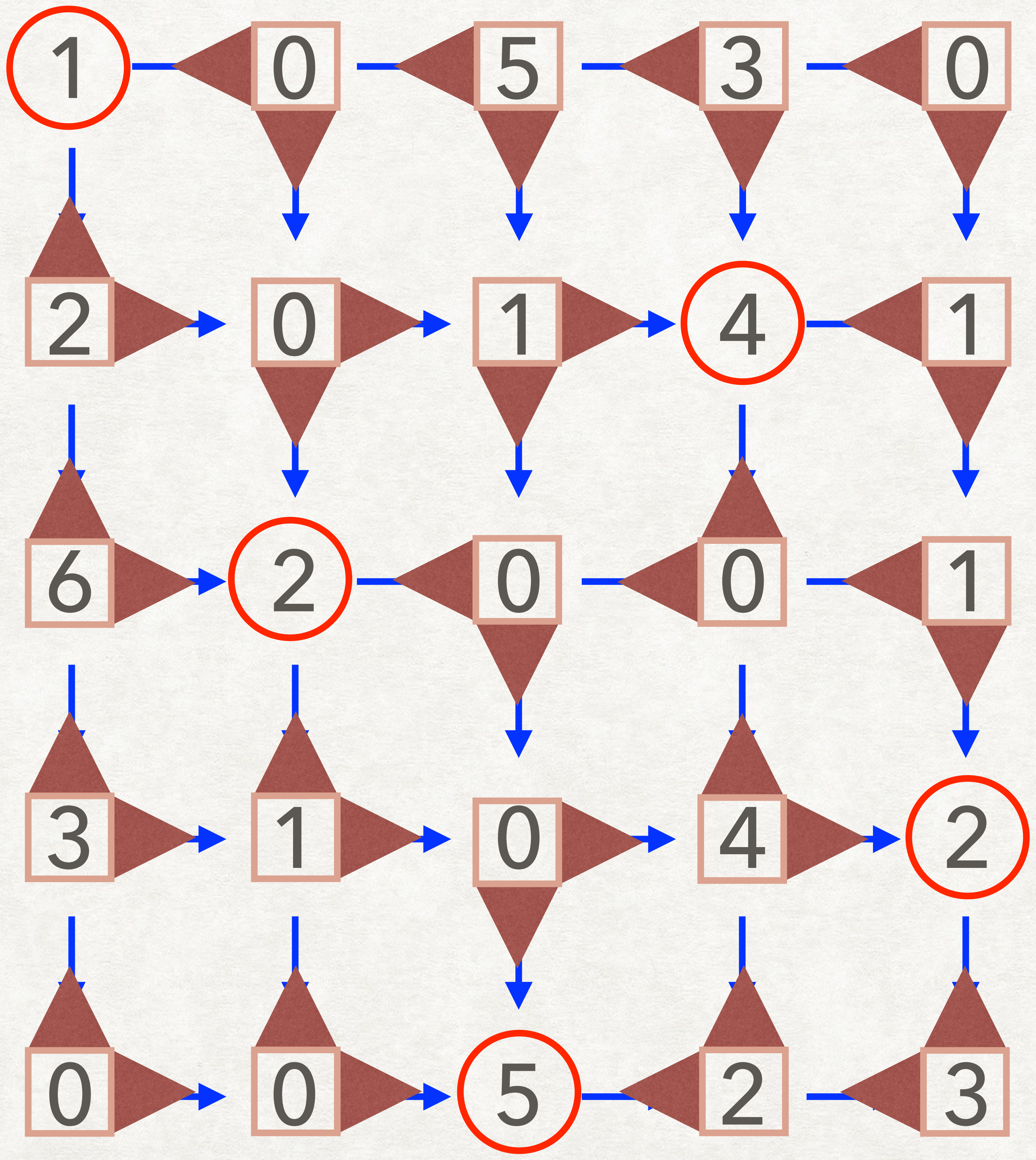


Tiles

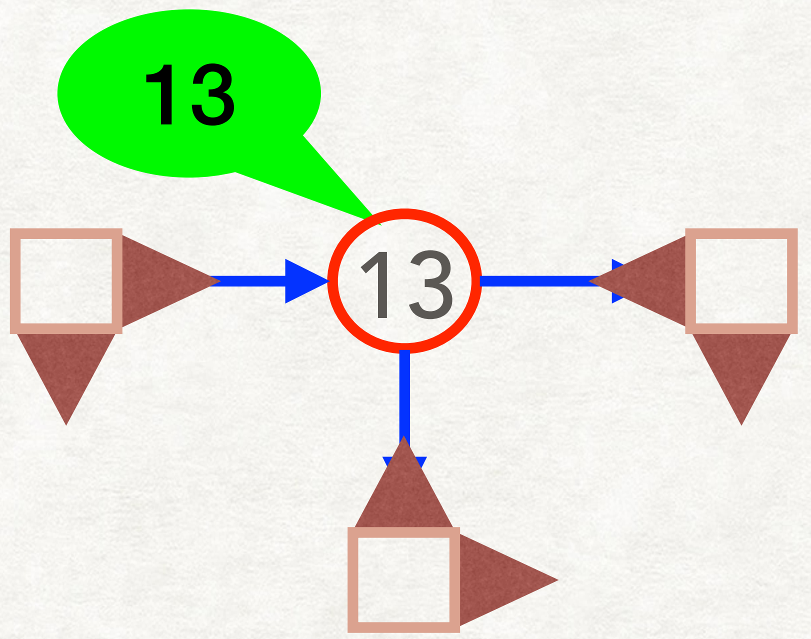


$$13 = 1 \to 1 \to 0 \to 1$$

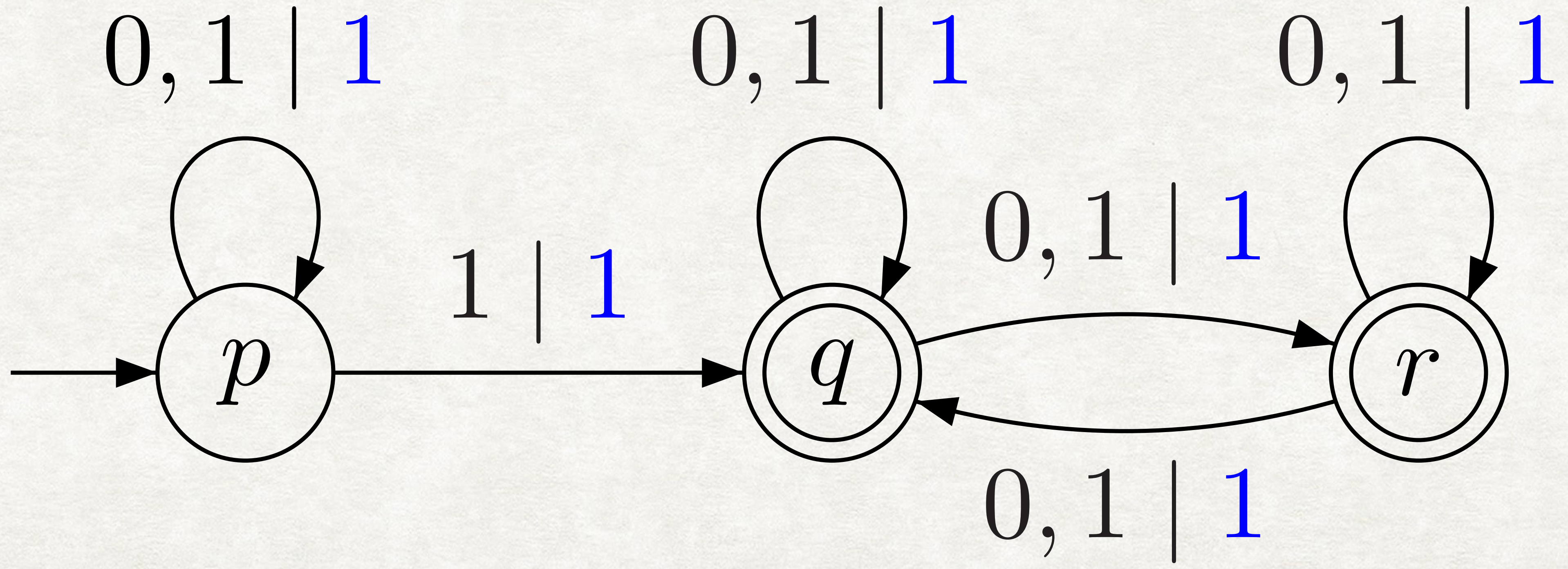
WEIGHTED TILING SYSTEM



Tiles



$$13 = 1 \to 1 \to 0 \to 1$$



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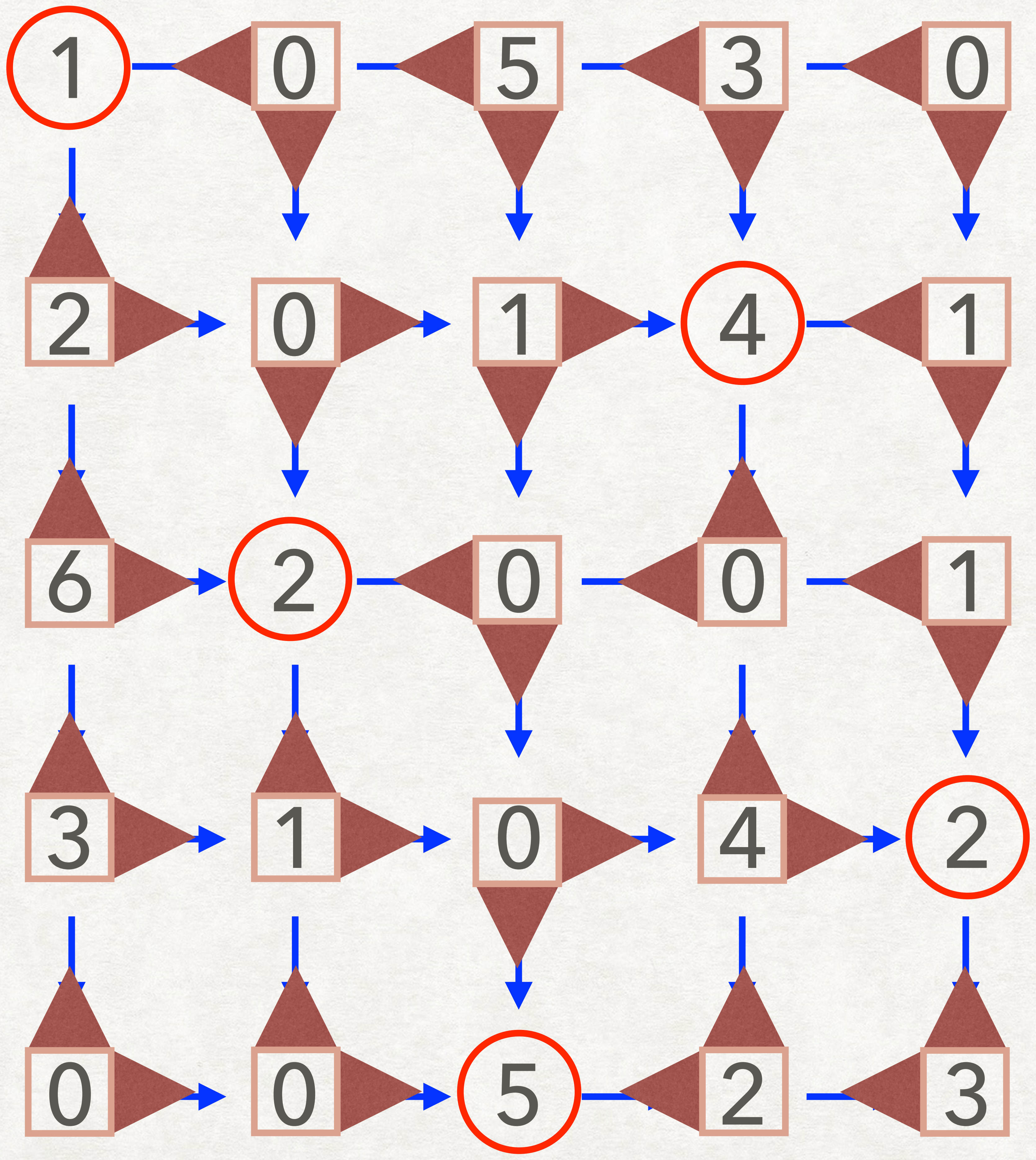
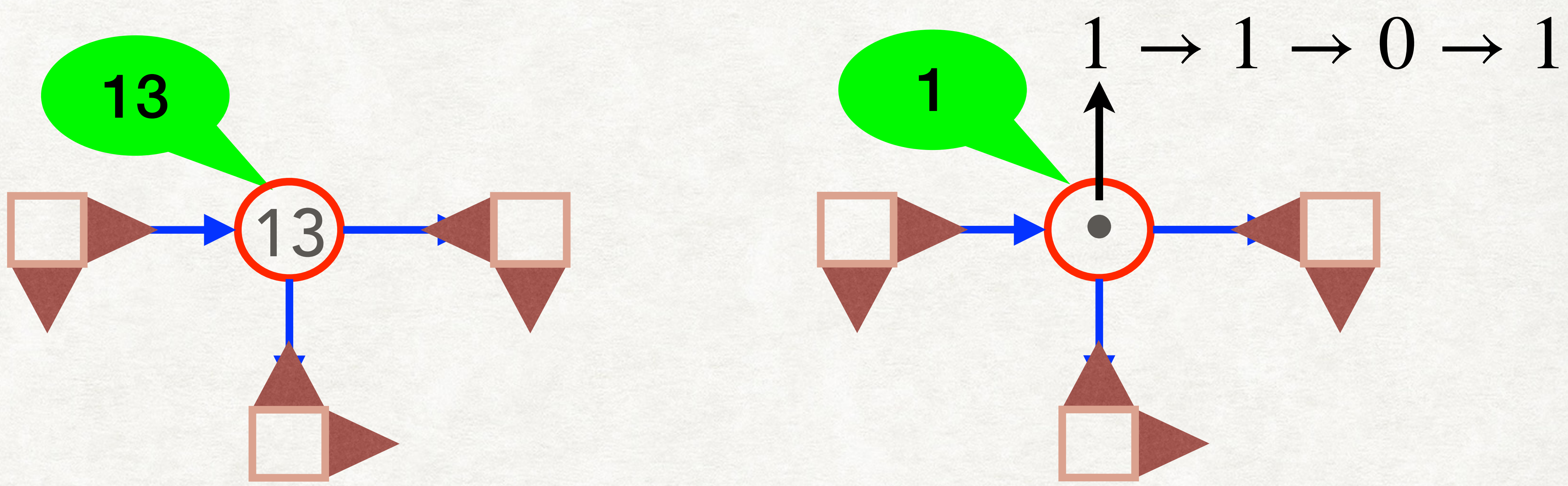
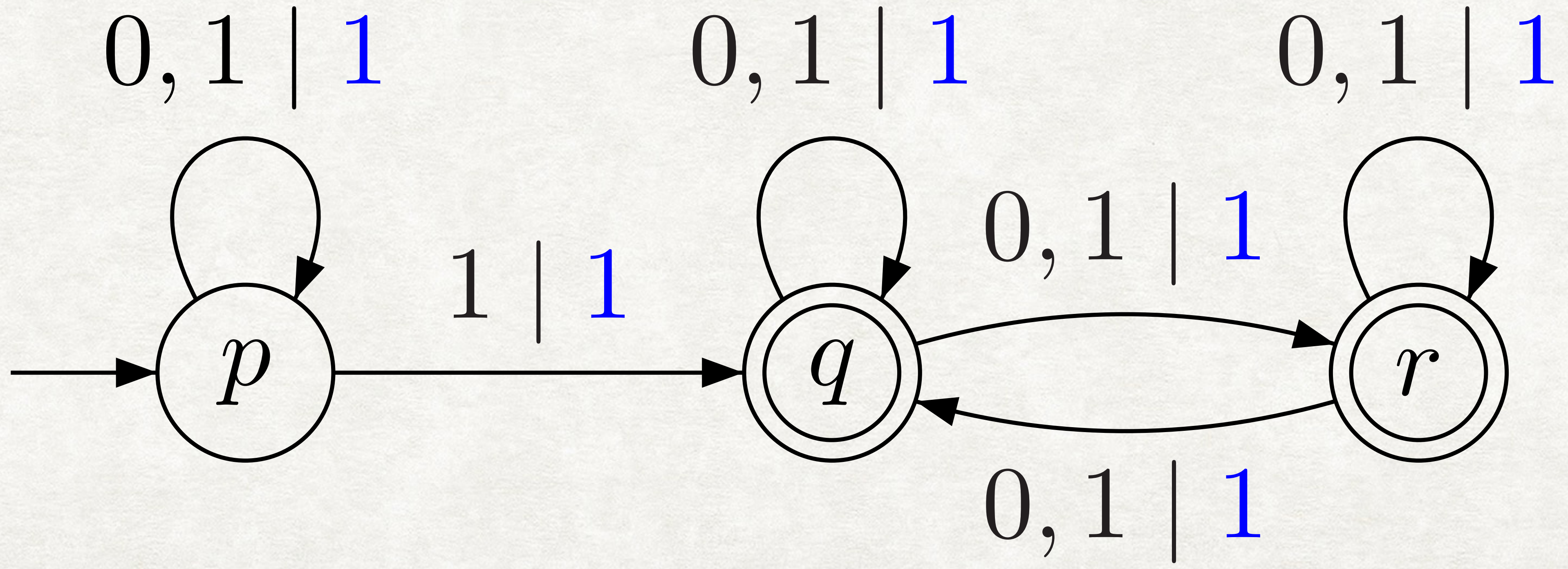


Tiles



$$13 = 1 \to 1 \to 0 \to 1$$



EVALUATION PROBLEM

- Input: G a (Σ, Γ)-graph and $\mathcal{T}$ a weighted tiling system (WTS) (weights in binary)
- Problem: Compute $[\![\mathcal{T}]\!](G)$

EVALUATION PROBLEM

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in binary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary graphs
Boolean	$(\{0,1\}, \vee, \wedge, 0, 1)$	NP-complete

EVALUATION PROBLEM

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in binary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary graphs
Boolean	$(\{0,1\}, \vee, \wedge, 0, 1)$	NP-complete
Natural	$(\mathbb{N}, +, \times, 0, 1)$	#P-complete

EVALUATION PROBLEM

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in binary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary graphs
Boolean	$(\{0,1\}, \vee, \wedge, 0, 1)$	NP-complete
Natural	$(\mathbb{N}, +, \times, 0, 1)$	#P-complete

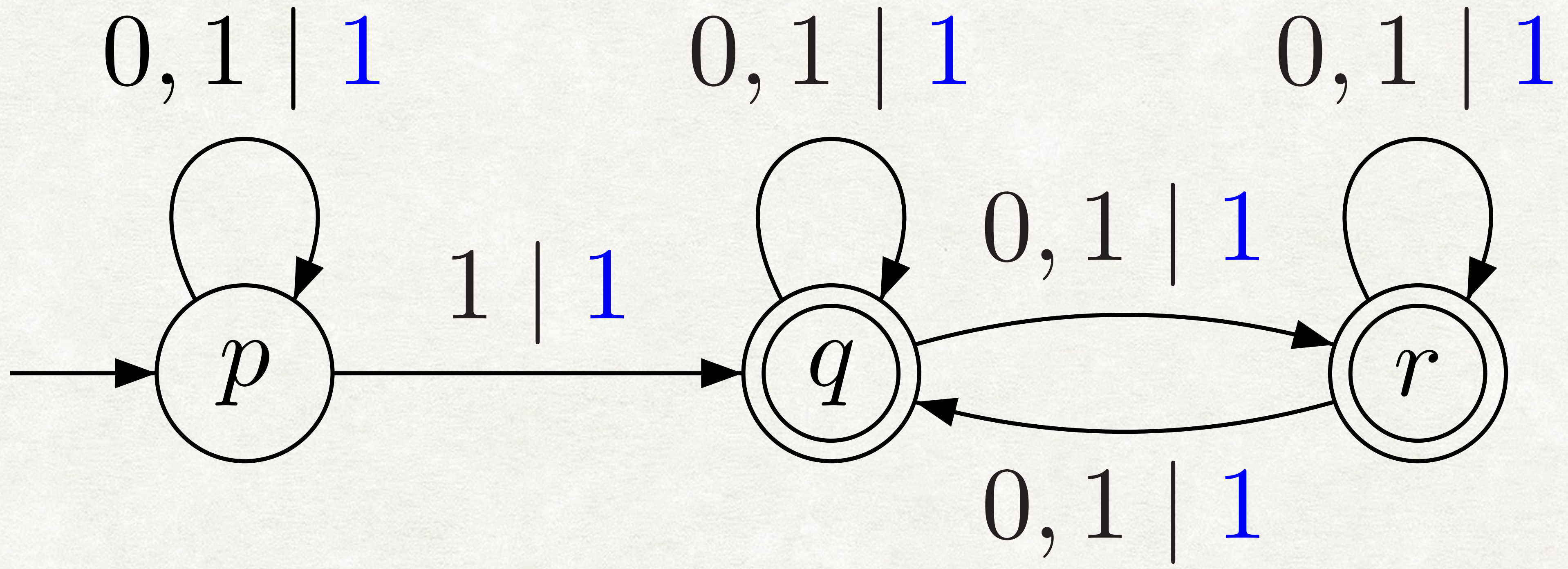
COUNTING ACCEPTING RUNS

f is in #P if $f(x) = \#\mathscr{M}(x)$ number of accepting runs of a PTime TM $\mathscr{M}$ on input x

EVALUATION OVER $(\mathbb{N}, +, \times, 0, 1)$ IN #P:

Construct a PTime Turing machine $\mathscr{M}$ such that $\#\mathscr{M}(\mathscr{T}, G) = [\![\mathscr{T}]\!](G)$

- Guess $\rho : V_G \to Q$ from vertices of G to states of $\mathscr{T}$
- Compute $N = \text{weight}(\rho)$ N in binary on a working tape
- Run $\mathscr{A}$ on N



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#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	•	-	+	•	-	+	•
x_2	•	•	-	+	•	•	-
x_3	-	+	+	-	•	-	-
x_4	-	•	•	•	+	+	•
x_5	+	-	•	+	•	-	+

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	•	-	+	•	-	+	•
x_2	•	•	-	+	•	•	-
x_3	-	+	+	-	•	-	-
x_4	-	•	•	•	+	+	•
x_5	+	-	•	+	•	-	+

#-SAT

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$C_4 = x_2 \vee \neg x_3 \vee x_5$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	•	-	+	•	-	+	•
x_2	•	•	-	+	•	•	-
x_3	-	+	+	-	•	-	-
x_4	-	•	•	•	+	+	•
x_5	+	-	•	+	•	-	+

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top$ •	-	+	•	-	+	•
x_2	$\bot$ •	•	-	+	•	•	-
x_3	$\top$ -	+	+	-	•	-	-
x_4	$\bot$ -	•	•	•	+	+	•
x_5	$\top$ +	-	•	+	•	-	+

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$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top$ •	$\top$ -	$\top$ +	$\top$ •	$\top$ -	$\top$ +	$\top$ •
x_2	$\bot$ •	•	-	+	•	•	-
x_3	$\top$ -	+	+	-	•	-	-
x_4	$\bot$ -	•	•	•	+	+	•
x_5	$\top$ +	-	•	+	•	-	+

#-SAT

$$\varphi = \bigwedge_i C_i$$

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G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top_{\bullet}$	$\top_{-}$	$\top_{+}$	$\top_{\bullet}$	$\top_{-}$	$\top_{+}$	$\top_{\bullet}$
x_2	$\bot_{\bullet}$	$\bot_{\bullet}$	$\bot_{-}$	$\bot_{+}$	$\bot_{\bullet}$	$\bot_{\bullet}$	$\bot_{-}$
x_3	$\top_{-}$	$\top_{+}$	$\top_{+}$	$\top_{-}$	$\top_{\bullet}$	$\top_{-}$	$\top_{-}$
x_4	$\bot_{-}$	$\bot_{\bullet}$	$\bot_{\bullet}$	$\bot_{\bullet}$	$\bot_{+}$	$\bot_{+}$	$\bot_{\bullet}$
x_5	$\top_{+}$	$\top_{-}$	$\top_{\bullet}$	$\top_{+}$	$\top_{\bullet}$	$\top_{-}$	$\top_{+}$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top$ •	$\top$ -	$\top$ +	$\top$ •	$\top$ -	$\top$ +	$\top$ •
x_2	$\bot$ •	$\bot$ •	$\bot$ -	$\bot$ +	$\bot$ •	$\bot$ •	$\bot$ -
x_3	$\top$ -	$\top$ +	$\top$ +	$\top$ -	$\top$ •	$\top$ -	$\top$ -
x_4	$\bot$ -	$\bot$ •	$\bot$ •	$\bot$ •	$\bot$ +	$\bot$ +	$\bot$ •
x_5	$\top$ +	$\top$ -	$\top$ •	$\top$ +	$\top$ •	$\top$ -	$\top$ +

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top$ • $\perp$	$\top$ -	$\top$ +	$\top$ •	$\top$ -	$\top$ +	$\top$ •
x_2	$\perp$ • $\perp$	$\perp$ •	$\perp$ -	$\perp$ +	$\perp$ •	$\perp$ •	$\perp$ -
x_3	$\top$ - $\perp$	$\top$ +	$\top$ +	$\top$ -	$\top$ •	$\top$ -	$\top$ -
x_4	$\perp$ - $\top$	$\perp$ •	$\perp$ •	$\perp$ •	$\perp$ +	$\perp$ +	$\perp$ •
x_5	$\top$ + $\top$	$\top$ -	$\top$ •	$\top$ +	$\top$ •	$\top$ -	$\top$ +

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top \bullet \bot$	$\top - \bot$	$\top +$	$\top \bullet$	$\top -$	$\top +$	$\top \bullet$
x_2	$\bot \bullet \bot$	$\bot \bullet \bot$	$\bot -$	$\bot +$	$\bot \bullet$	$\bot \bullet$	$\bot -$
x_3	$\top - \bot$	$\top + \top$	$\top +$	$\top -$	$\top \bullet$	$\top -$	$\top -$
x_4	$\bot - \top$	$\bot \bullet \top$	$\bot \bullet$	$\bot \bullet$	$\bot +$	$\bot +$	$\bot \bullet$
x_5	$\top + \top$	$\top - \top$	$\top \bullet$	$\top +$	$\top \bullet$	$\top -$	$\top +$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \lor \neg x_3 \lor x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top \bullet \bot$	$\top - \bot$	$\top + \top$	$\top \bullet$	$\top -$	$\top +$	$\top \bullet$
x_2	$\bot \bullet \bot$	$\bot \bullet \bot$	$\bot - \top$	$\bot +$	$\bot \bullet$	$\bot \bullet$	$\bot -$
x_3	$\top - \bot$	$\top + \top$	$\top + \top$	$\top -$	$\top \bullet$	$\top -$	$\top -$
x_4	$\bot - \top$	$\bot \bullet \top$	$\bot \bullet \top$	$\bot \bullet$	$\bot +$	$\bot +$	$\bot \bullet$
x_5	$\top + \top$	$\top - \top$	$\top \bullet \top$	$\top +$	$\top \bullet$	$\top -$	$\top +$

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$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top$ • $\perp$	$\top$ - $\perp$	$\top$ + $\top$	$\top$ • $\perp$	$\top$ -	$\top$ +	$\top$ •
x_2	$\perp$ • $\perp$	$\perp$ • $\perp$	$\perp$ - $\top$	$\perp$ + $\perp$	$\perp$ •	$\perp$ •	$\perp$ -
x_3	$\top$ - $\perp$	$\top$ + $\top$	$\top$ + $\top$	$\top$ - $\perp$	$\top$ •	$\top$ -	$\top$ -
x_4	$\perp$ - $\top$	$\perp$ • $\top$	$\perp$ • $\top$	$\perp$ • $\perp$	$\perp$ +	$\perp$ +	$\perp$ •
x_5	$\top$ + $\top$	$\top$ - $\top$	$\top$ • $\top$	$\top$ + $\top$	$\top$ •	$\top$ -	$\top$ +

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top \bullet \bot$	$\top - \bot$	$\top + \top$	$\top \bullet \bot$	$\top - \bot$	$\top +$	$\top \bullet$
x_2	$\bot \bullet \bot$	$\bot \bullet \bot$	$\bot - \top$	$\bot + \bot$	$\bot \bullet \bot$	$\bot \bullet$	$\bot -$
x_3	$\top - \bot$	$\top + \top$	$\top + \top$	$\top - \bot$	$\top \bullet \bot$	$\top -$	$\top -$
x_4	$\bot - \top$	$\bot \bullet \top$	$\bot \bullet \top$	$\bot \bullet \bot$	$\bot + \bot$	$\bot +$	$\bot \bullet$
x_5	$\top + \top$	$\top - \top$	$\top \bullet \top$	$\top + \top$	$\top \bullet \bot$	$\top -$	$\top +$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \lor \neg x_3 \lor x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top \bullet \perp$	$\top - \perp$	$\top + \top$	$\top \bullet \perp$	$\top - \perp$	$\top + \top$	$\top \bullet$
x_2	$\perp \bullet \perp$	$\perp \bullet \perp$	$\perp - \top$	$\perp + \perp$	$\perp \bullet \perp$	$\perp \bullet \top$	$\perp -$
x_3	$\top - \perp$	$\top + \top$	$\top + \top$	$\top - \perp$	$\top \bullet \perp$	$\top - \top$	$\top -$
x_4	$\perp - \top$	$\perp \bullet \top$	$\perp \bullet \top$	$\perp \bullet \perp$	$\perp + \perp$	$\perp + \top$	$\perp \bullet$
x_5	$\top + \top$	$\top - \top$	$\top \bullet \top$	$\top + \top$	$\top \bullet \perp$	$\top - \perp$	$\top +$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top \bullet \bot$	$\top - \bot$	$\top + \top$	$\top \bullet \bot$	$\top - \bot$	$\top + \top$	$\top \bullet \bot$
x_2	$\bot \bullet \bot$	$\bot \bullet \bot$	$\bot - \top$	$\bot + \bot$	$\bot \bullet \bot$	$\bot \bullet \top$	$\bot - \top$
x_3	$\top - \bot$	$\top + \top$	$\top + \top$	$\top - \bot$	$\top \bullet \bot$	$\top - \top$	$\top - \top$
x_4	$\bot - \top$	$\bot \bullet \top$	$\bot \bullet \top$	$\bot \bullet \bot$	$\bot + \bot$	$\bot + \top$	$\bot \bullet \top$
x_5	$\top + \top$	$\top - \top$	$\top \bullet \top$	$\top + \top$	$\top \bullet \bot$	$\top - \bot$	$\top + \bot$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top\ \bullet\ \perp$	$\top\ -\ \perp$	$\top\ +\ \top$	$\top\ \bullet\ \perp$	$\top\ -\ \perp$	$\top\ +\ \top$	$\top\ \bullet\ \perp$
x_2	$\perp\ \bullet\ \perp$	$\perp\ \bullet\ \perp$	$\perp\ -\ \top$	$\perp\ +\ \perp$	$\perp\ \bullet\ \perp$	$\perp\ \bullet\ \top$	$\perp\ -\ \top$
x_3	$\top\ -\ \perp$	$\top\ +\ \top$	$\top\ +\ \top$	$\top\ -\ \perp$	$\top\ \bullet\ \perp$	$\top\ -\ \top$	$\top\ -\ \top$
x_4	$\perp\ -\ \top$	$\perp\ \bullet\ \top$	$\perp\ \bullet\ \top$	$\perp\ \bullet\ \perp$	$\perp\ +\ \perp$	$\perp\ +\ \top$	$\perp\ \bullet\ \top$
x_5	$\top\ +\ \top$	$\top\ -\ \top$	$\top\ \bullet\ \top$	$\top\ +\ \top$	$\top\ \bullet\ \perp$	$\top\ -\ \perp$	$\top\ +\ \perp$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$C_4 = x_2 \vee \neg x_3 \vee x_5$

States: $\{ \bot , \top \}^2$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top\ \bullet\ \bot$	$\top\ -\ \bot$	$\top\ +\ \top$	$\top\ \bullet\ \bot$	$\top\ -\ \bot$	$\top\ +\ \top$	$\top\ \bullet\ \bot$
x_2	$\bot\ \bullet\ \bot$	$\bot\ \bullet\ \bot$	$\bot\ -\ \top$	$\bot\ +\ \bot$	$\bot\ \bullet\ \bot$	$\bot\ \bullet\ \top$	$\bot\ -\ \top$
x_3	$\top\ -\ \bot$	$\top\ +\ \top$	$\top\ +\ \top$	$\top\ -\ \bot$	$\top\ \bullet\ \bot$	$\top\ -\ \top$	$\top\ -\ \top$
x_4	$\bot\ -\ \top$	$\bot\ \bullet\ \top$	$\bot\ \bullet\ \top$	$\bot\ \bullet\ \bot$	$\bot\ +\ \bot$	$\bot\ +\ \top$	$\bot\ \bullet\ \top$
x_5	$\top\ +\ \top$	$\top\ -\ \top$	$\top\ \bullet\ \top$	$\top\ +\ \top$	$\top\ \bullet\ \bot$	$\top\ -\ \bot$	$\top\ +\ \bot$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$C_4 = x_2 \vee \neg x_3 \vee x_5$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top \bullet \bot$	$\top - \bot$	$\top + \top$	$\top \bullet \bot$	$\top - \bot$	$\top + \top$	$\top \bullet \bot$
x_2	$\bot \bullet \bot$	$\bot \bullet \bot$	$\bot - \top$	$\bot + \bot$	$\bot \bullet \bot$	$\bot \bullet \top$	$\bot - \top$
x_3	$\top - \bot$	$\top + \top$	$\top + \top$	$\top - \bot$	$\top \bullet \bot$	$\top - \top$	$\top - \top$
x_4	$\bot - \top$	$\bot \bullet \top$	$\bot \bullet \top$	$\bot \bullet \bot$	$\bot + \bot$	$\bot + \top$	$\bot \bullet \top$
x_5	$\top + \top$	$\top - \top$	$\top \bullet \top$	$\top + \top$	$\top \bullet \bot$	$\top - \bot$	$\top + \bot$

States: $\{ \bot , \top \}^2$

weights:

- all valid tiles have weight 1
- Except lower right corner

$$\text{weight} = \begin{cases} 1 & \text{if } \top \\ 0 & \text{if } \bot \end{cases}$$

#-SAT

$$\varphi = \bigwedge_i C_i$$

$$C_4 = x_2 \vee \neg x_3 \vee x_5$$

G_φ	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	$\top$ • $\perp$	$\top$ - $\perp$	$\top$ + $\top$	$\top$ • $\perp$	$\top$ - $\perp$	$\top$ + $\top$	$\top$ • $\perp$
x_2	$\perp$ • $\perp$	$\perp$ • $\perp$	$\perp$ - $\top$	$\perp$ + $\perp$	$\perp$ • $\perp$	$\perp$ • $\top$	$\perp$ - $\top$
x_3	$\top$ - $\perp$	$\top$ + $\top$	$\top$ + $\top$	$\top$ - $\perp$	$\top$ • $\perp$	$\top$ - $\top$	$\top$ - $\top$
x_4	$\perp$ - $\top$	$\perp$ • $\top$	$\perp$ • $\top$	$\perp$ • $\perp$	$\perp$ + $\perp$	$\perp$ + $\top$	$\perp$ • $\top$
x_5	$\top$ + $\top$	$\top$ - $\top$	$\top$ • $\top$	$\top$ + $\top$	$\top$ • $\perp$	$\top$ - $\perp$	$\top$ + $\perp$

States: $\{\perp, \top\}^2$

weights:

- all valid tiles have weight 1
- Except lower right corner

$$\text{weight} = \begin{cases} 1 & \text{if } \top \\ 0 & \text{if } \perp \end{cases}$$

$[\![\mathscr{T}]\!](G_\varphi)$ = number of satisfying assignments

EVALUATION PROBLEM

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary
Boolean	$(\{0,1\}, \vee, \wedge, 0, 1)$	NP-complete
Natural	$(\mathbb{N}, +, \times, 0, 1)$	#P-complete

COUNTING ACCEPTING RUNS

f is in #P if $f(x) = \#\mathscr{M}(x)$ number of accepting runs of a PTime TM $\mathscr{M}$ on input x

- Permanent is #P-complete (even for {0,1}-matrices)
- #SAT is #P-complete

EVALUATION PROBLEM

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in binary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary graphs
Boolean	$(\{0,1\}, \vee, \wedge, 0, 1)$	NP-complete
Natural	$(\mathbb{N}, +, \times, 0, 1)$	#P-complete
Integer	$(\mathbb{Z}, +, \times, 0, 1)$	GapP-complete
Rational	$(\mathbb{Q}, +, \times, 0, 1)$	

COUNTING ACCEPTING RUNS

DIFFERENCE ACCEPTING/REJECTING RUNS

f is in #P if $f(x) = \#\mathscr{M}(x)$ number of accepting runs of a PTime TM $\mathscr{M}$ on input x

f is in GapP if $f(x) = \#\mathscr{M}(x) - \#\overline{\mathscr{M}}(x)$ (accepting - rejecting) runs of a PTime TM $\mathscr{M}$ on input x

EVALUATION PROBLEM: TROPICAL SEMIRINGS

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in unary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary graphs
max-plus-N	$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$	
max-plus-Z	$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$	

EVALUATION PROBLEM: TROPICAL SEMIRINGS

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in unary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary graphs
max-plus-N	$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$	$FP^{NP[\log]}$ – complete
max-plus-Z	$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$	

$f \in FP^{NP[\log]}$ if $f(x)$ computed by a PTime TM $\mathscr{M}$ with $\log |x|$ many queries to an NP machine

EVALUATION PROBLEM: TROPICAL SEMIRINGS

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in unary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Upper bound: $(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

- Compute $m = |V_G| \times$ minweight and $M = |V_G| \times$ maxweight
- Binary search in $[m, M]$:

  call an NP machine to check if there is some $\rho : V_G \to Q$ with weight$(\rho) \geq k$

$f \in FP^{NP[\log]}$ if $f(x)$ computed by a PTime TM $\mathscr{M}$ with $\log |x|$ many queries to an NP machine

EVALUATION PROBLEM: TROPICAL SEMIRINGS

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in unary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Upper bound: $(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

- Compute $m = |V_G| \times$ minweight and $M = |V_G| \times$ maxweight
- Binary search in $[m, M]$:

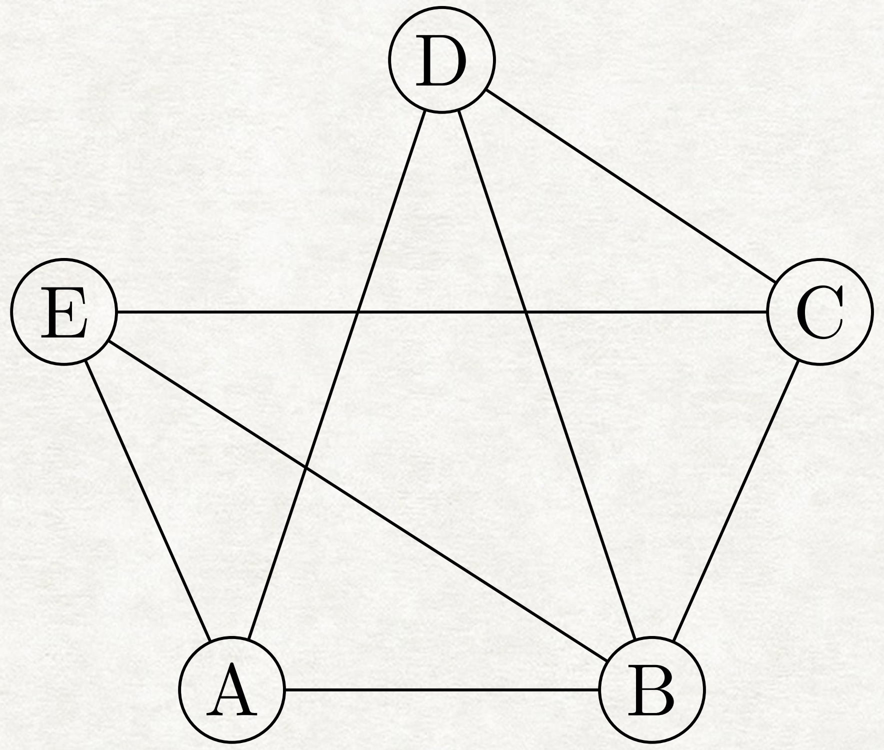
  call an NP machine to check if there is some $\rho : V_G \to Q$ with weight$(\rho) \geq k$

Lower bound:

- Clique number is $FP^{NP[\log]}$ complete

$f \in FP^{NP[\log]}$ if $f(x)$ computed by a PTime TM $\mathscr{M}$ with $\log |x|$ many queries to an NP machine

COMPUTING THE CLIQUE NUMBER



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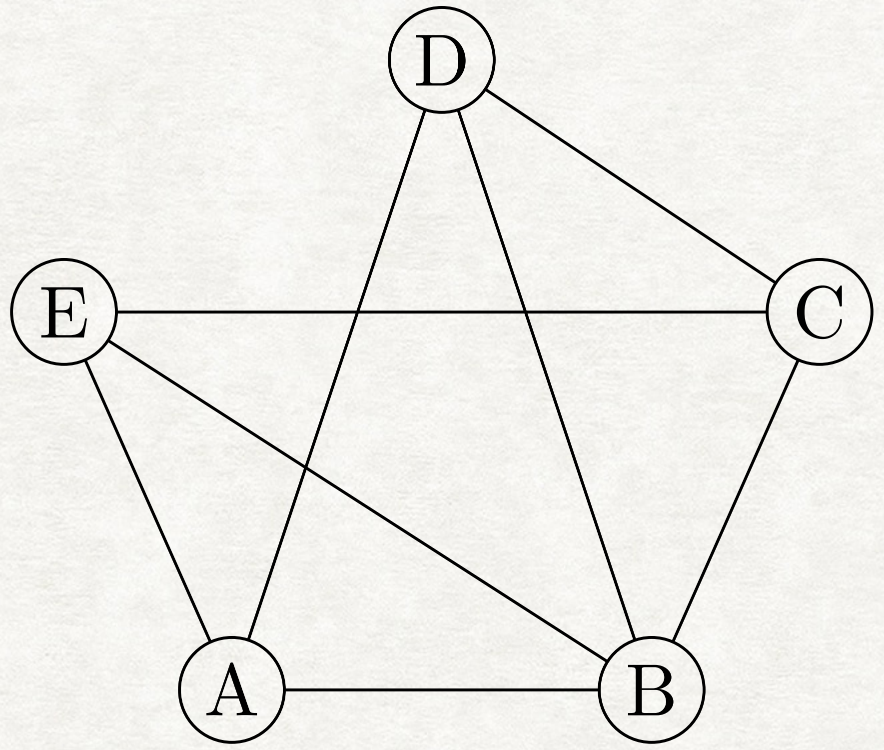

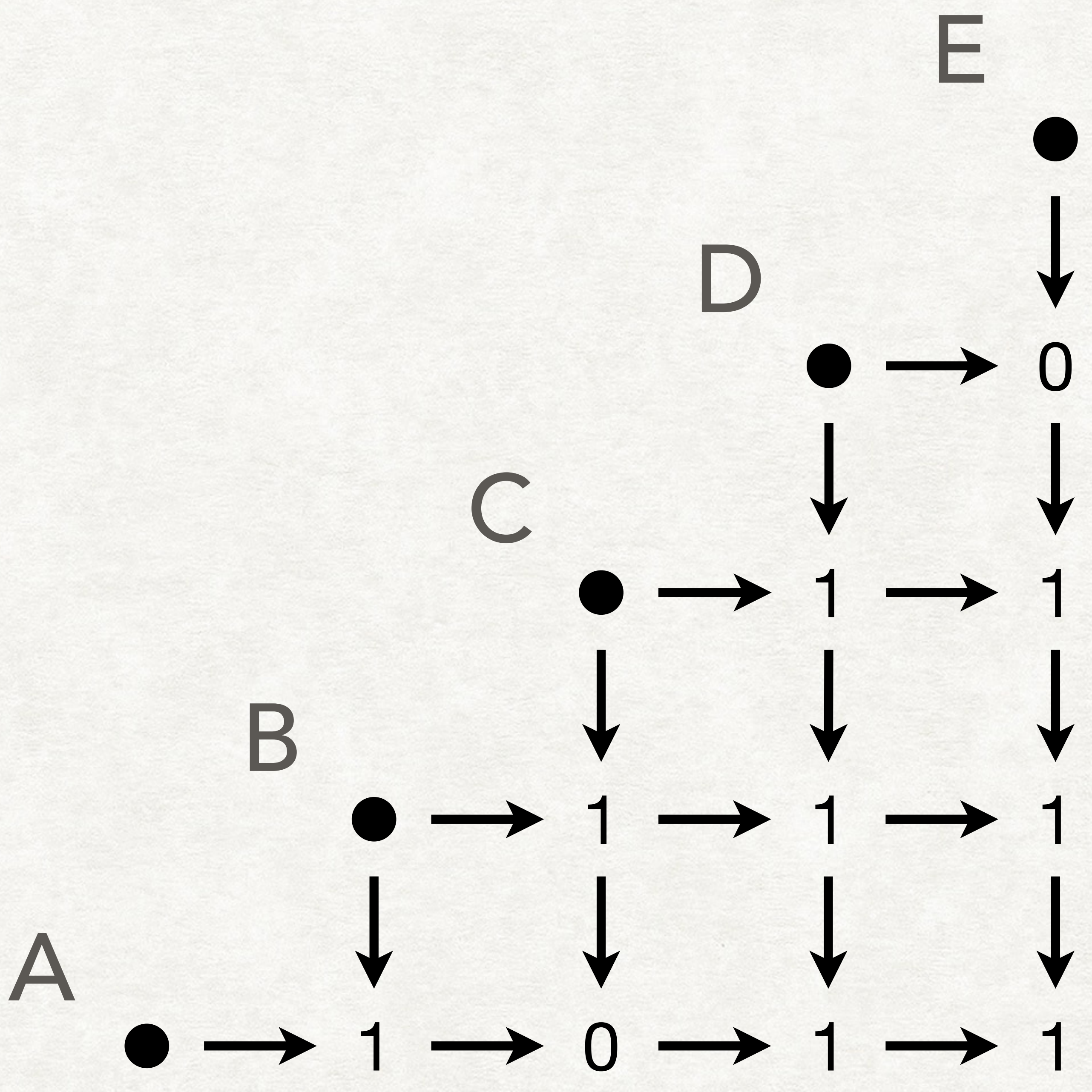

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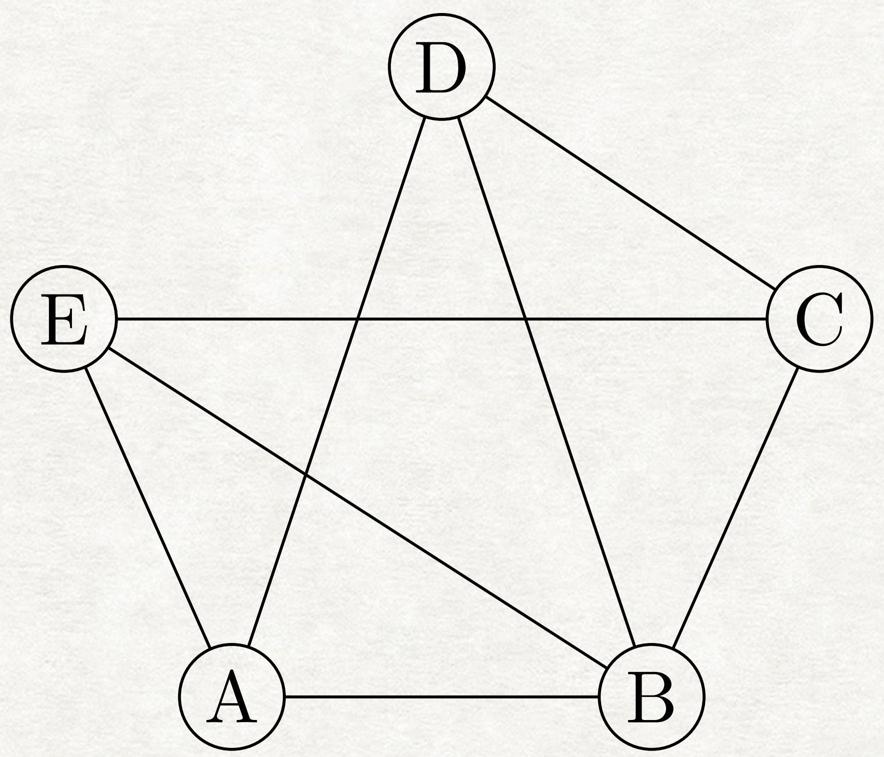

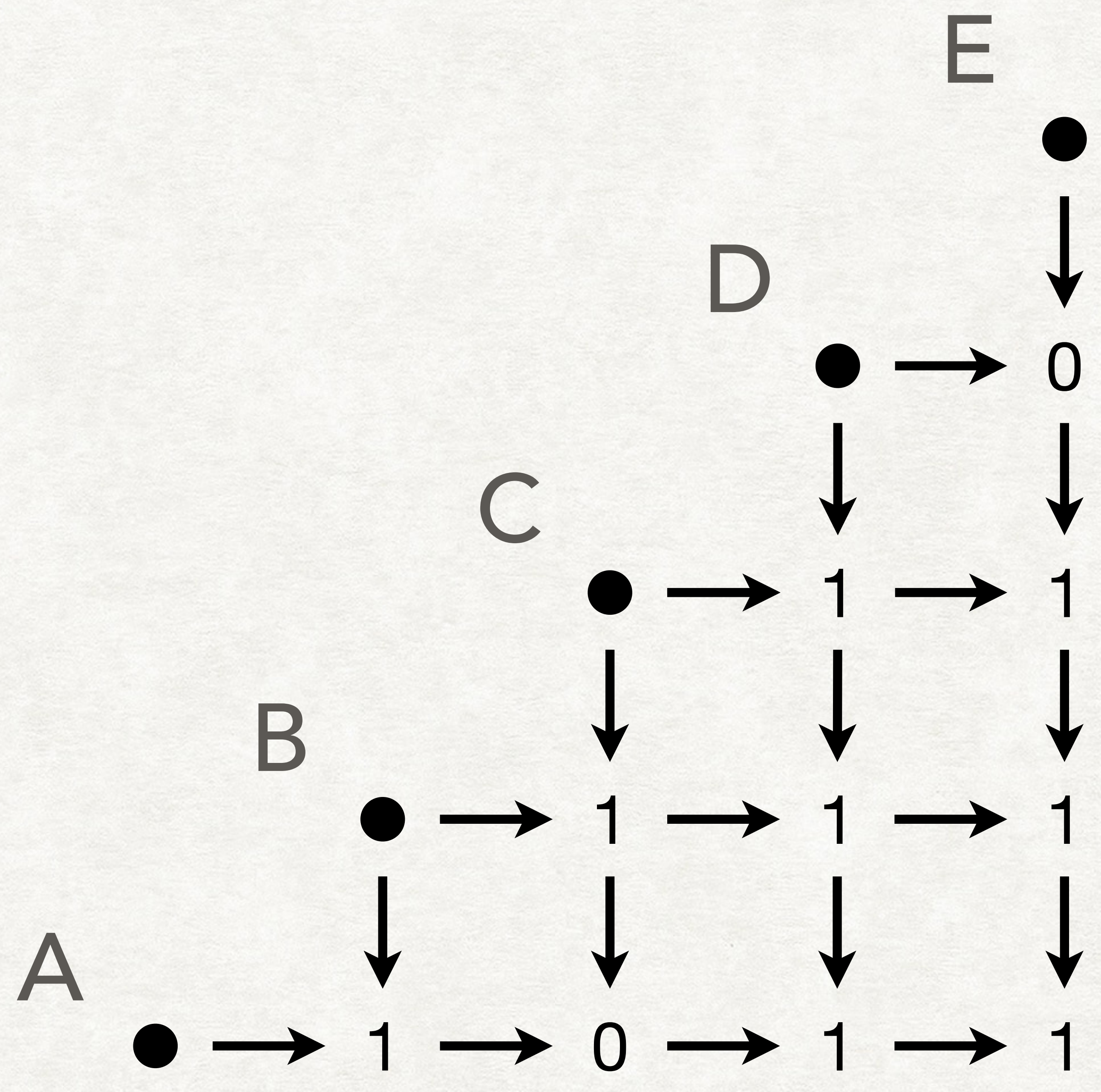

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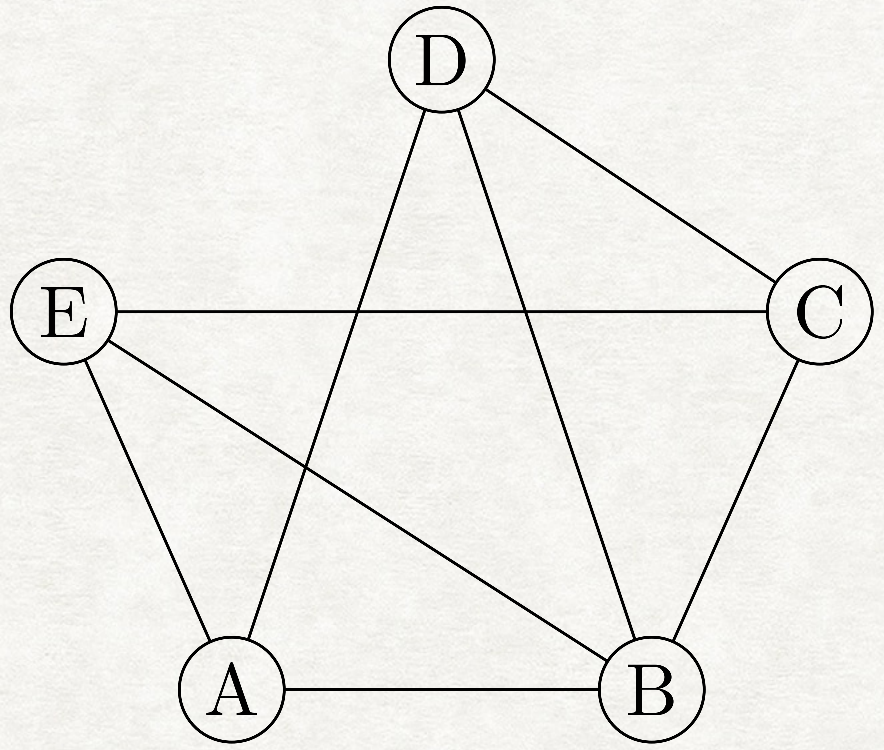

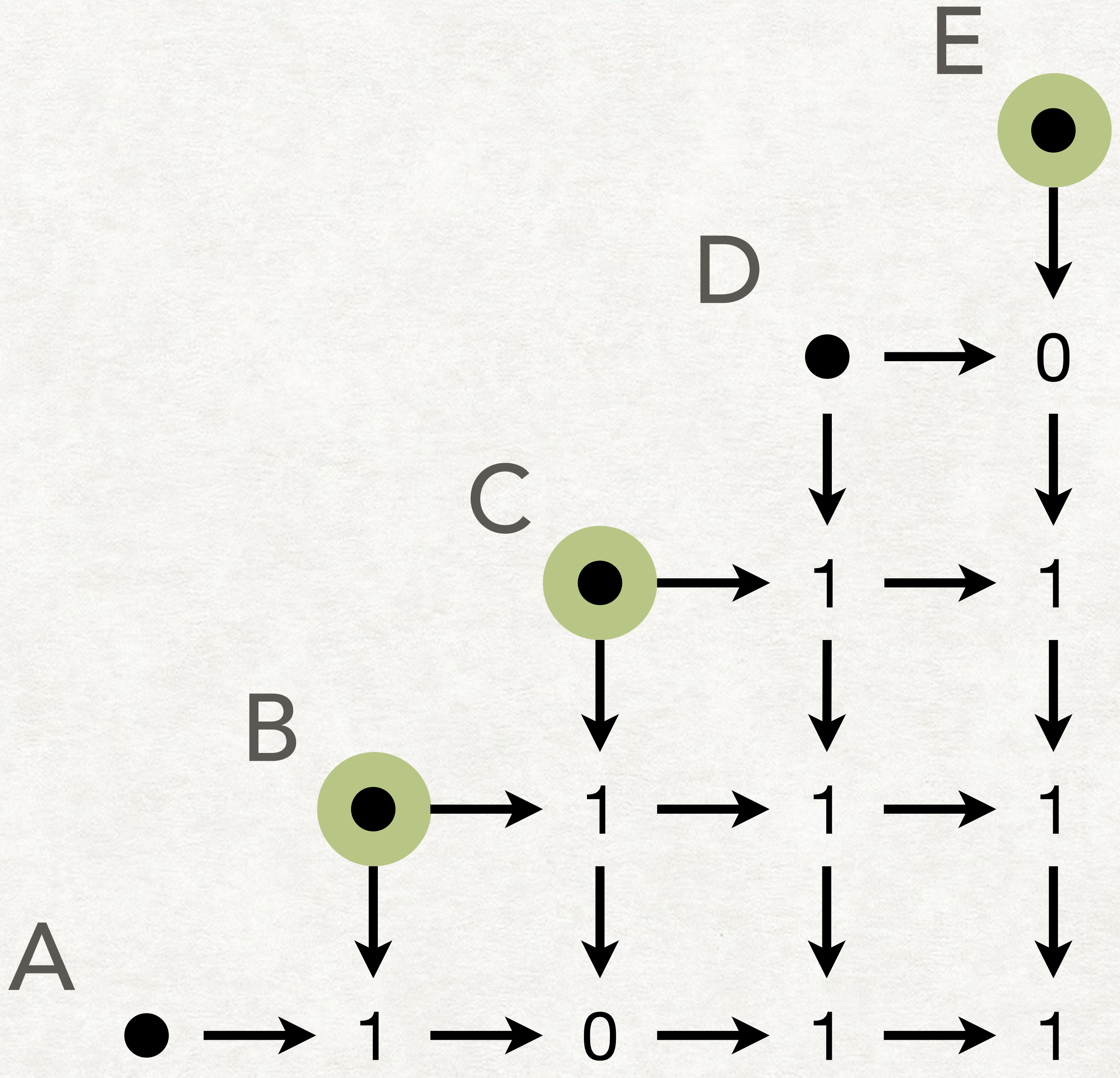

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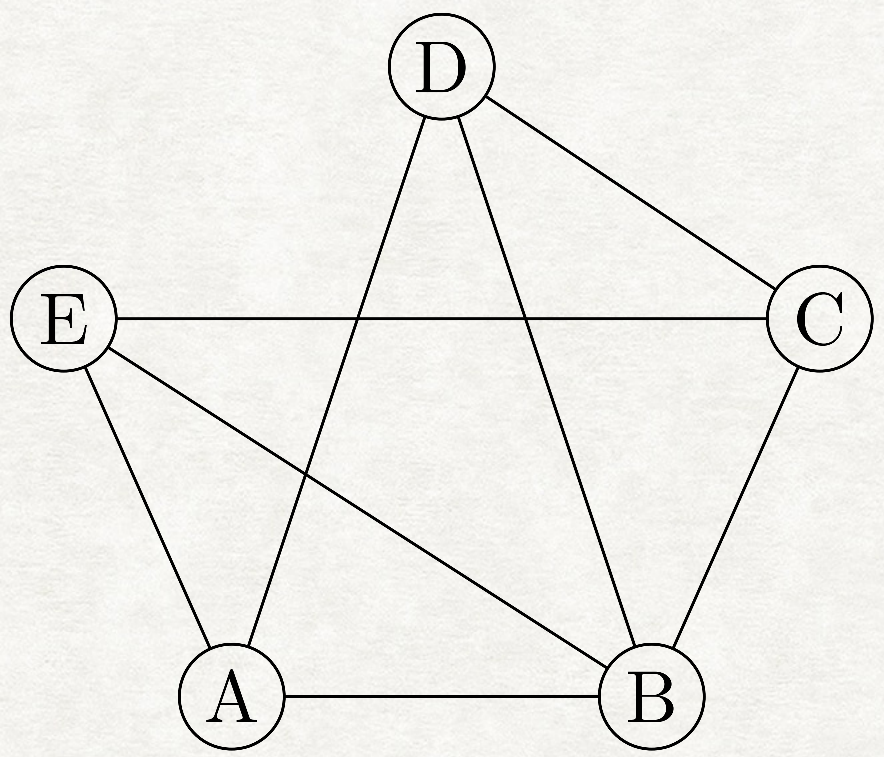

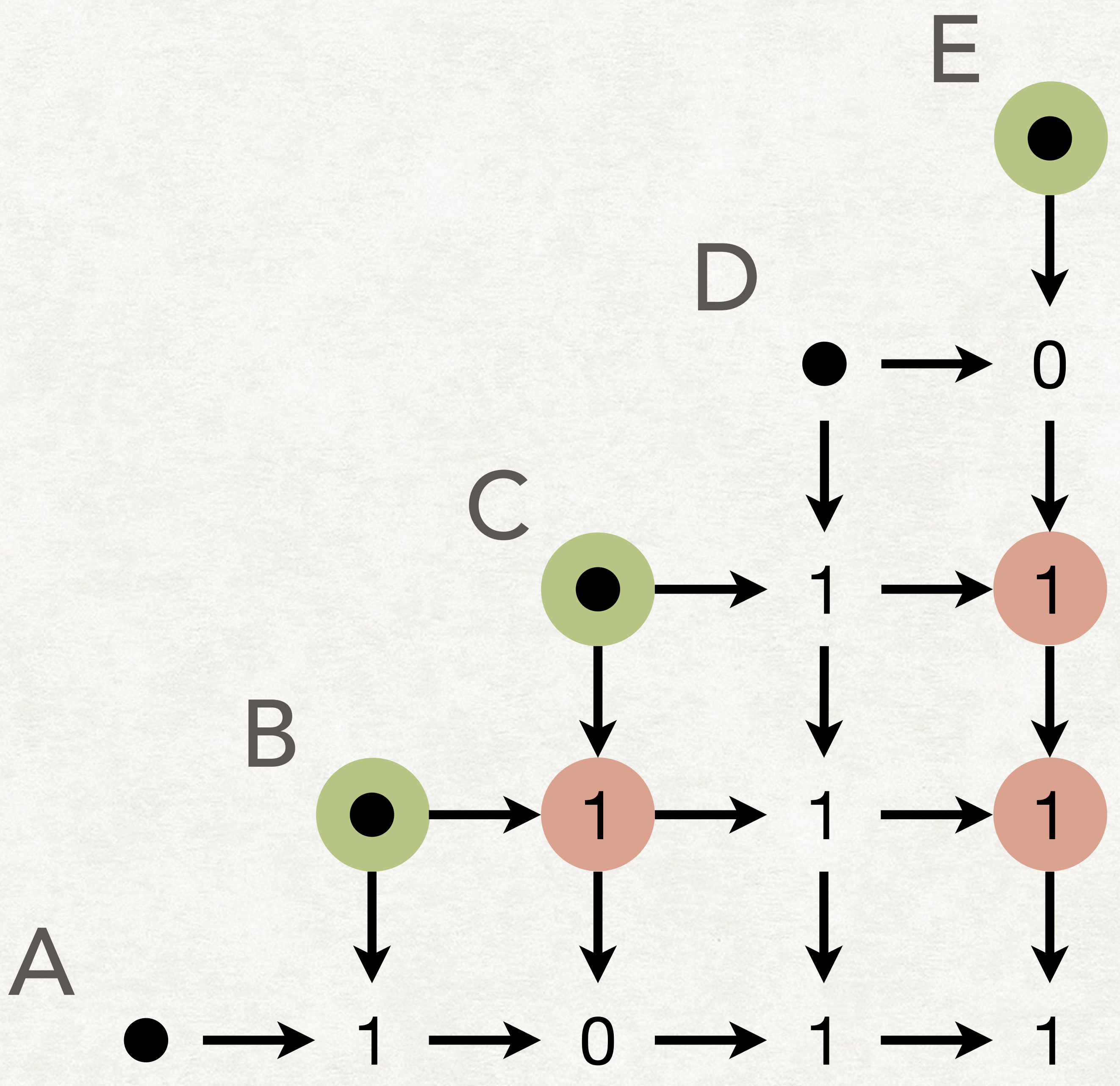

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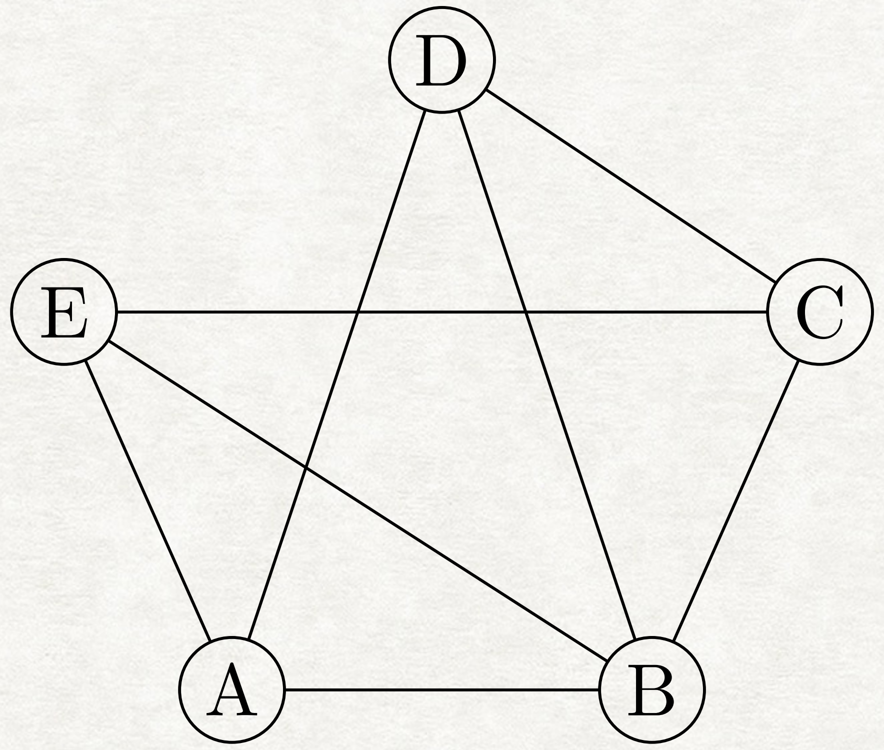

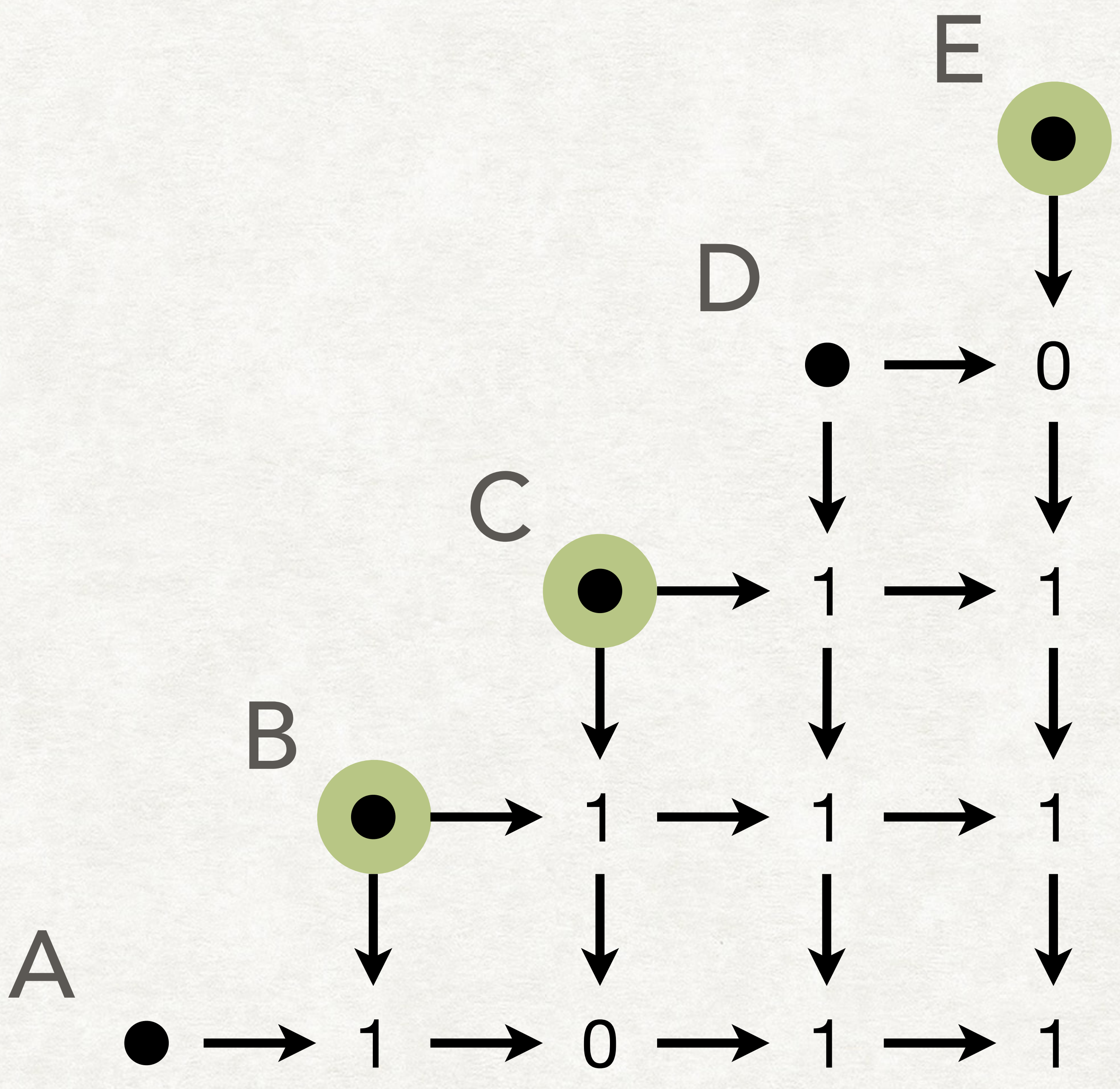

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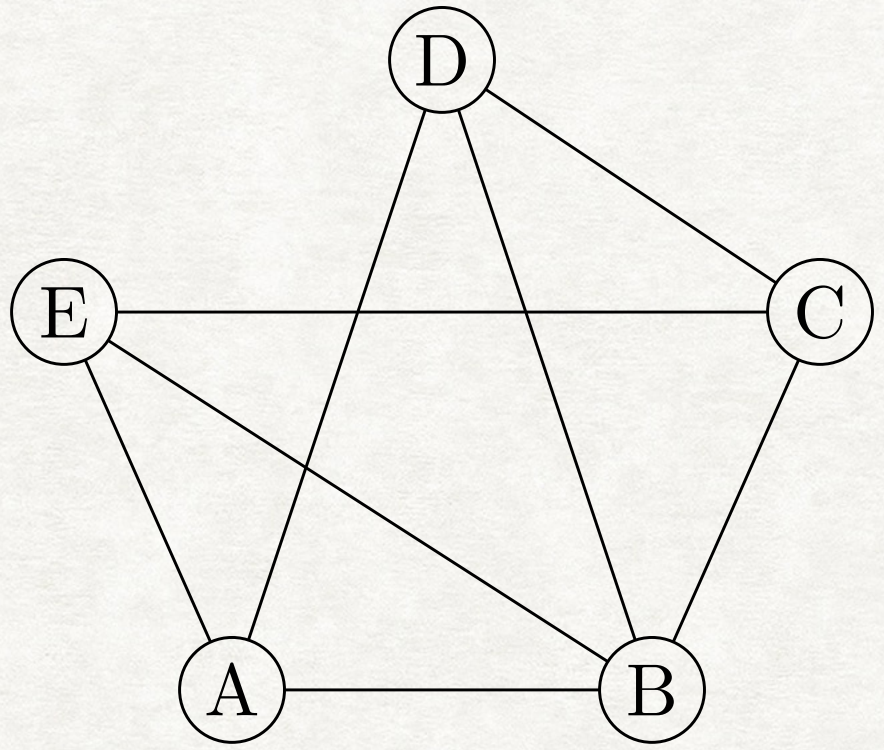

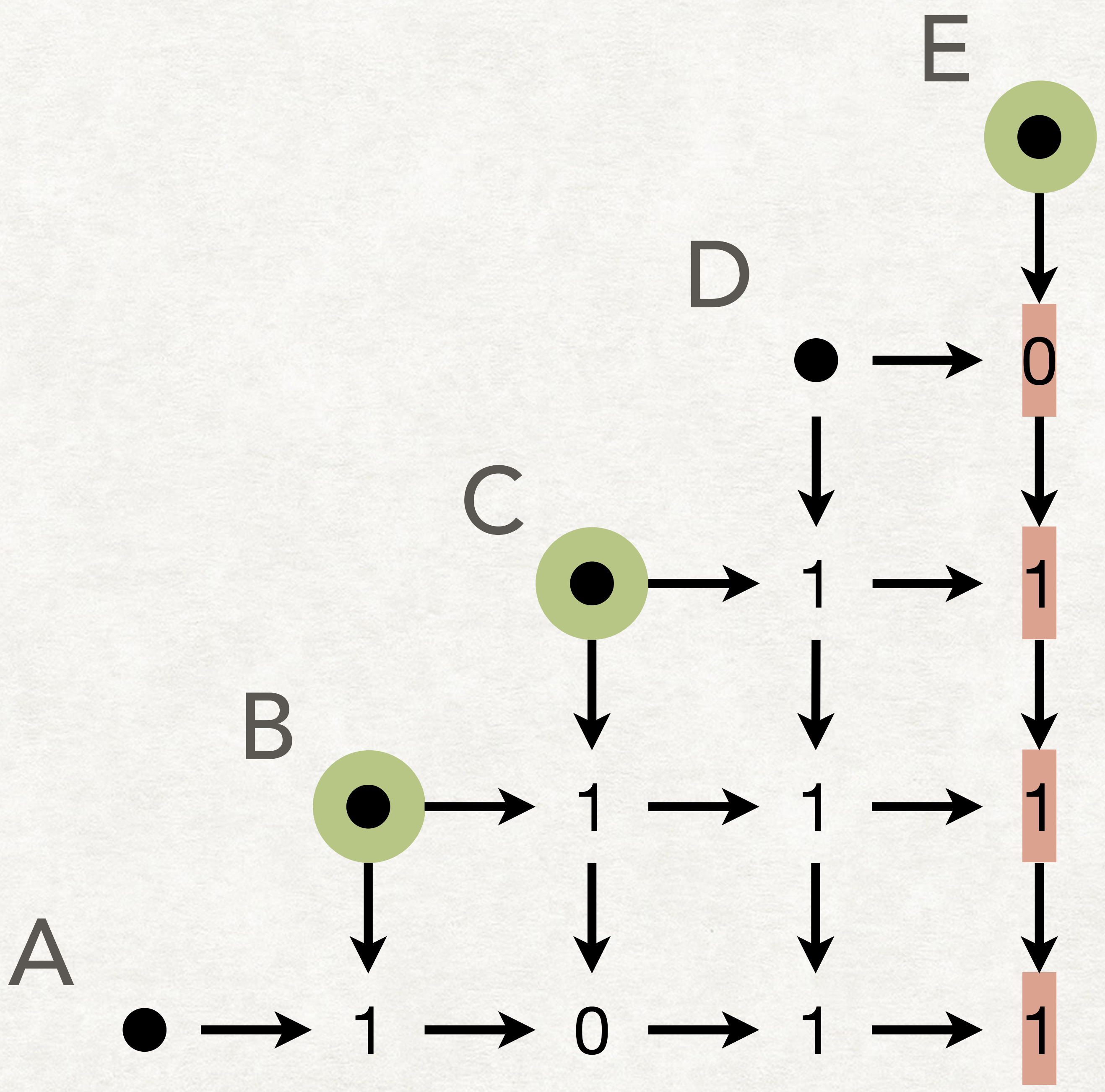

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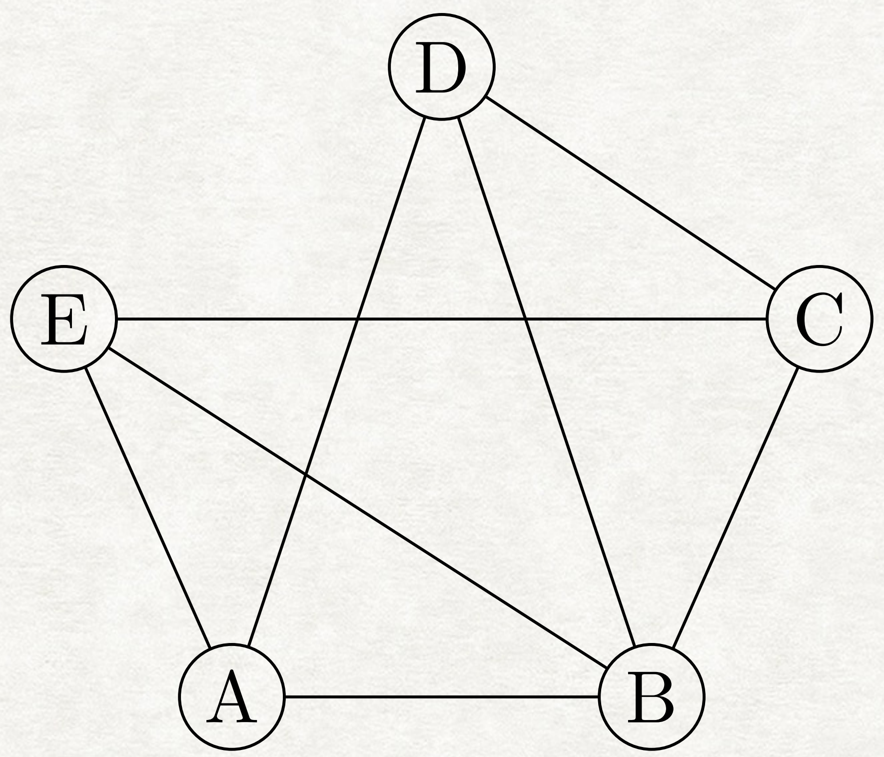

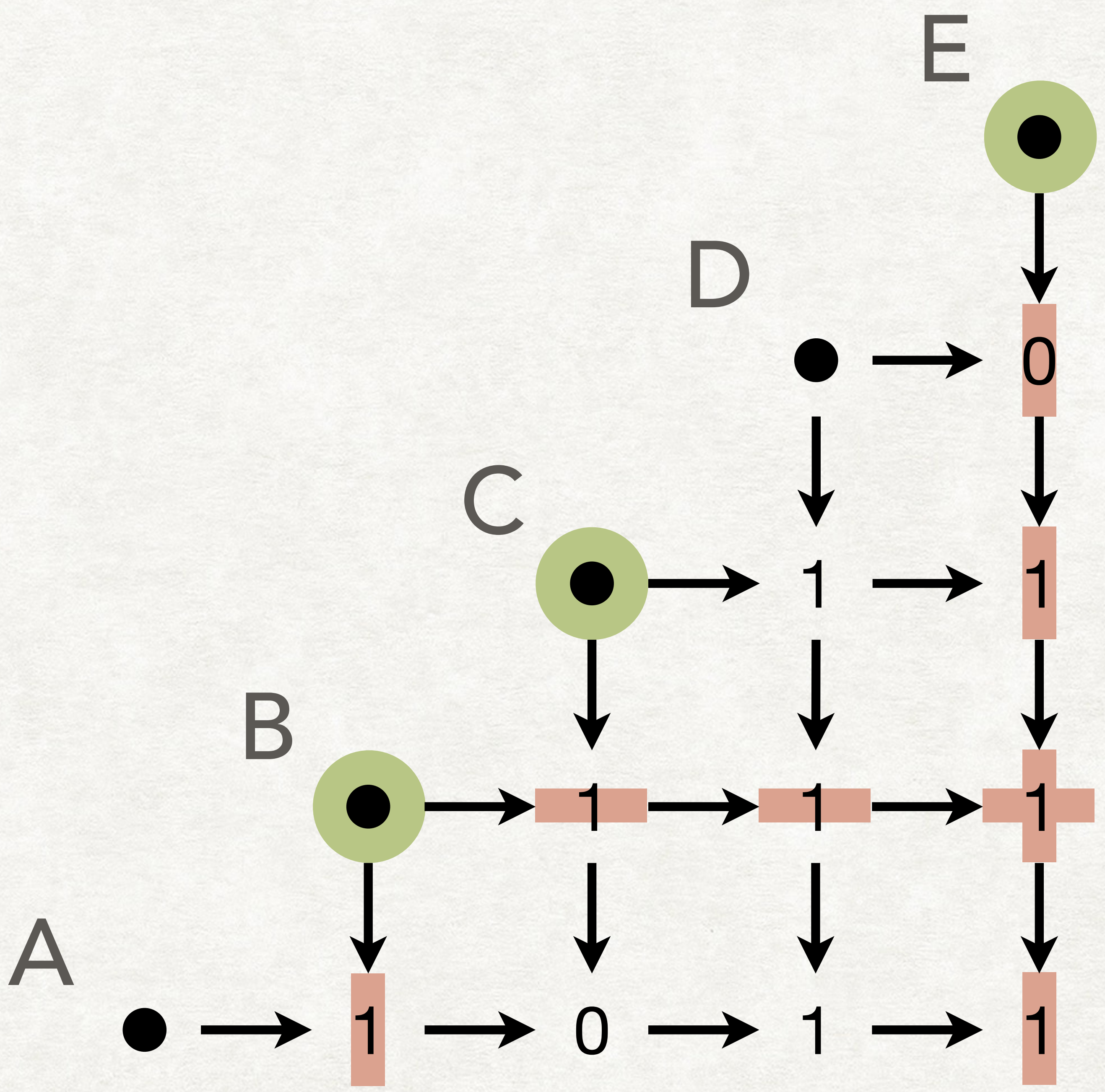

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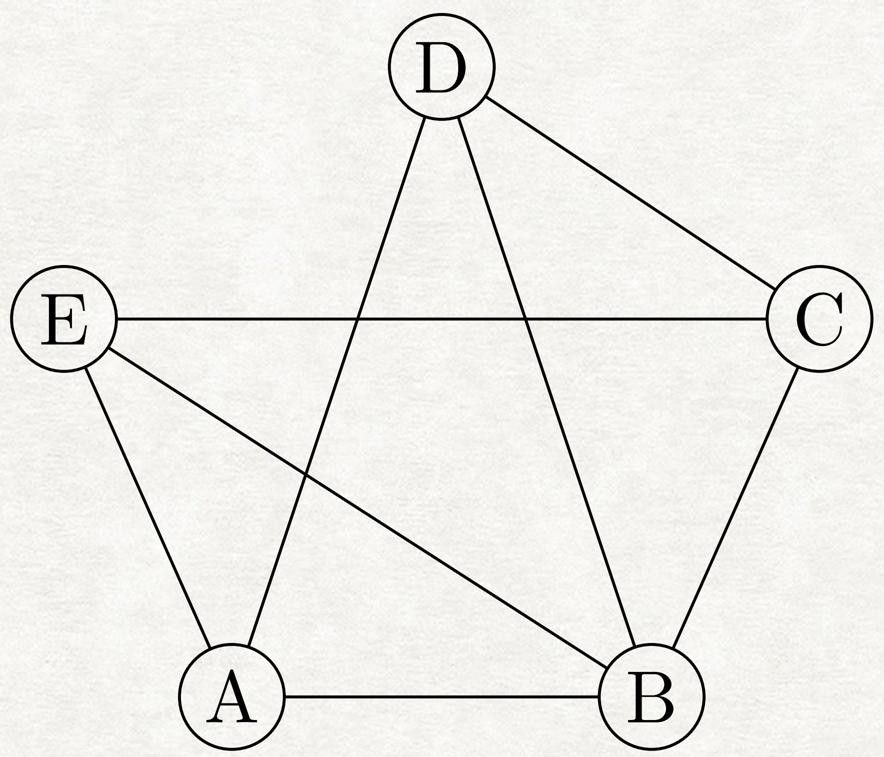

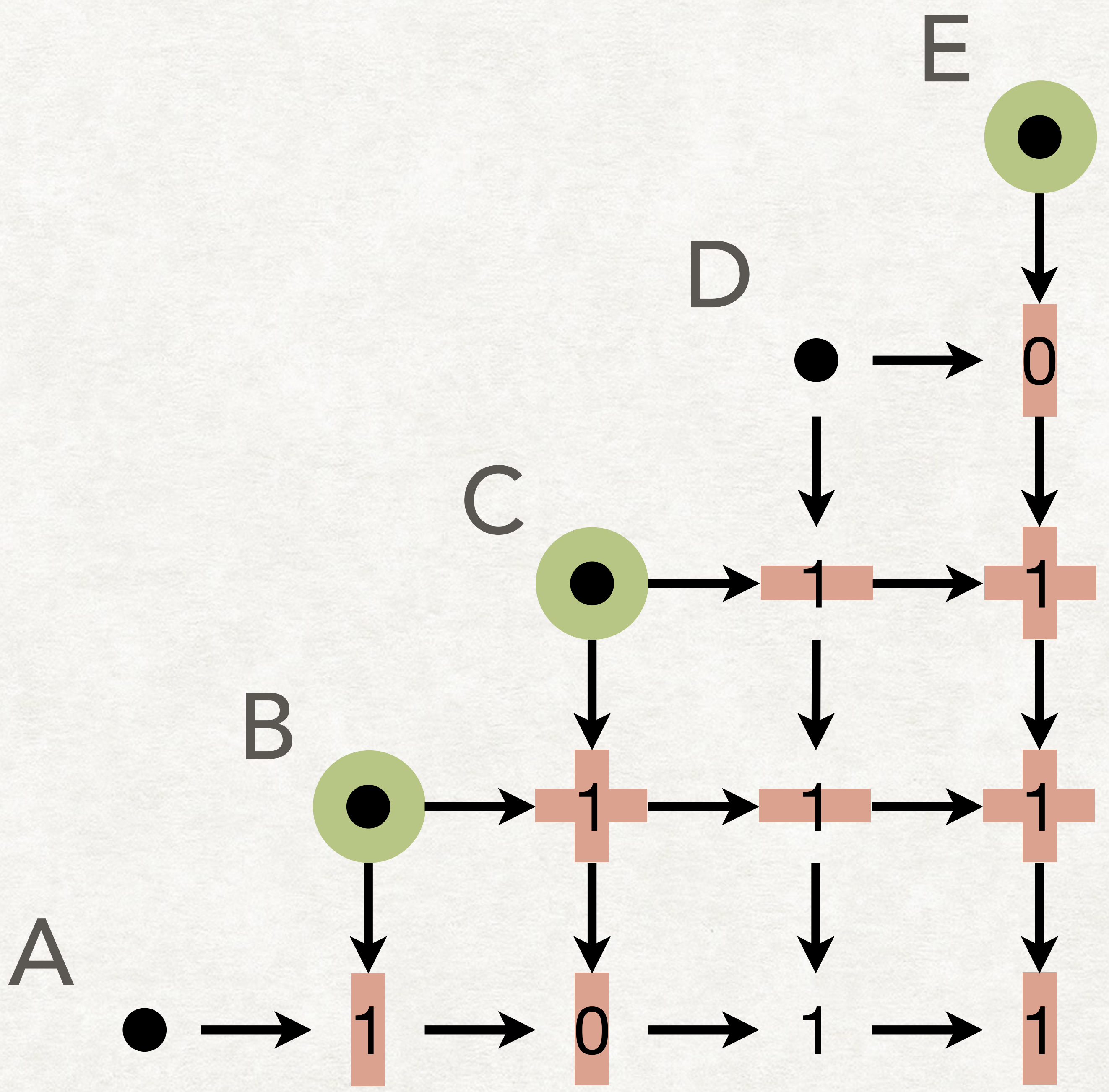

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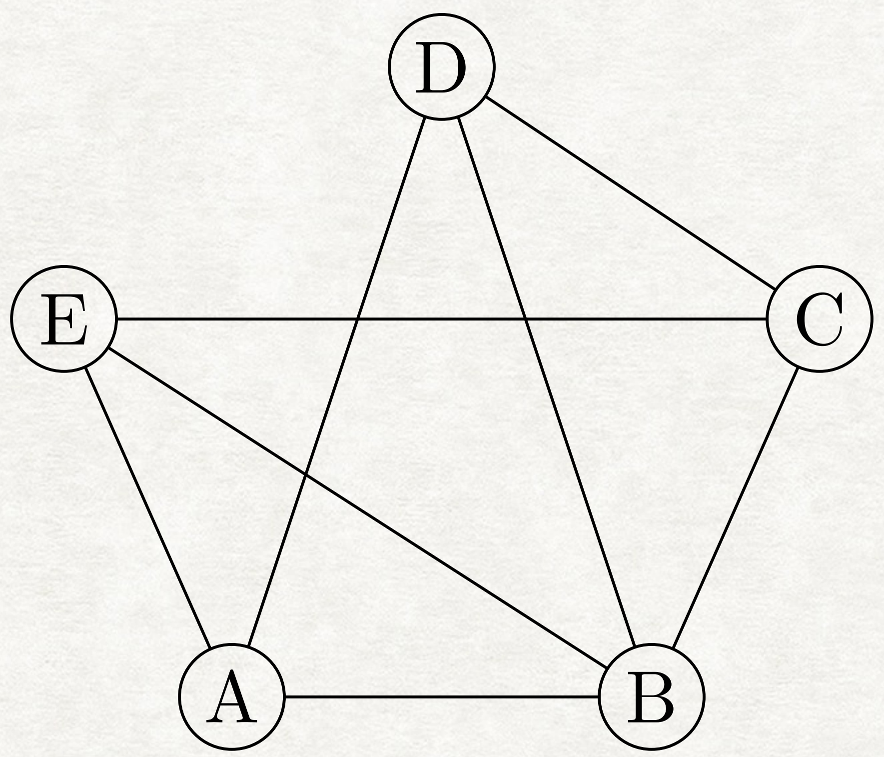
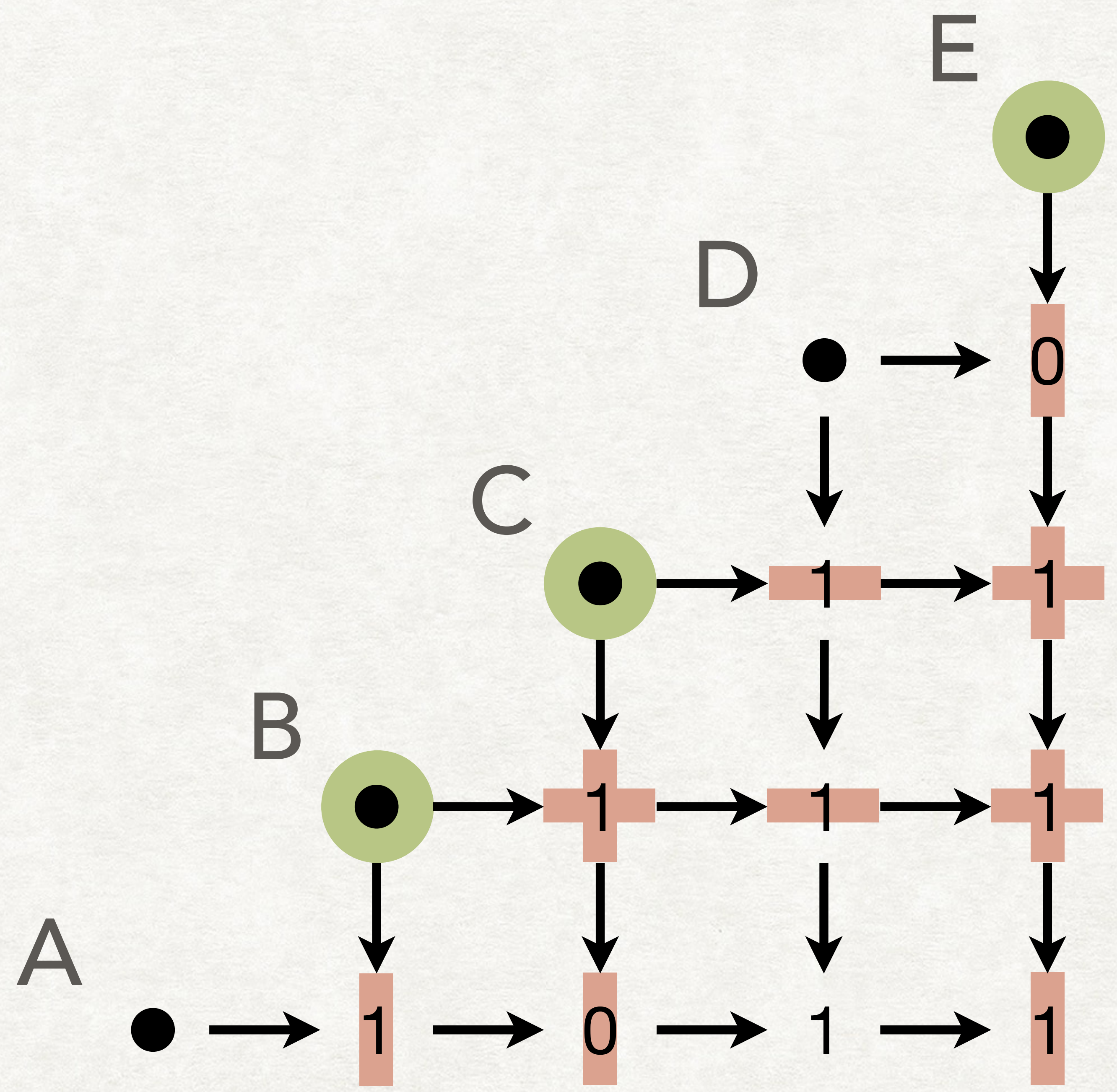
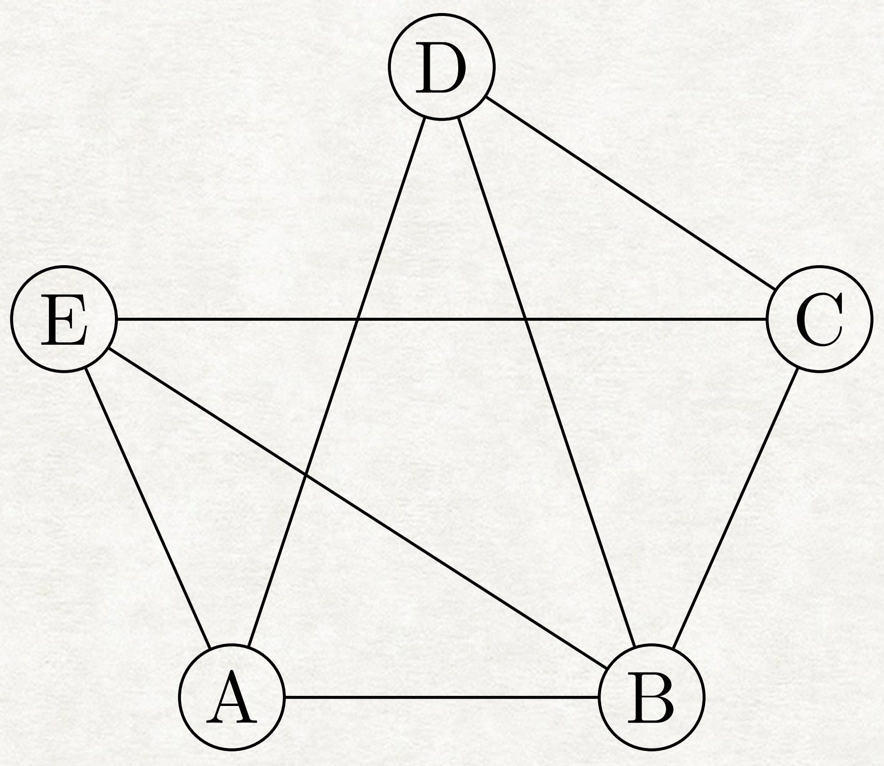
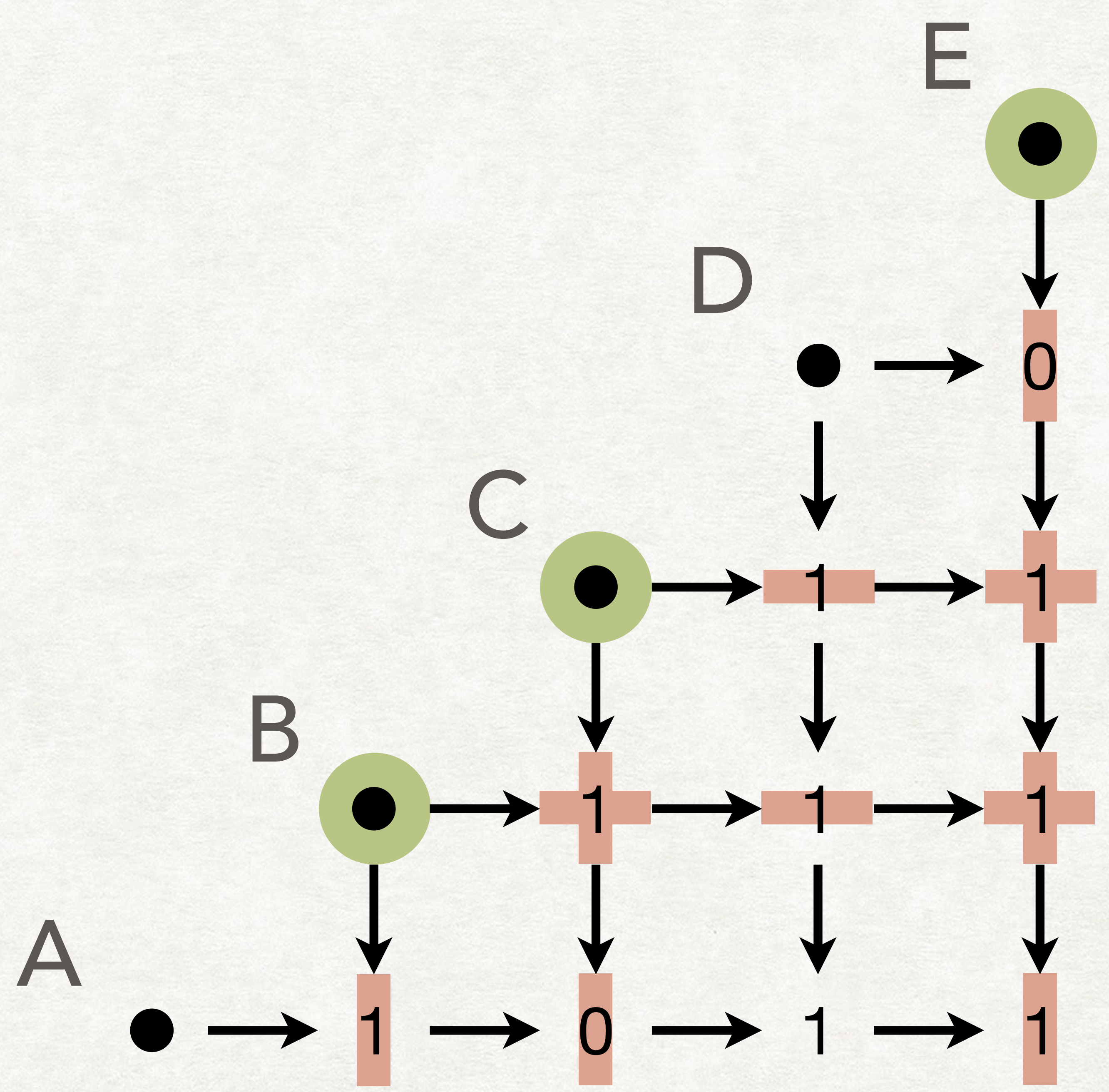




COMPUTING THE CLIQUE NUMBER

$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$





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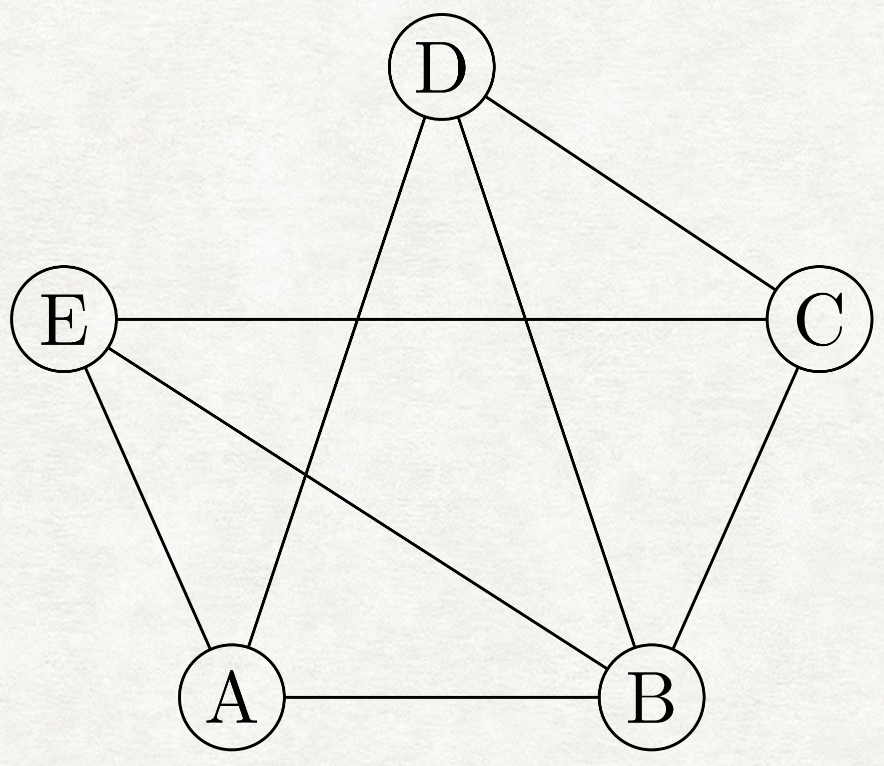
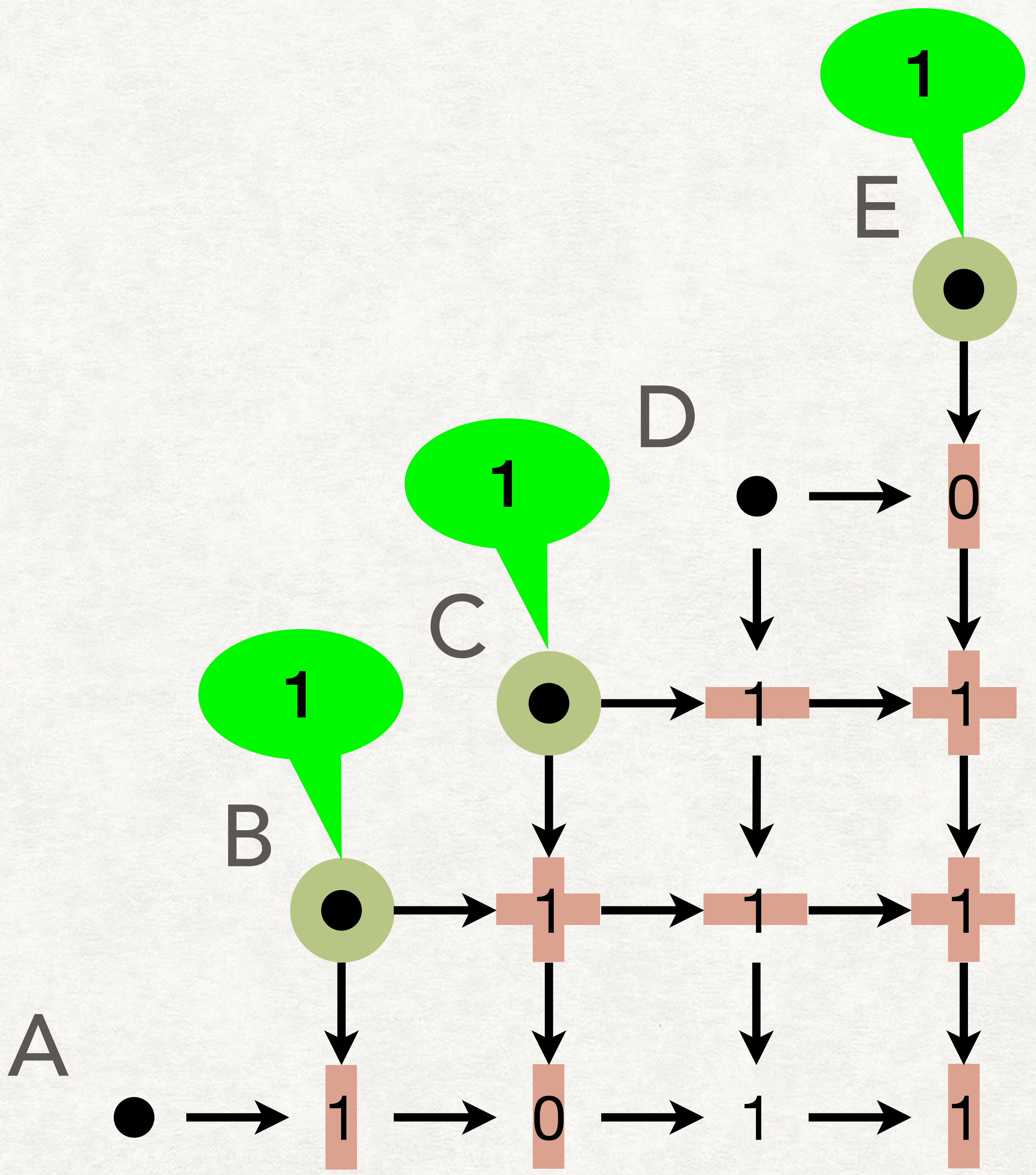
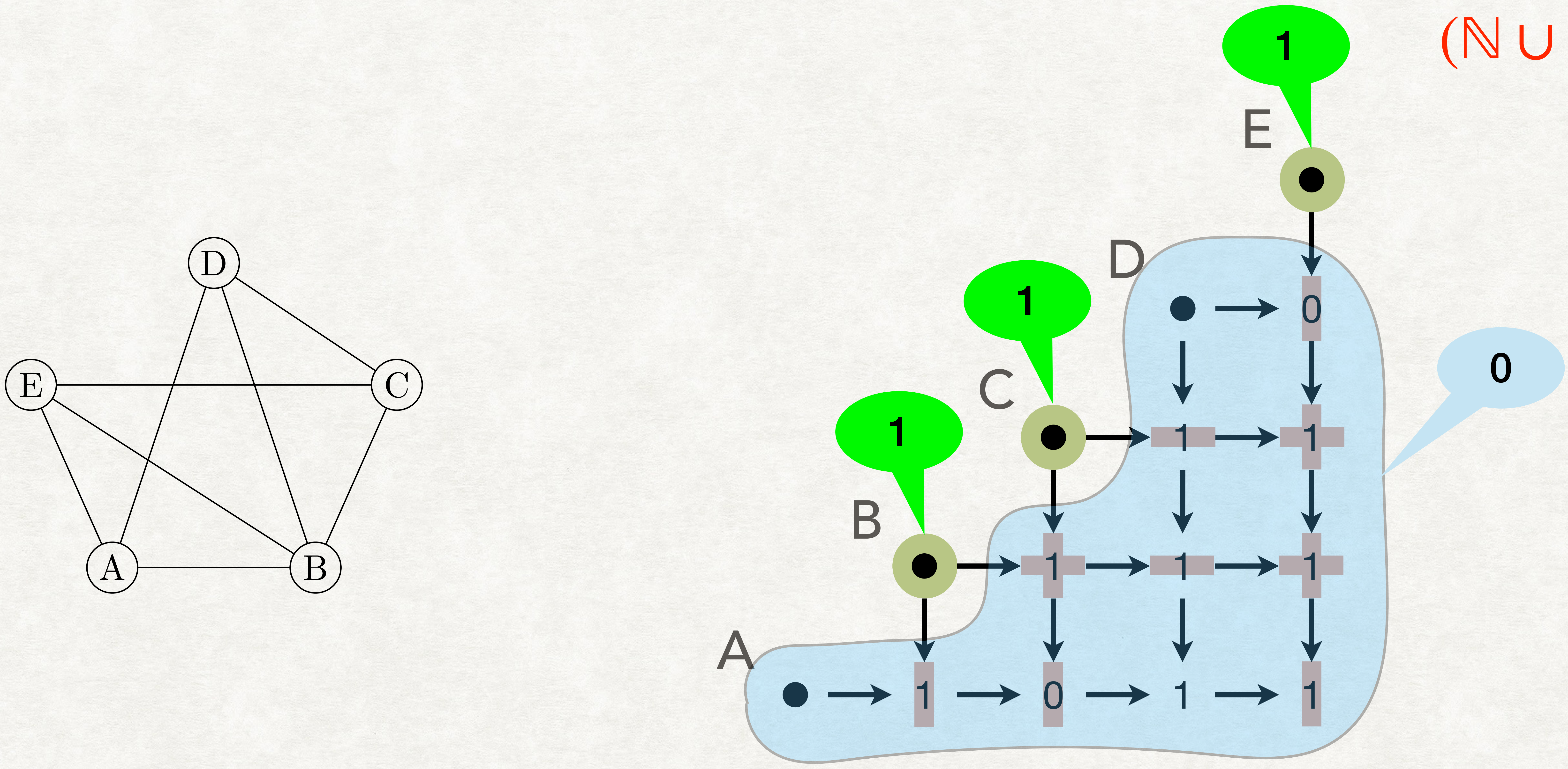




$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

COMPUTING THE CLIQUE NUMBER

$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$



EVALUATION PROBLEM: TROPICAL SEMIRINGS

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS) (weights in unary)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Semiring		Arbitrary graphs
max-plus-N	$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$	$FP^{NP[\log]}$ – complete
max-plus-Z	$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$	
min-plus-N	$(\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$	
min-plus-Z	$(\mathbb{Z} \cup \{+\infty\}, \min, +, +\infty, 0)$	

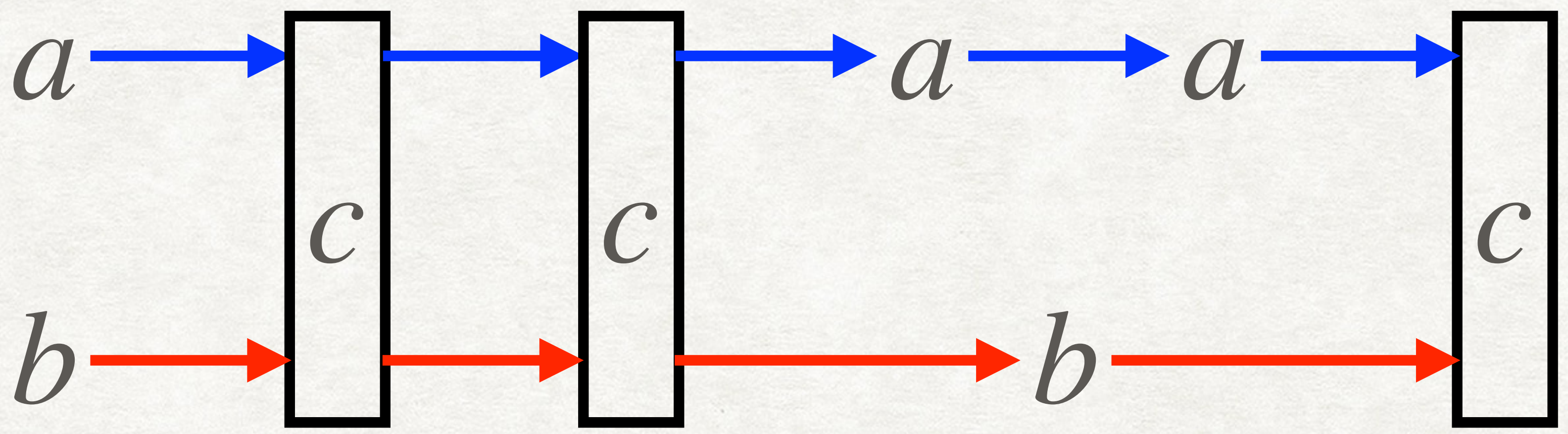
$f \in FP^{NP[\log]}$ if $f(x)$ computed by a PTime TM $\mathscr{M}$ with $\log|x|$ many queries to an NP machine

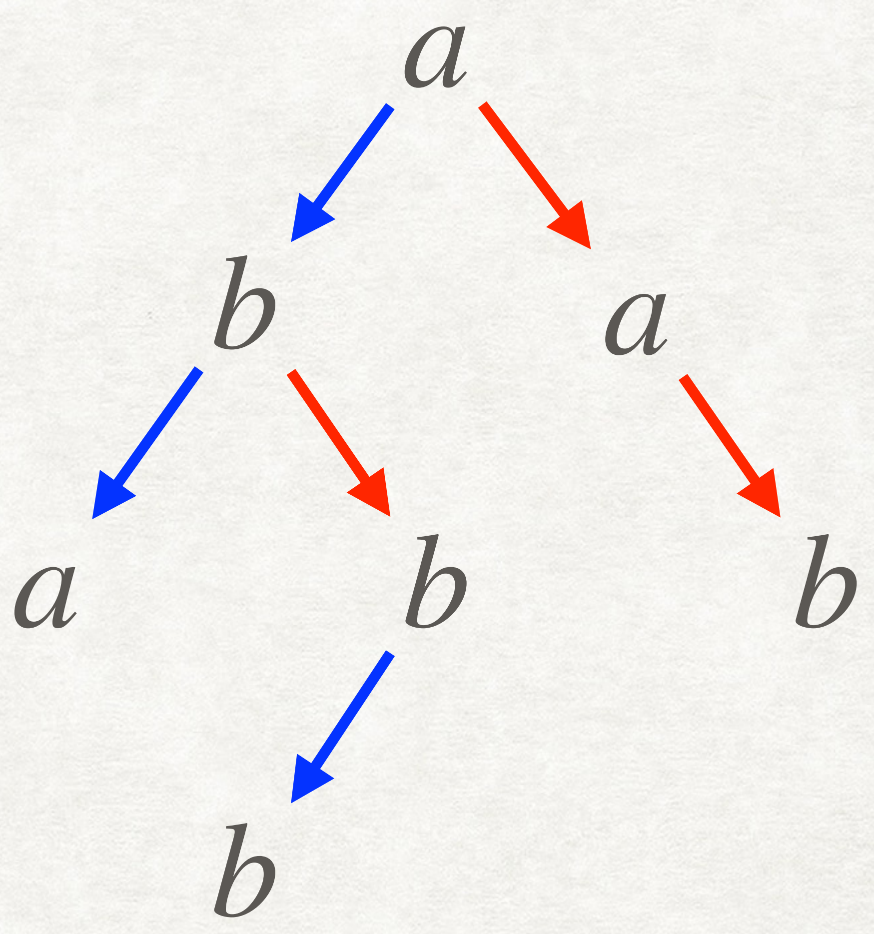
EVALUATION PROBLEM: BOUNDED TREE-WIDTH

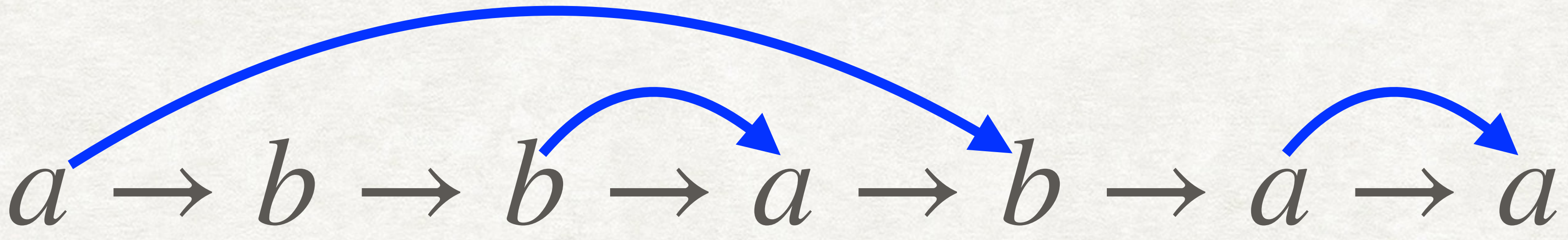
- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

- Words
- Trees
- Mazurkiewicz traces
- Message sequence charts
- Nested words
- Multiply nested words
- Pictures (grids)

$$a \to b \to b \to a \to b \to a \to a$$







EVALUATION PROBLEM: BOUNDED TREE-WIDTH

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

- Words
- Trees
- Mazurkiewicz traces
- Message sequence charts
- Nested words
- Multiply nested words
- Pictures (grids)

BOUNDED APPROXIMATIONS

EVALUATION PROBLEM: BOUNDED TREE-WIDTH

- Input: G a (Σ, Γ)-graph and $\mathscr{T}$ a weighted tiling system (WTS)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

Evaluation with fixed bound on tree-width

- Linear in G
- Polynomial in $\mathscr{T}$

- Words
- Trees
- Mazurkiewicz traces
- Message sequence charts
- Nested words
- Multiply nested words
- Pictures (grids)

BOUNDED APPROXIMATIONS

EVALUATION PROBLEM: BOUNDED TREE-WIDTH

- Input: G a (Σ, Γ)-graph of tree-width $\leq k$ and $\mathscr{T}$ a weighted tiling system (WTS)
- Problem: Compute $[\![\mathscr{T}]\!](G)$

- Get a tree decomposition using Bodlander's algorithm $\mathcal{O}(|G|)$
- Extract a tree term τ $\mathcal{O}(|G|)$
- Construct a weighted tree automaton $\mathscr{A}$ from $\mathscr{T}$ s.t. $[\![\mathscr{A}]\!](\tau) = [\![\mathscr{T}]\!](G)$ $|\mathscr{T}|^{\mathcal{O}(k)}$
  - Guess the state for each node of G (leaf of τ)
  - Maintain the tile for every "active" node
  - Weight of a tile given when a node becomes "inactive"
  - Weight of all other transitions is 1.
- Compute $[\![\mathscr{A}]\!](\tau)$ $\mathcal{O}(|\tau| \cdot |\mathscr{A}|)$

SUMMARY

<table>
<tr><th colspan="2">Semiring</th><th>Arbitrary</th><th>Bounded Tree-Width</th></tr>
<tr><td>Boolean</td><td>$(\{0,1\}, \vee , \wedge ,0,1)$</td><td>NP-complete</td><td rowspan="8">Linear in the graph G

Polynomial in the WTS</td></tr>
<tr><td>Natural</td><td>$(\mathbb{N}, + , \times ,0,1)$</td><td>#P-complete</td></tr>
<tr><td>Integer</td><td>$(\mathbb{Z}, + , \times ,0,1)$</td><td rowspan="2">GapP-complete</td></tr>
<tr><td>Rational</td><td>$(\mathbb{Q}, + , \times ,0,1)$</td></tr>
<tr><td>max-plus-N</td><td>$(\mathbb{N} \cup \{-\infty\}, \max , + , - \infty,0)$</td><td rowspan="4">$FP^{NP[\log]}$ – complete
weights in unary</td></tr>
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SUMMARY

Hardness holds even with a fixed WTS

<table>
<tr><th colspan="2">Semiring</th><th>Arbitrary</th><th>Bounded Tree-Width</th></tr>
<tr><td>Boolean</td><td>$(\{0,1\}, \vee, \wedge, 0, 1)$</td><td>NP-complete</td><td rowspan="8">Linear in the graph G

Polynomial in the WTS</td></tr>
<tr><td>Natural</td><td>$(\mathbb{N}, +, \times, 0, 1)$</td><td>#P-complete</td></tr>
<tr><td>Integer</td><td>$(\mathbb{Z}, +, \times, 0, 1)$</td><td rowspan="2">GapP-complete</td></tr>
<tr><td>Rational</td><td>$(\mathbb{Q}, +, \times, 0, 1)$</td></tr>
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- Decision problems: $[\![\mathcal{T}(G)]\!] \bowtie s$ with $\bowtie \in \{ >, \geq, <, \leq, =, \neq \}$ and $s \in \mathbb{S}$
- Constraint satisfaction problem (CSP) and quantitative extensions
  - Boolean CSP — existence of a solution
  - #CSP — number of solutions
  - valued-CSP — min cost of a solution

OPEN PROBLEMS

- Tropical semirings with weights encoded in binary?
- Parametrised complexity for bounded tree-width? FPT?
- Can we use techniques from quantitative CSP?