# Regular transducer expressions for two-way deterministic transducers

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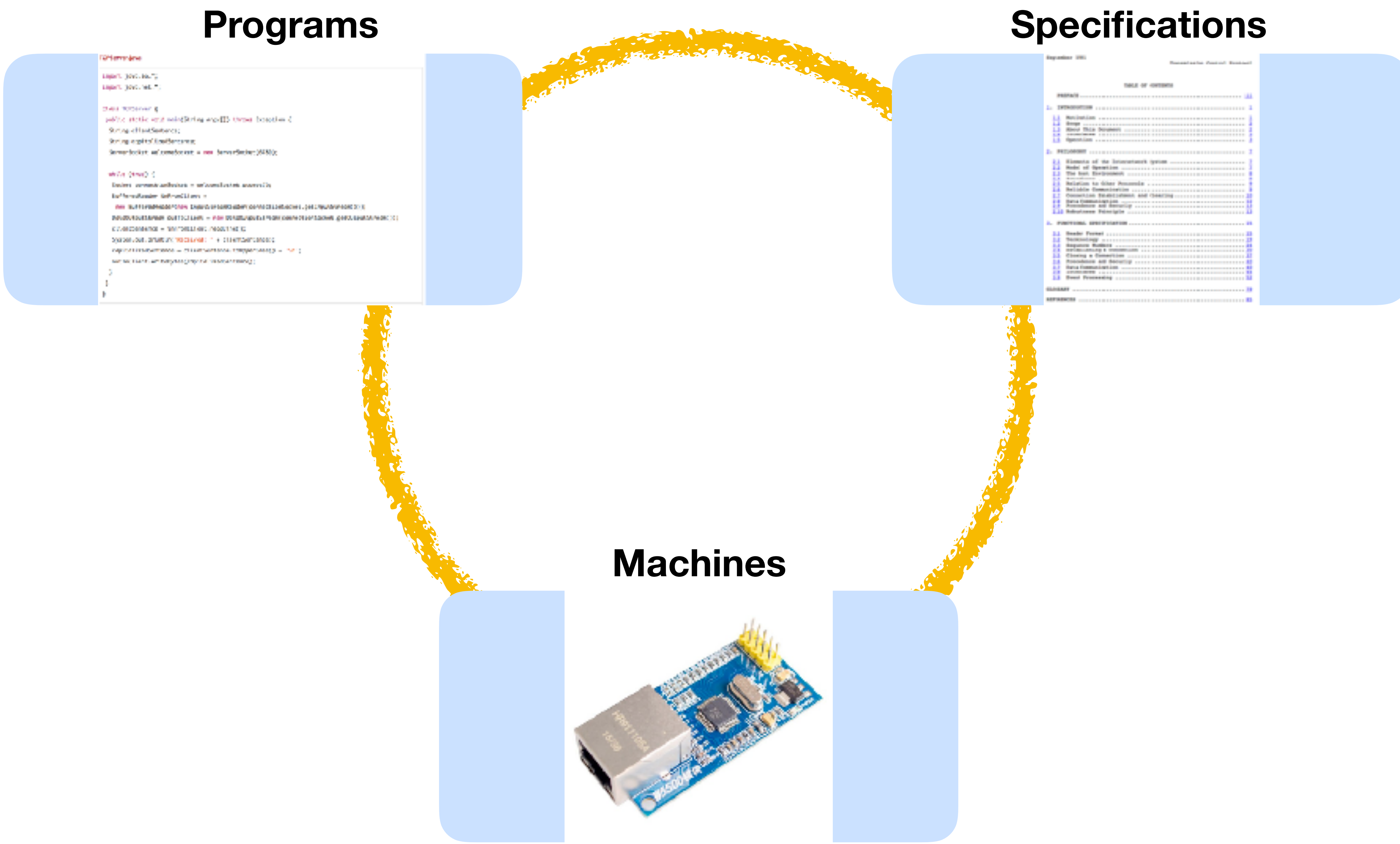
Joint work with Vrunda Dave and S. Krishna
IIT Bombay

**LICS'18**
**Also on ArXiv**

# The fundamental Triptych



# The fundamental Triptych

**Programs**

Structured, Compositional
Blocks, Sequences
Loops, Conditionals

**Specifications**

Logical: Is n prime?
Functional: encrypt, decrypt
Relational: French $\twoheadrightarrow$ English

**Machines**

Low level
Executable
Often sequential

# The fundamental Triptych

**Programs**

**Regular Expressions**

**Specifications**

**Monadic Second-order Logic**
**Linear Dynamic Logic**

**Sequential Systems**

**Kleene**

**Büchi**
**Elgot**
**Trakhtenbrot**

**Machines**

**Finite Automata**
**Deterministic**
**One-way**

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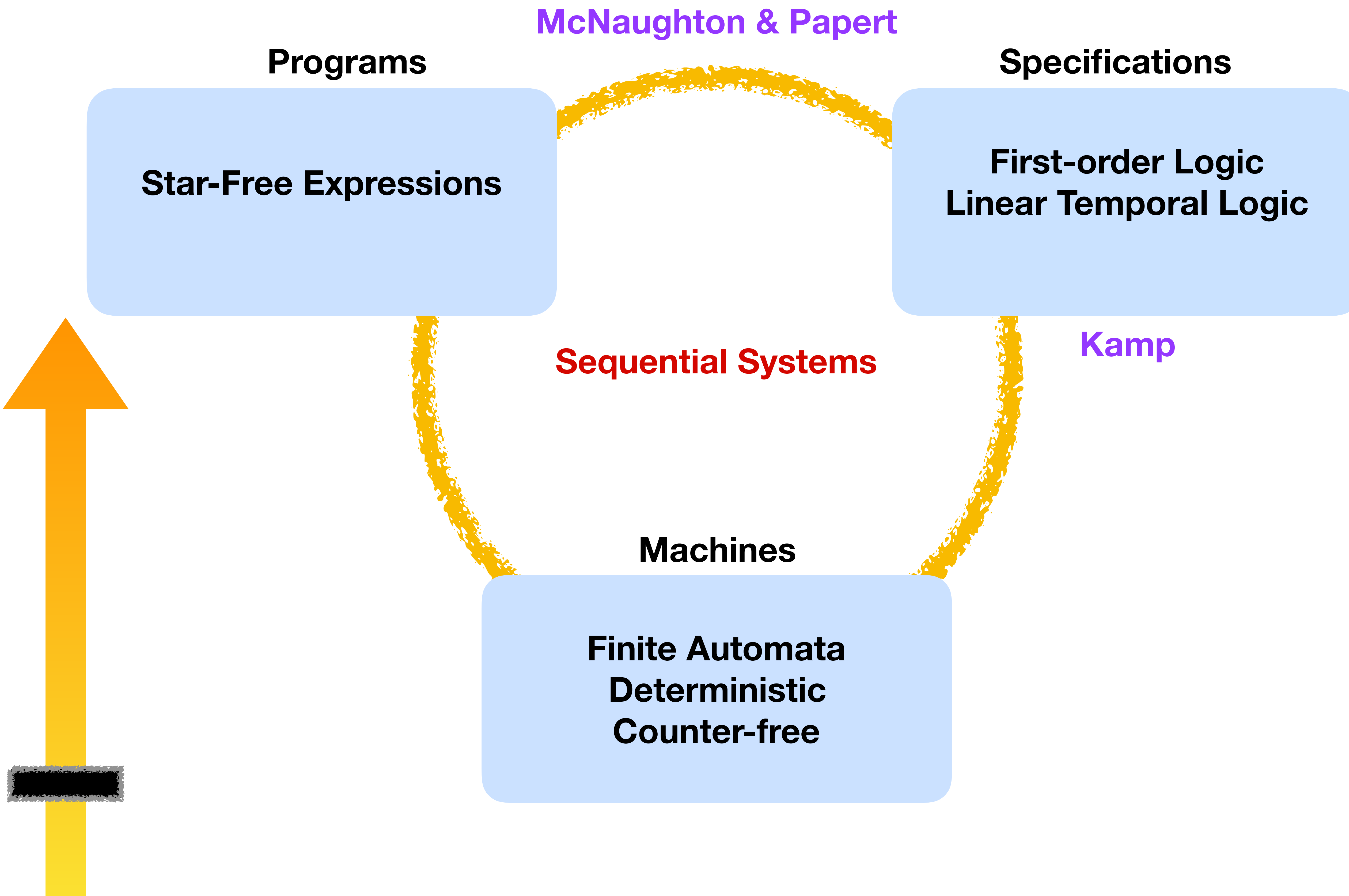

# The fundamental Triptych

**Programs**

**Context-Free Grammars**

**Specifications**

**Exists matching relation + MSO**

**Sequential Systems**

**Chomsky, Schützenberger**

**Lautemann & Schwentick & Thérien**

**Machines**

**Finite Automata non-deterministic with a stack**

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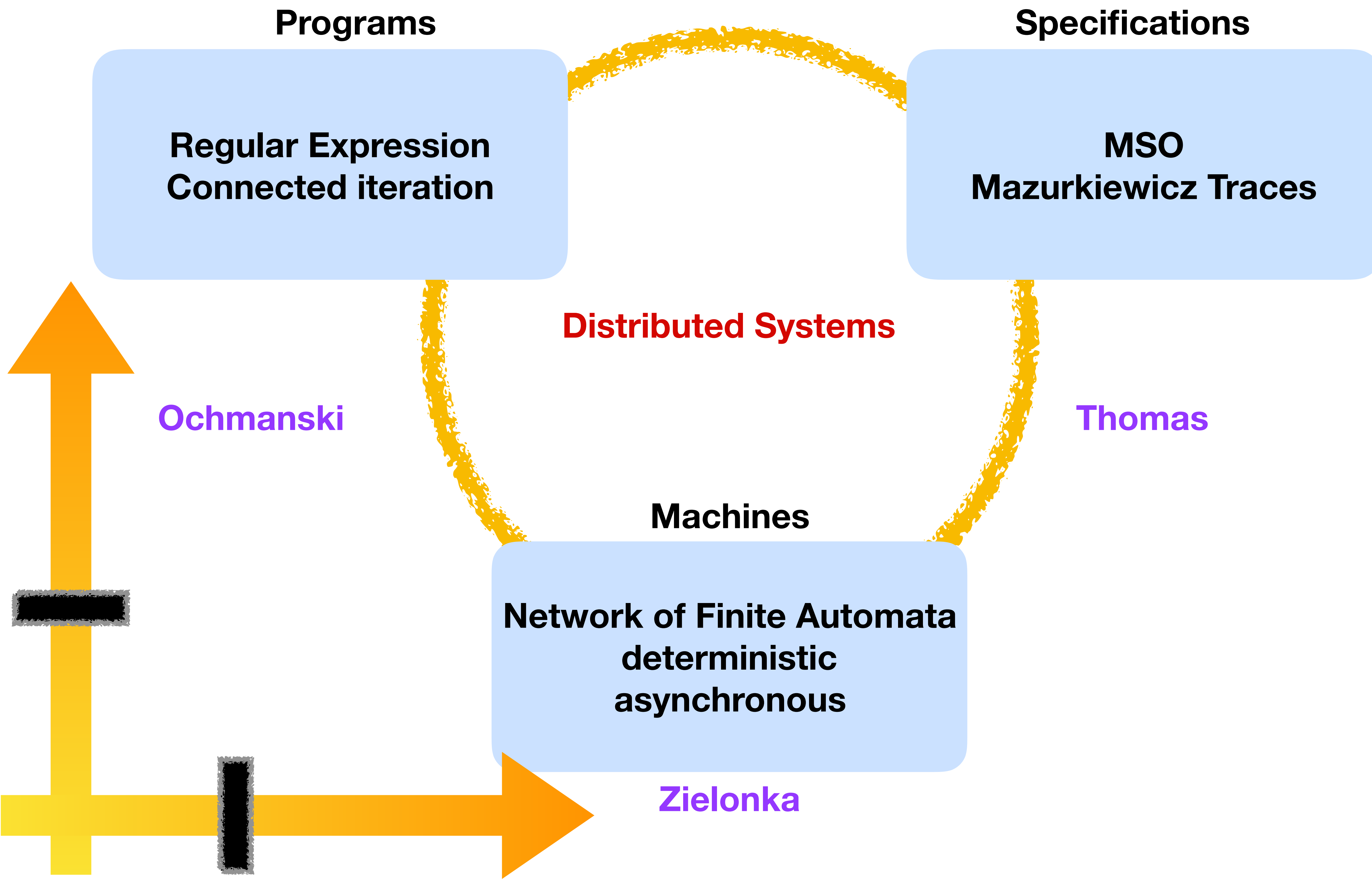

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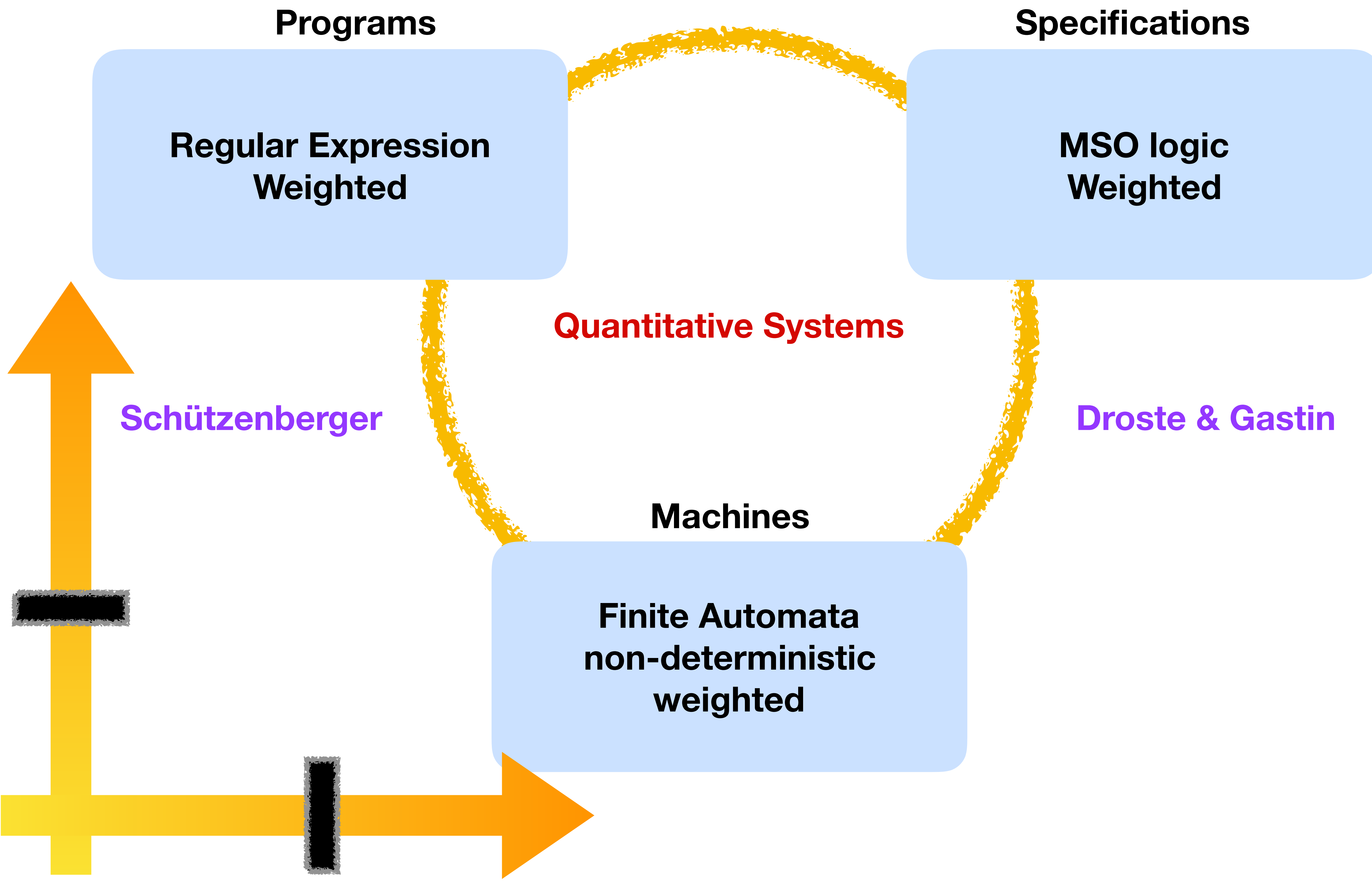

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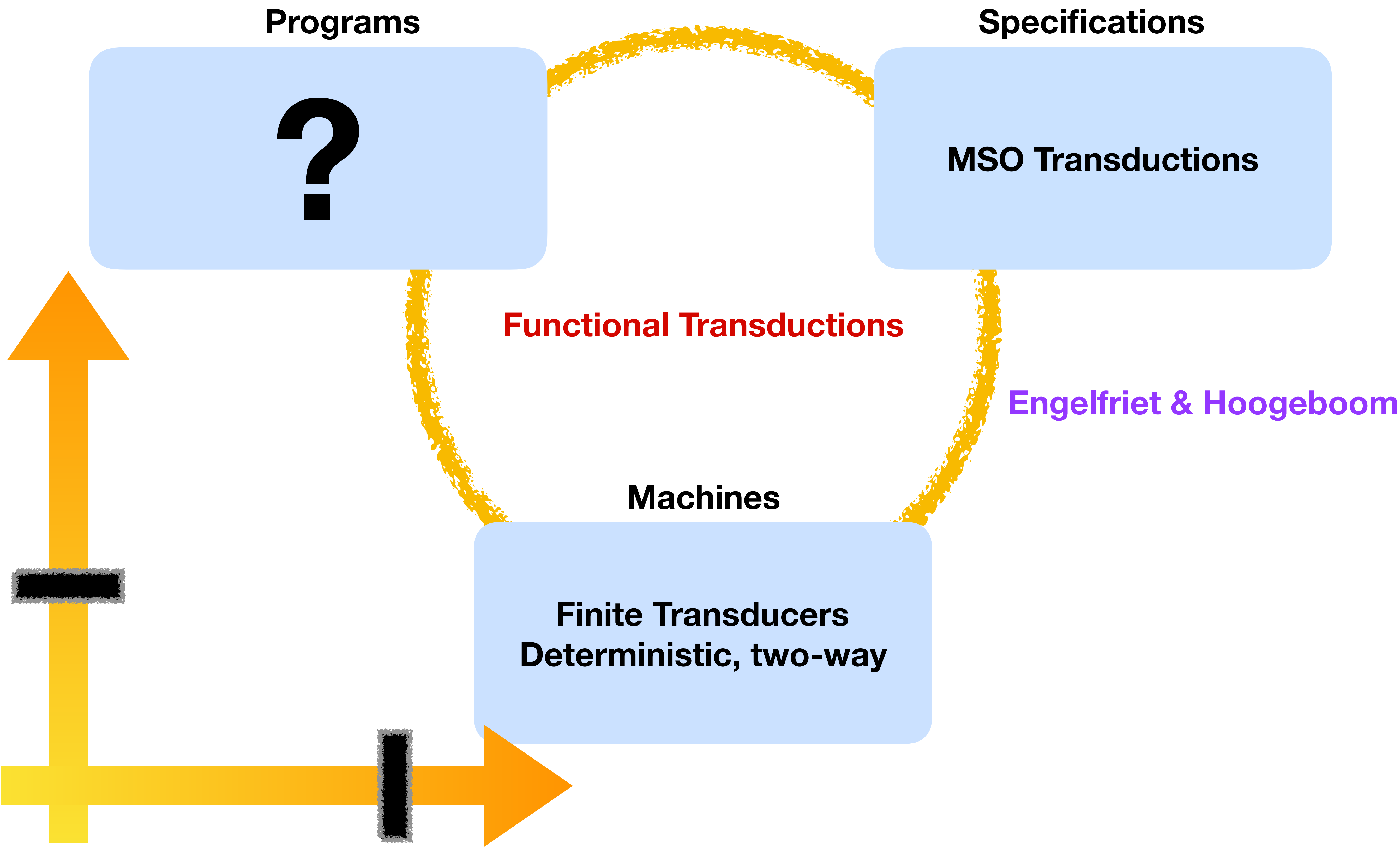

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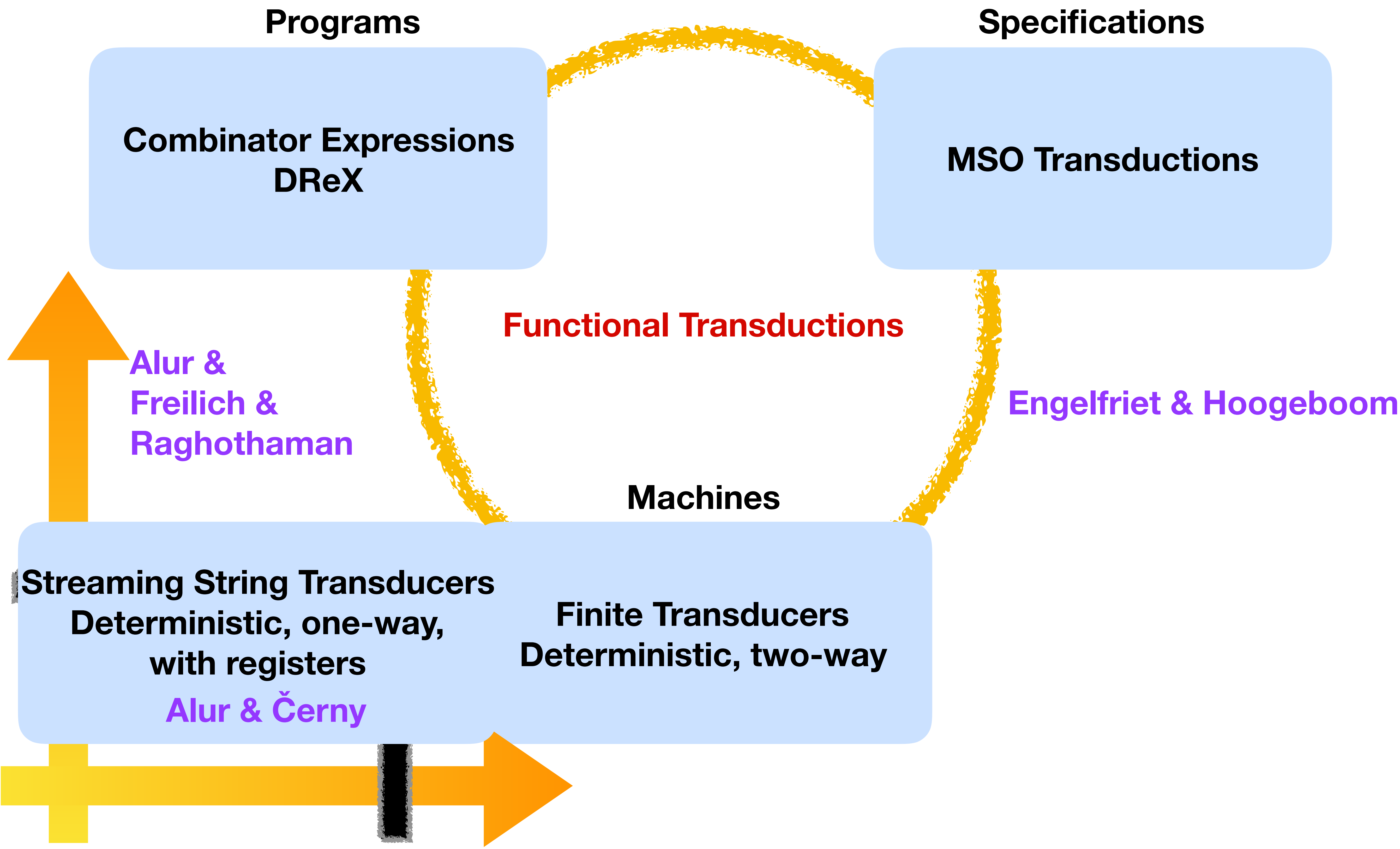

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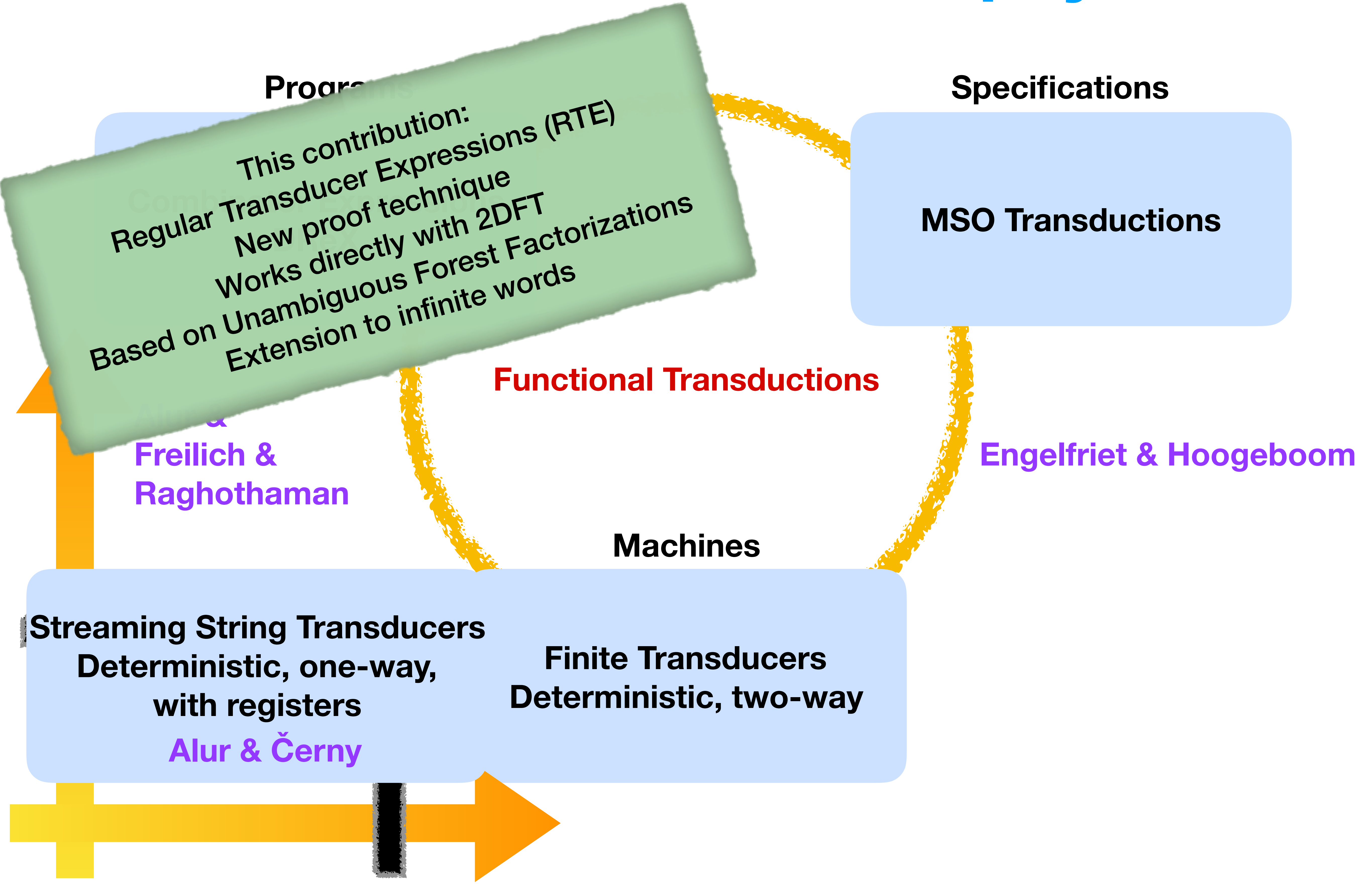


# Summary

- MSO Transductions
- 2-way Deterministic Transducers (2DFT)
- Regular Transducer Expressions (RTE)
- From RTE to 2DFT
- Transition Monoid
- Unambiguous Forest Factorization
- From 2DFT to RTE
- Conclusion

# Functional Transductions

$$f : \Sigma^* \to \Gamma^*$$

Running example: $\Sigma = \{\#, a\}$, $\Gamma = \{b, c\}$ and $\mathsf{dom}(f) = (\#a^+)^+\#$

$$f(\#a^{m_1}\#a^{m_2} \cdots \#a^{m_{k-1}}\#a^{m_k}\#) = c^{m_2}b^{m_1}c^{m_3}b^{m_2} \cdots c^{m_k}b^{m_{k-1}}$$

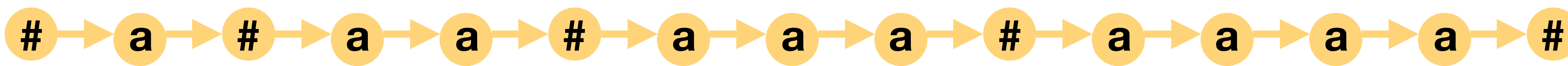
$$f(\#a^5\#a^2\#) = c^2b^5$$

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$

$$f(\#a\#) = \varepsilon$$

# MSO Transductions (Courcelle)

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$



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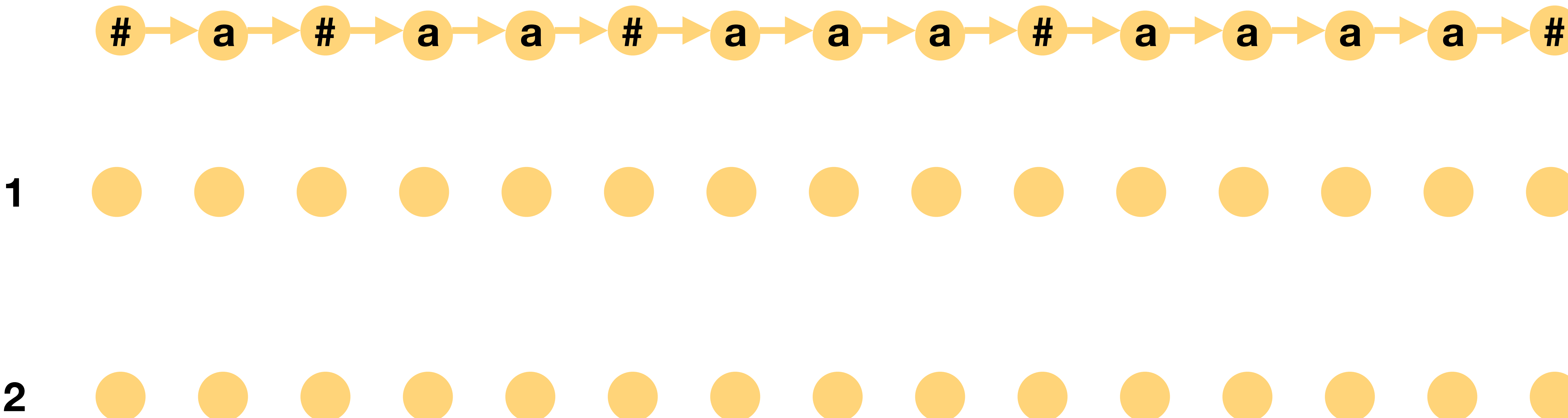

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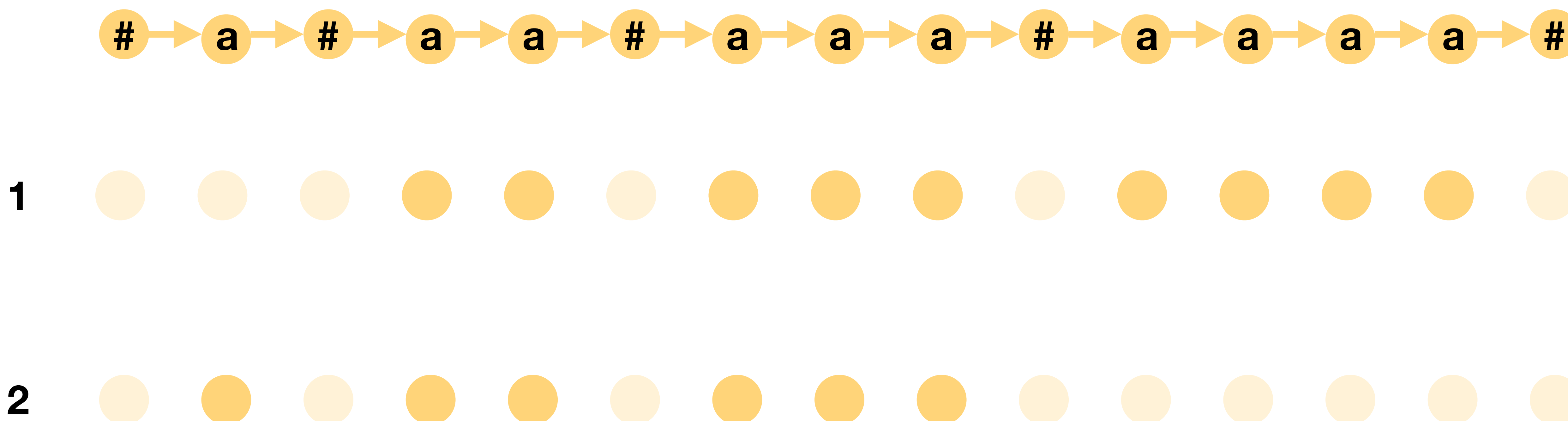


$$\varphi_1(x) = P_a(x) \wedge \exists z_1 < z_2 < x : \; P_\#(z_1) \wedge P_\#(z_2)$$

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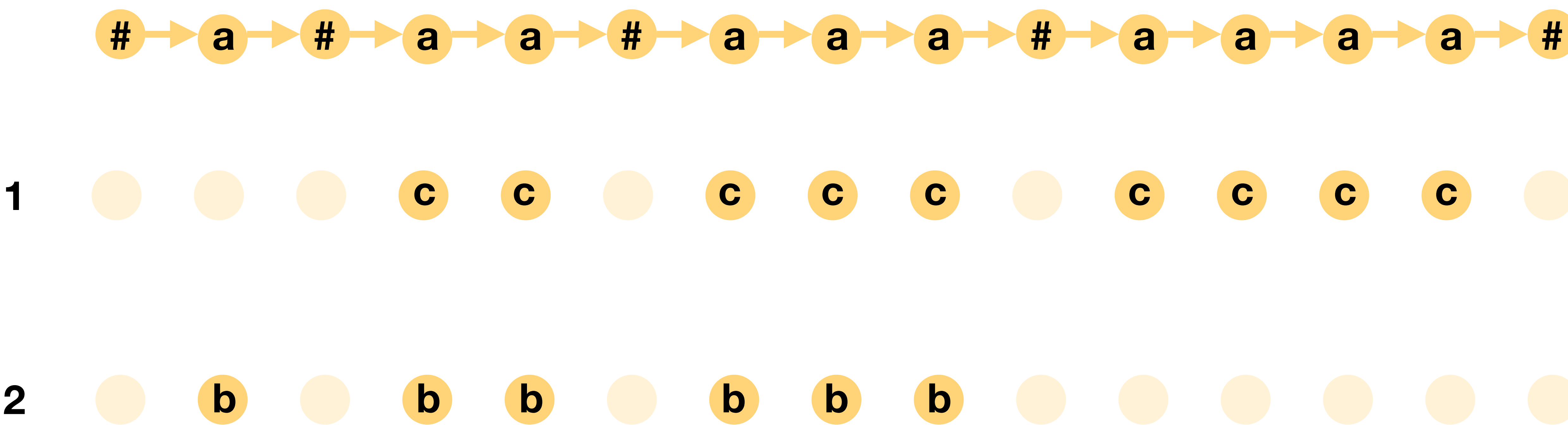


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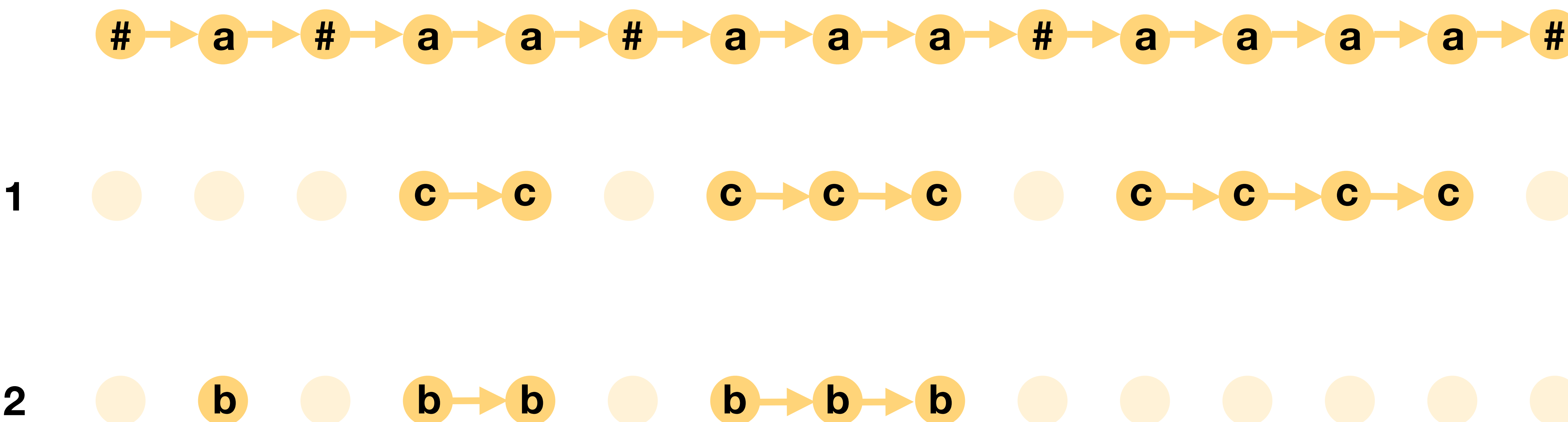


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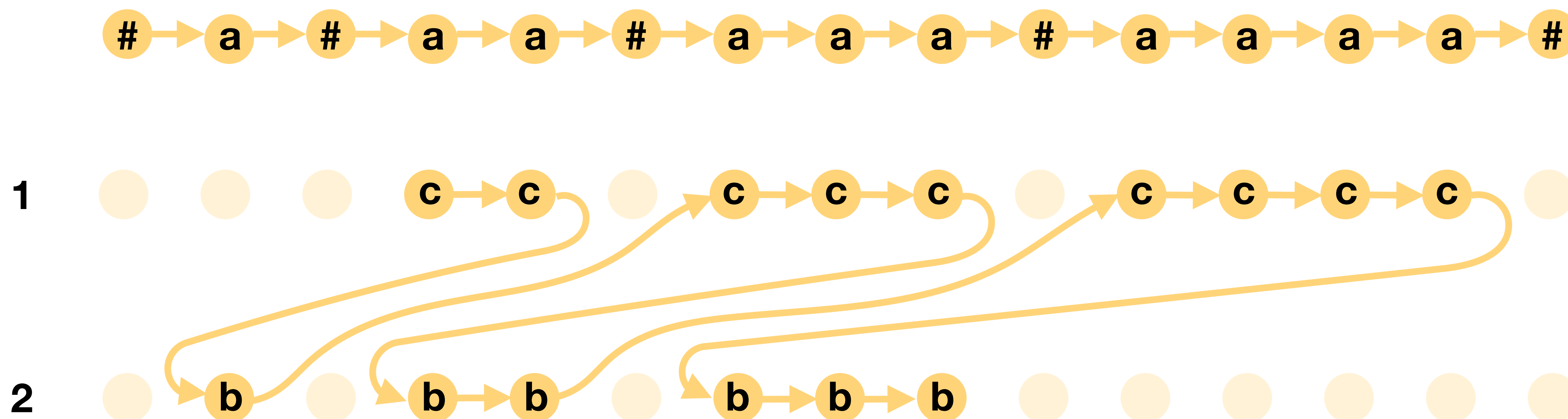


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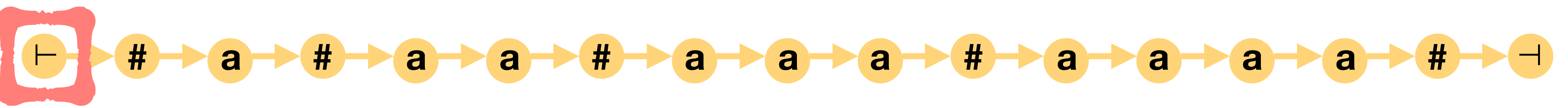
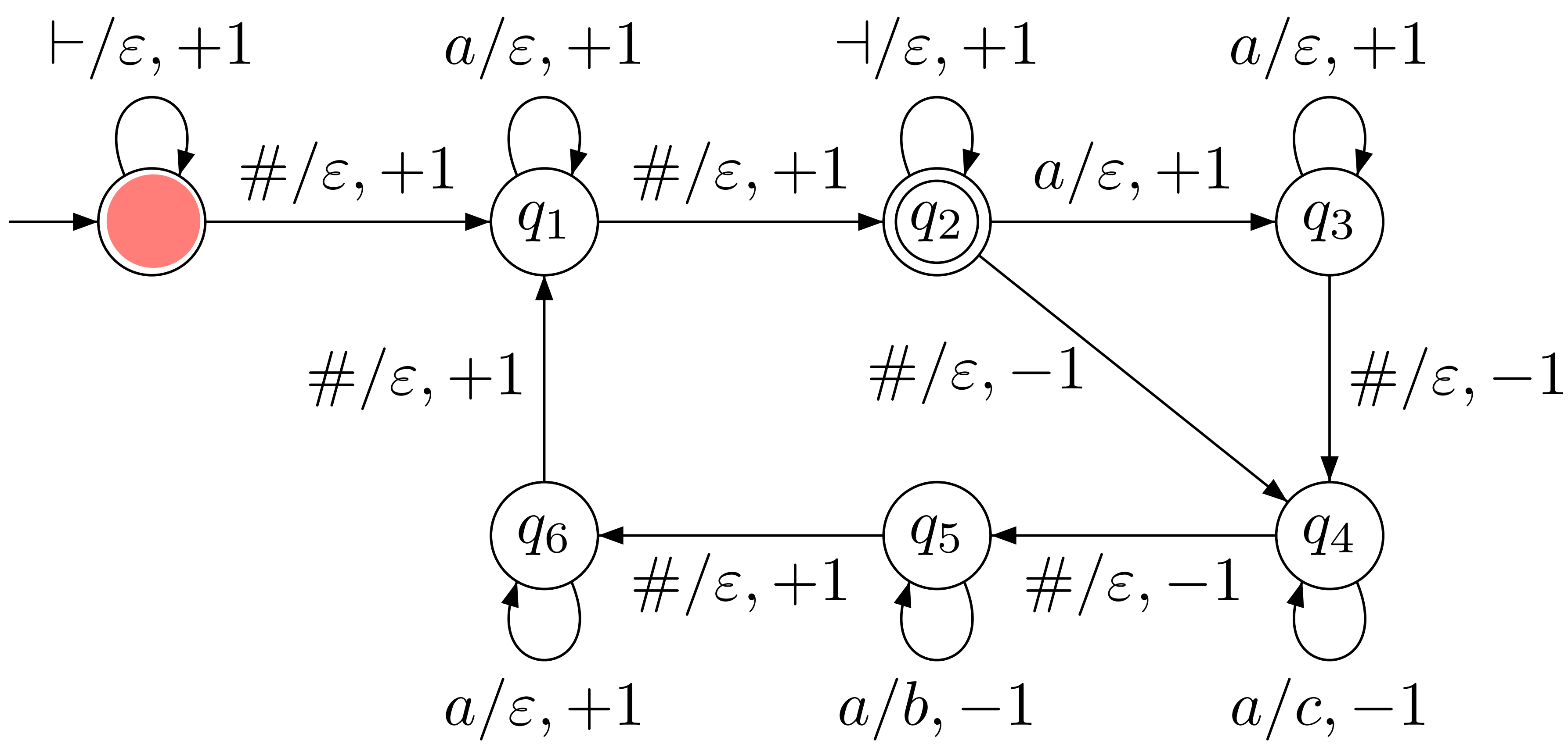
$$\mathsf{succ}_{1,2}(x,y) = \varphi_1(x) \wedge \varphi_2(y) \wedge \exists z_1 < z_2 < z_3, \ P_\#(z_1) \wedge P_\#(z_2) \wedge P_\#(z_3)$$
$$\wedge\, x \lessdot z_3 \wedge z_1 \lessdot y \wedge (\forall z, z_1 < z < z_3 \implies (z = z_2 \vee \neg P_\#(z)))$$

# 2-way Deterministic Transducers

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$



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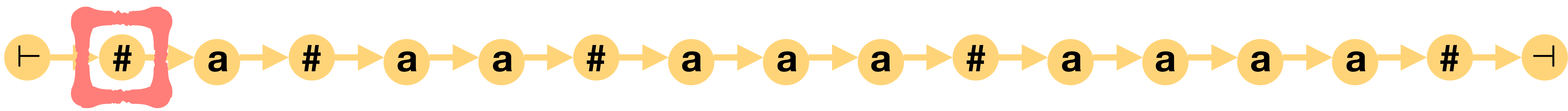


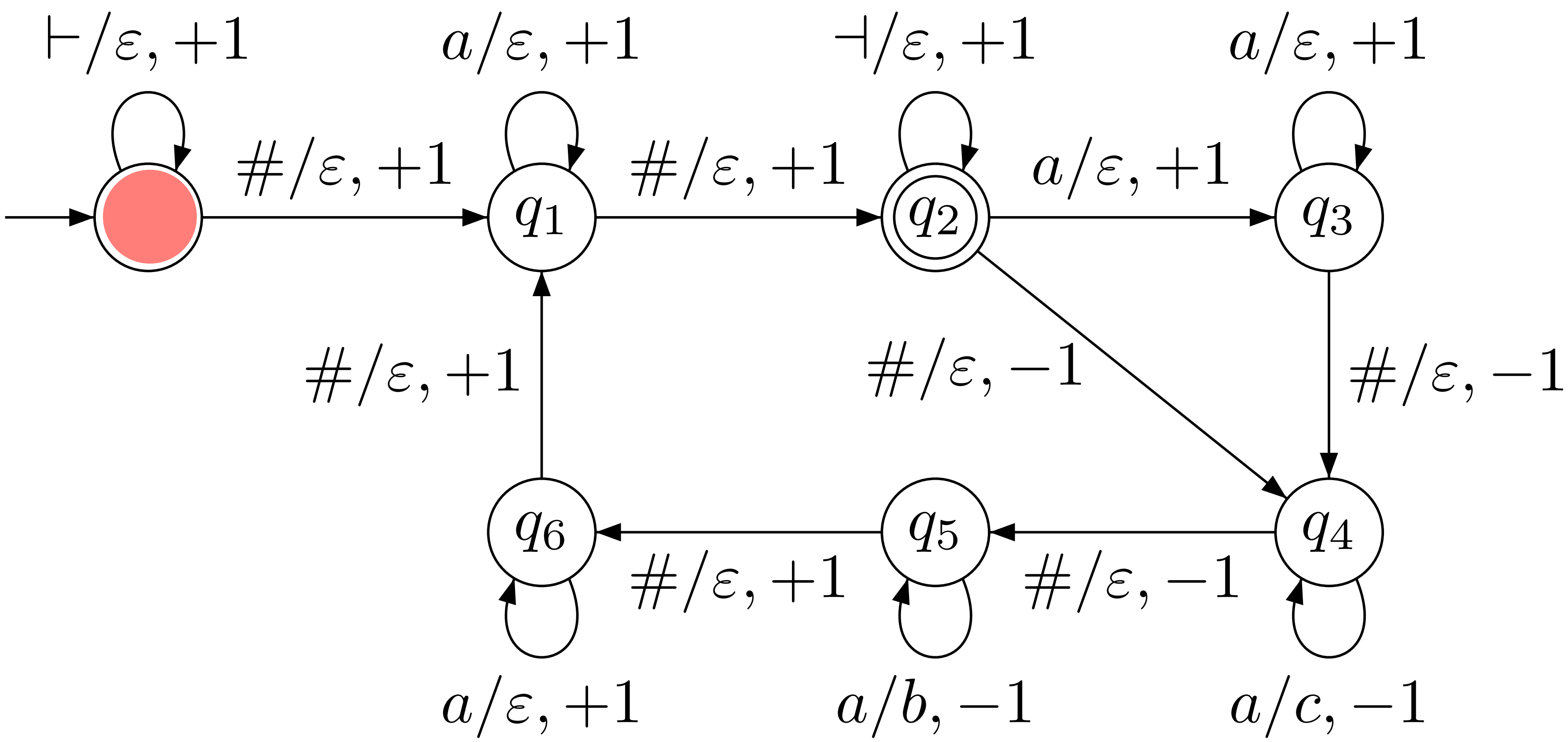

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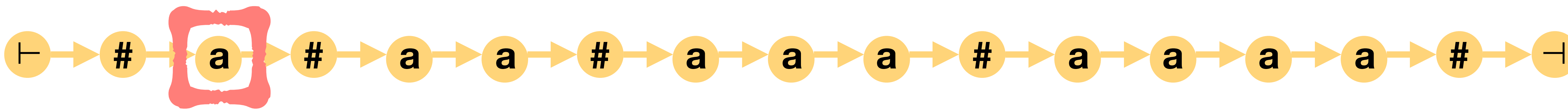


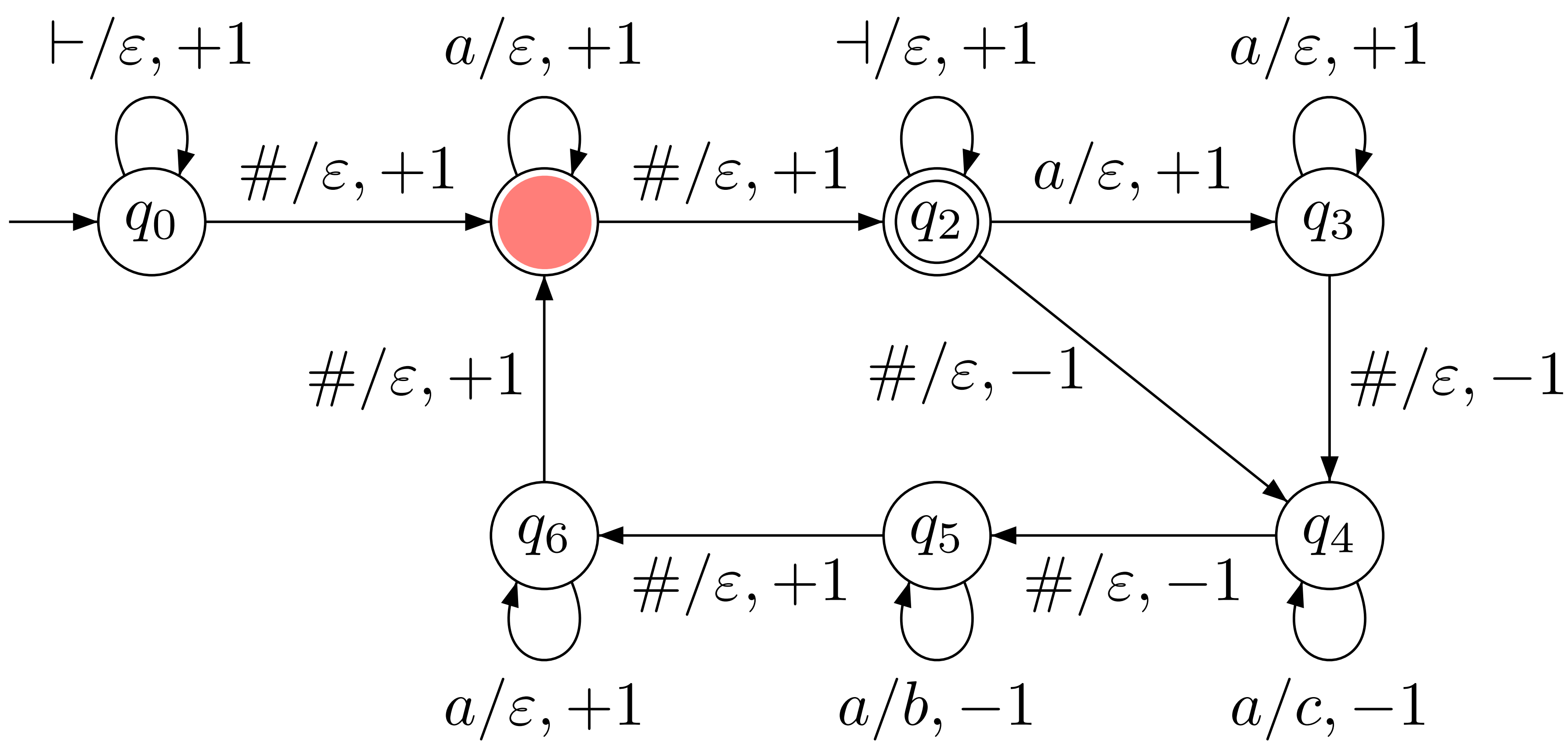

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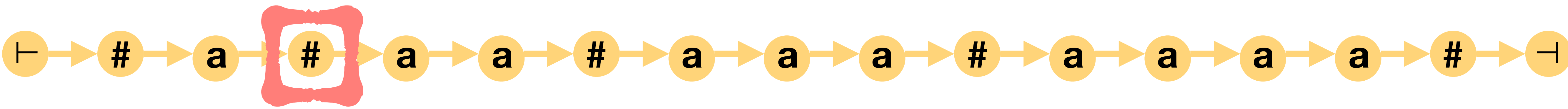


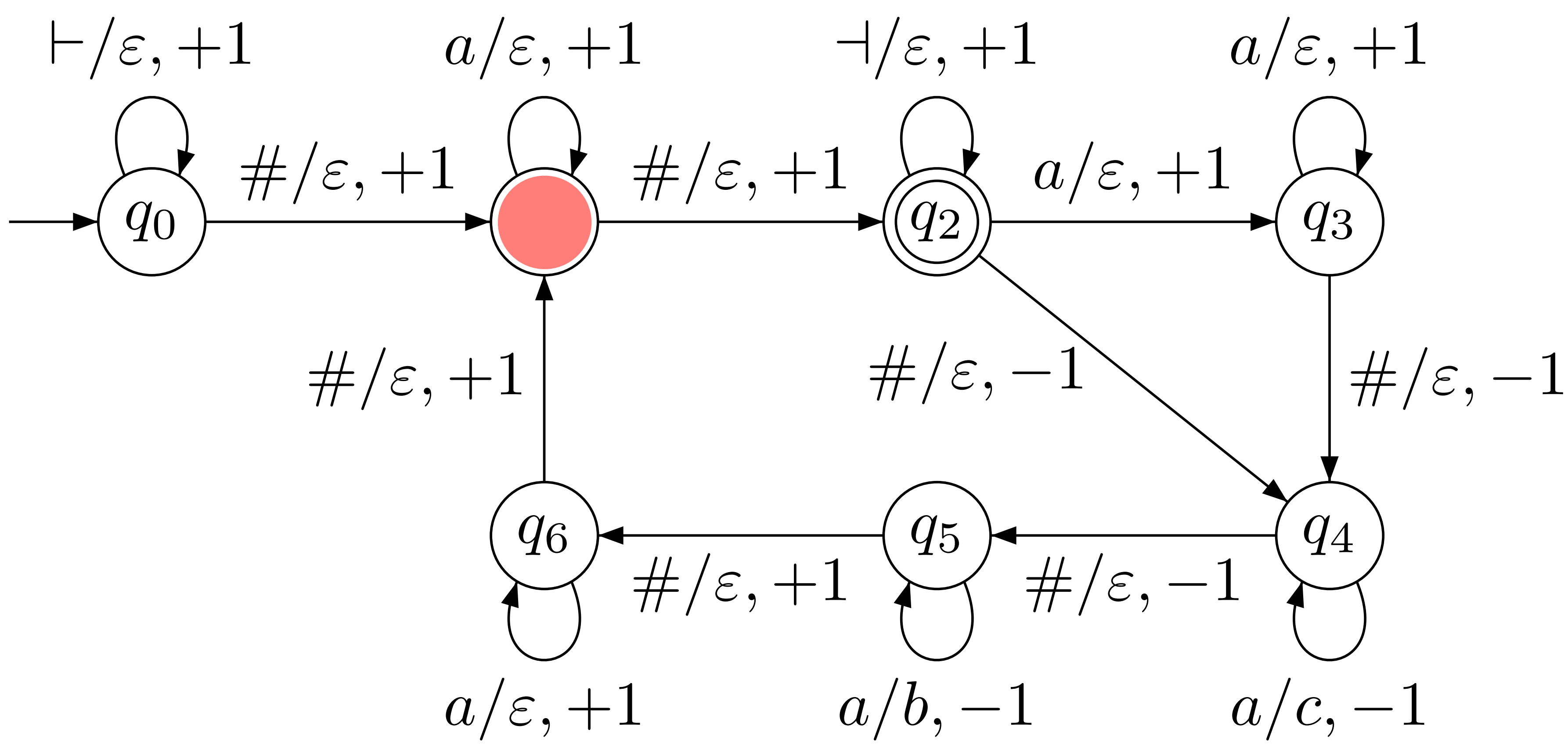

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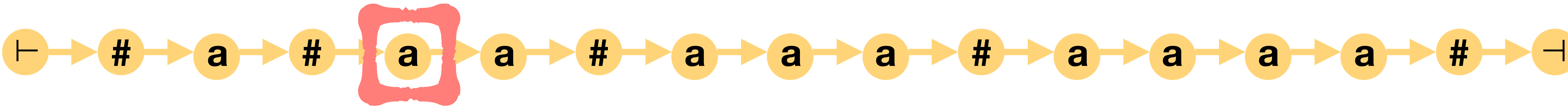


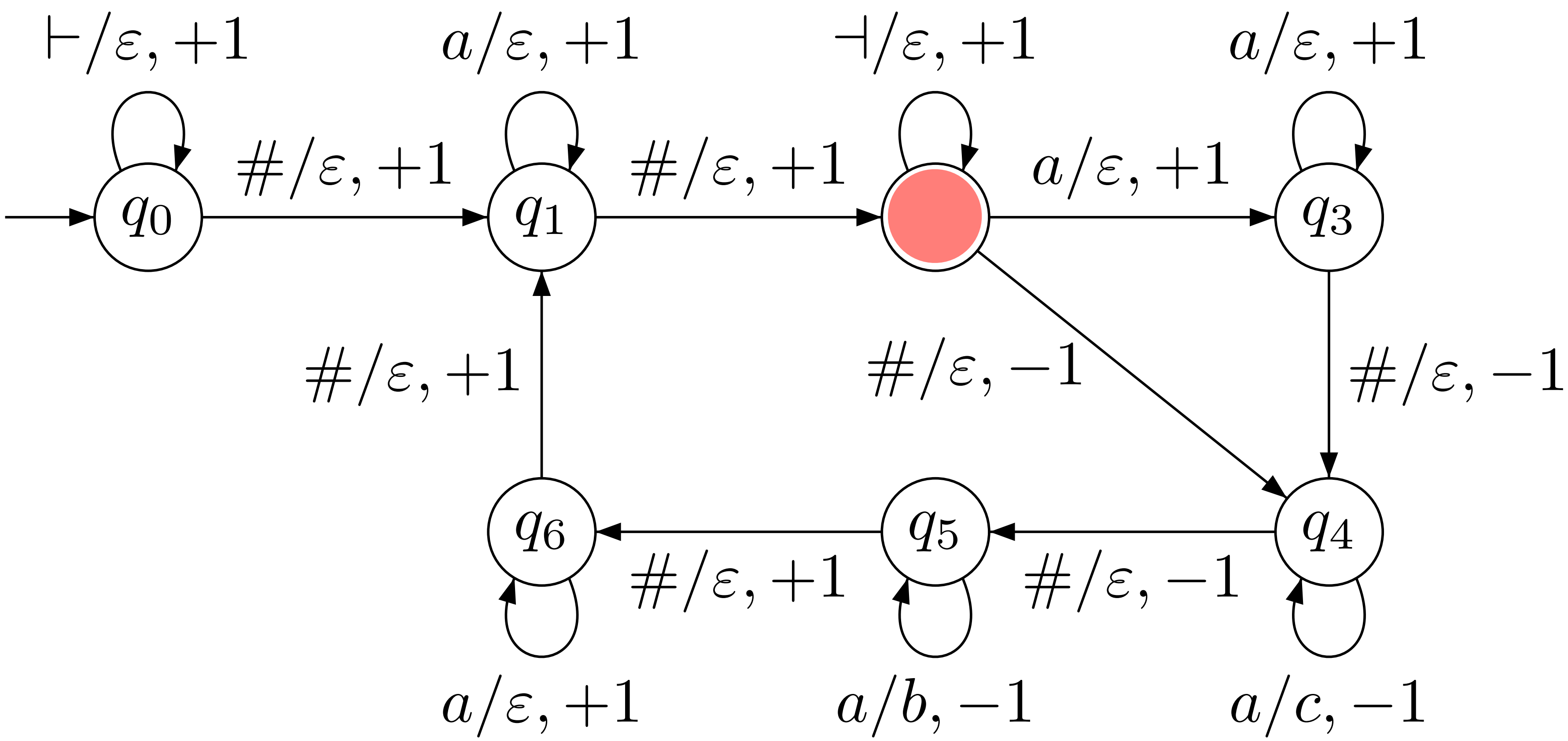


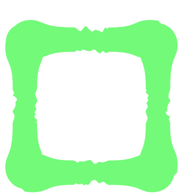

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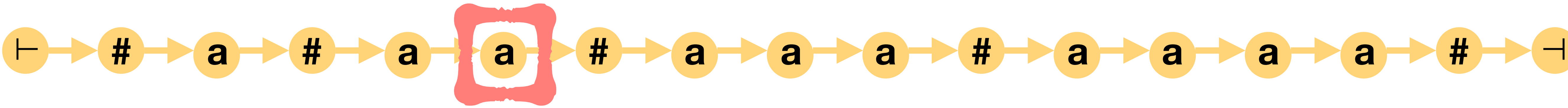


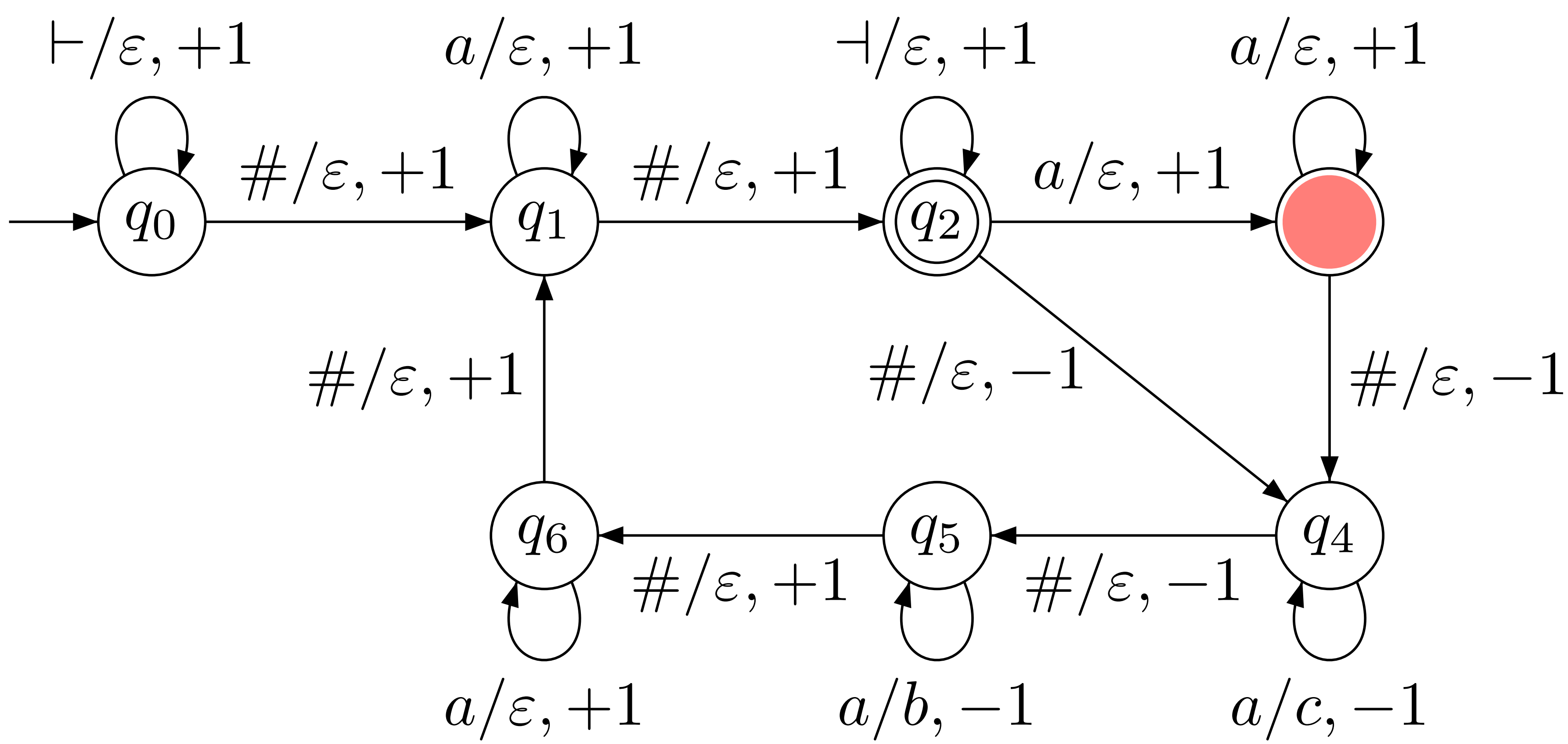

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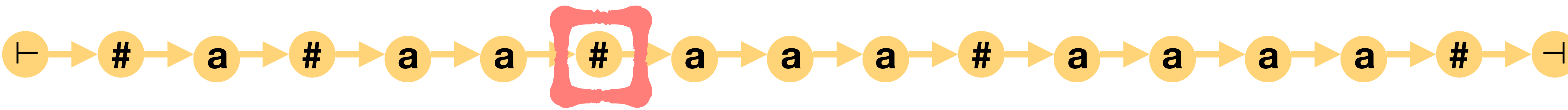


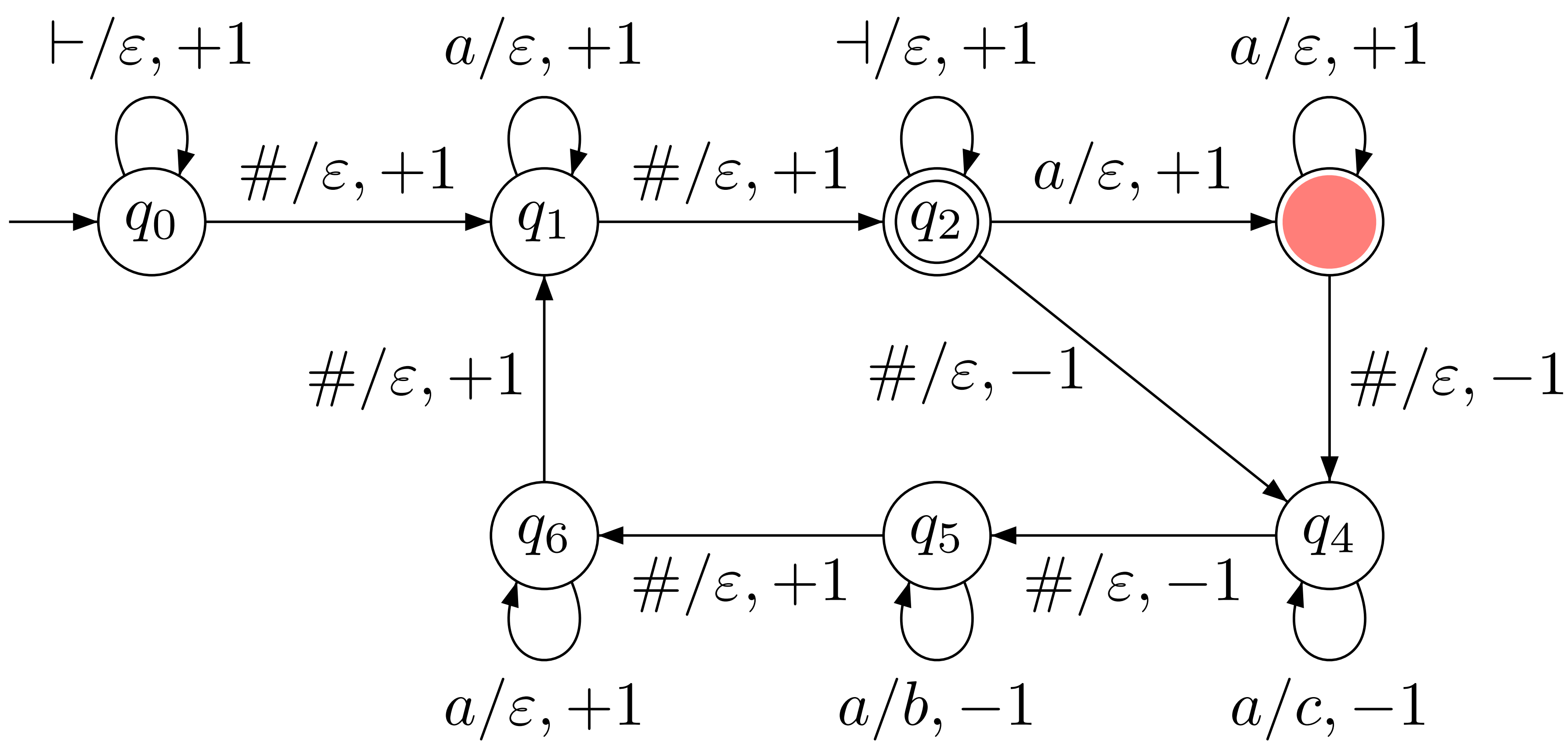

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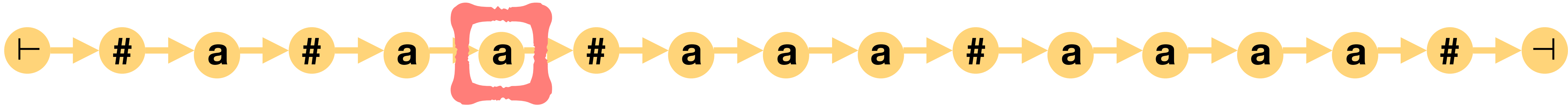


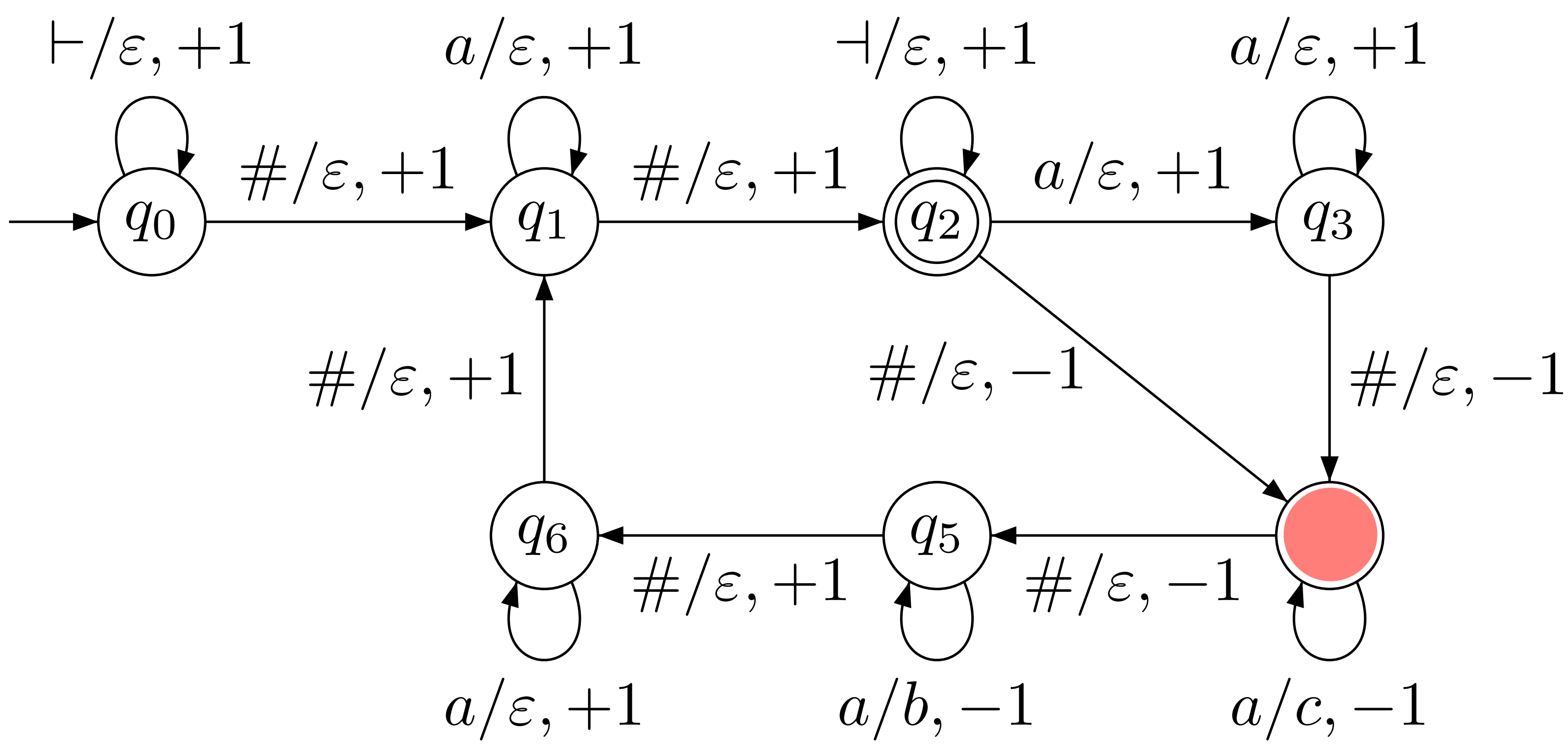

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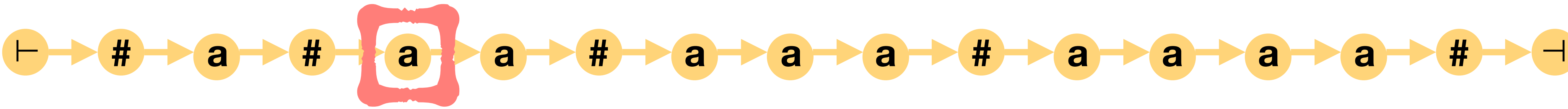


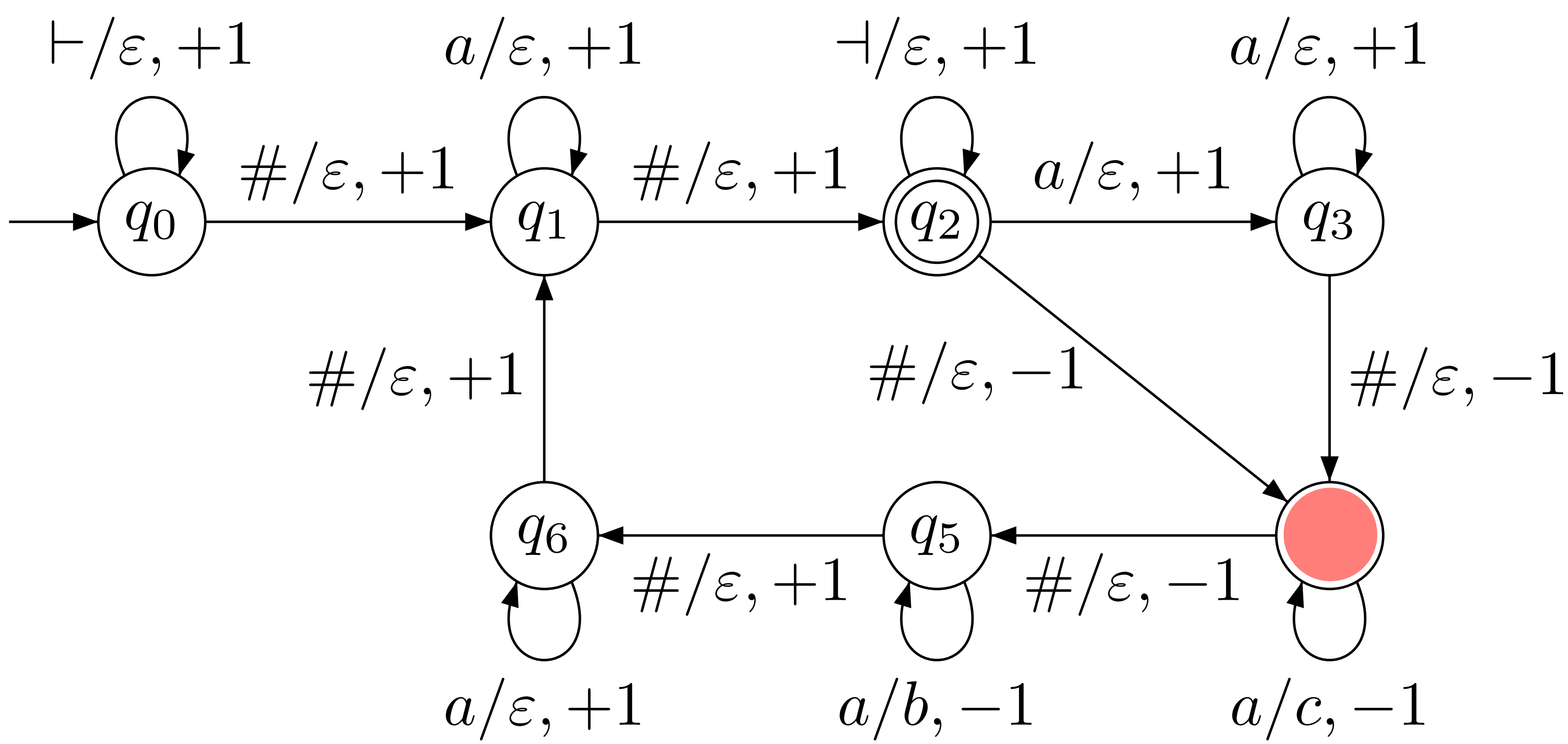


**c**

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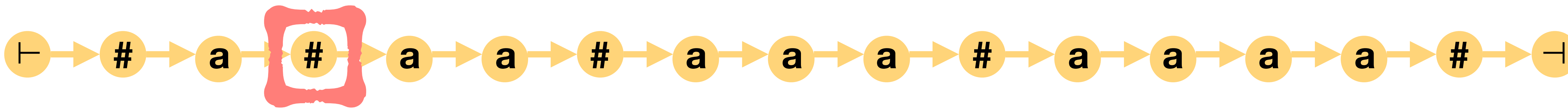


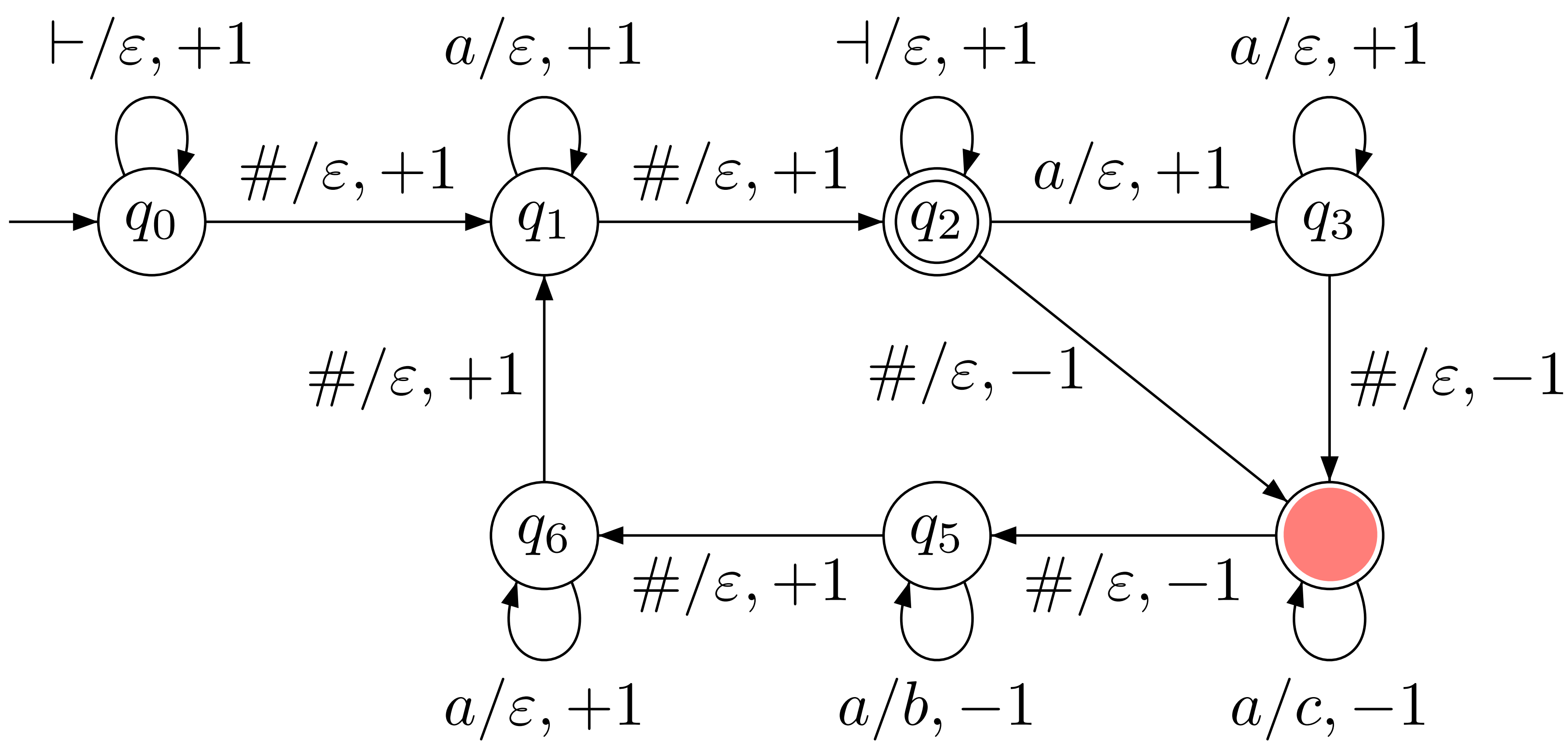


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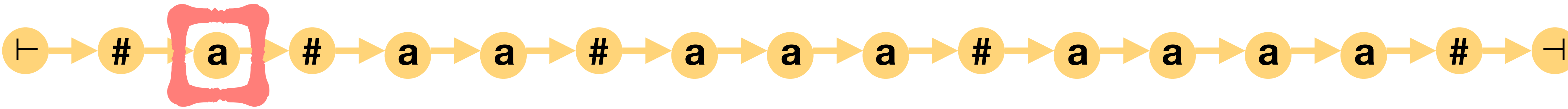


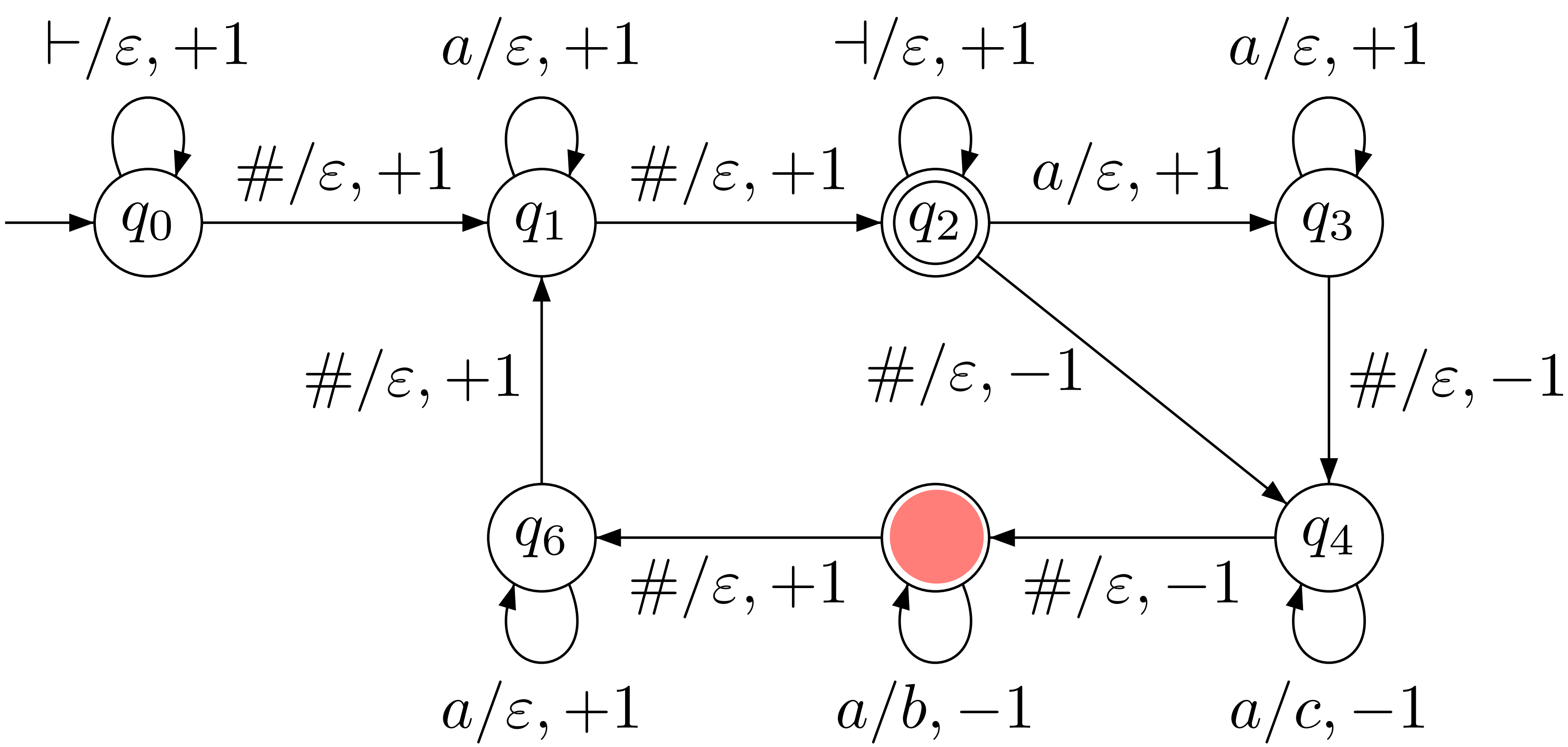


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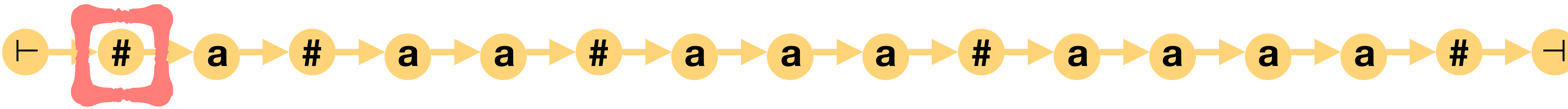


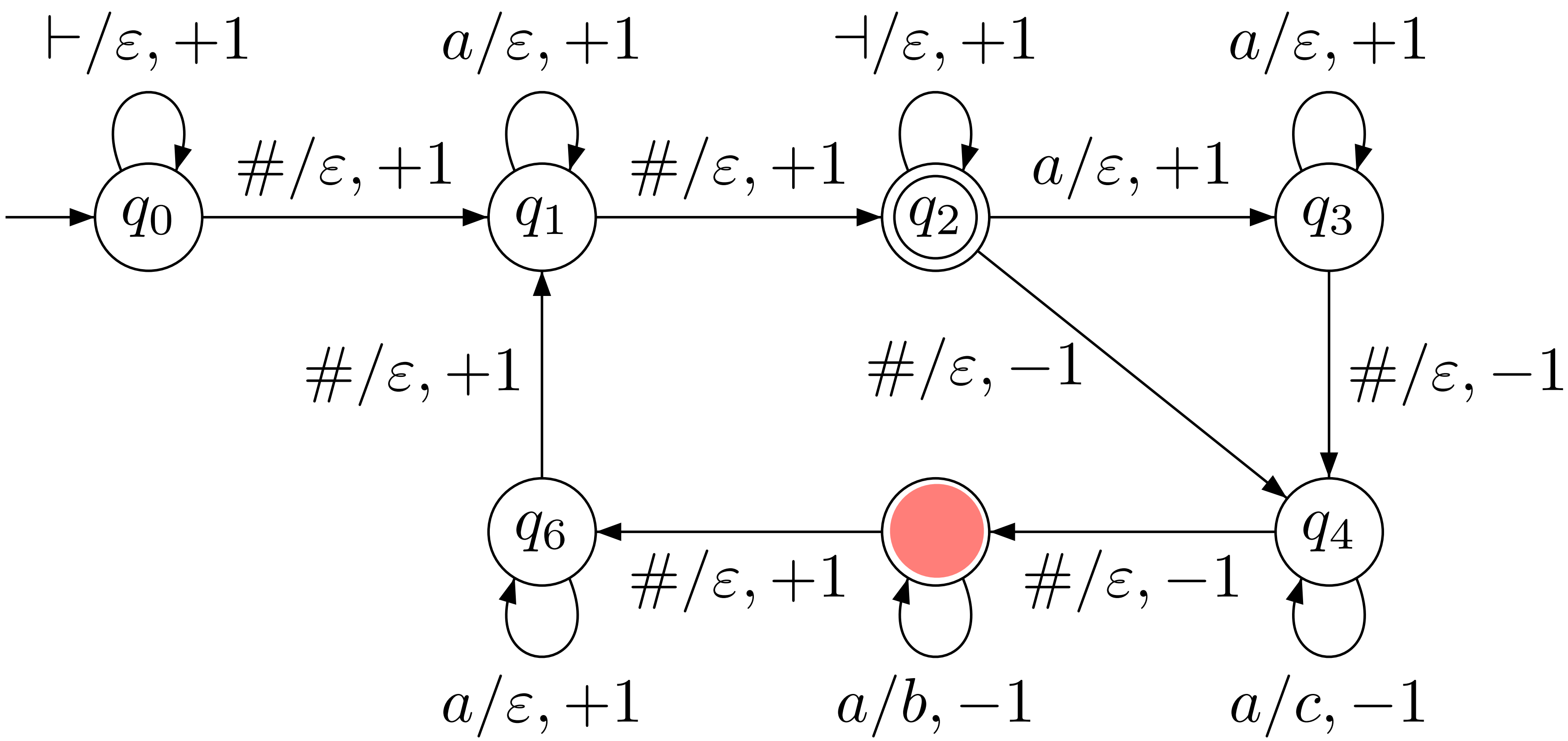


**c c b**

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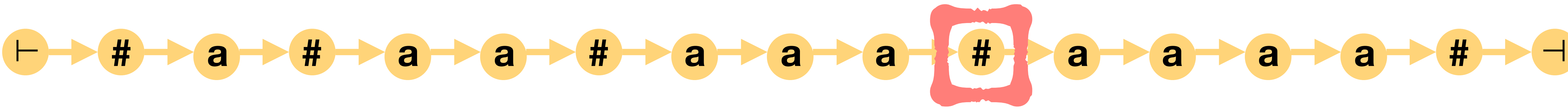


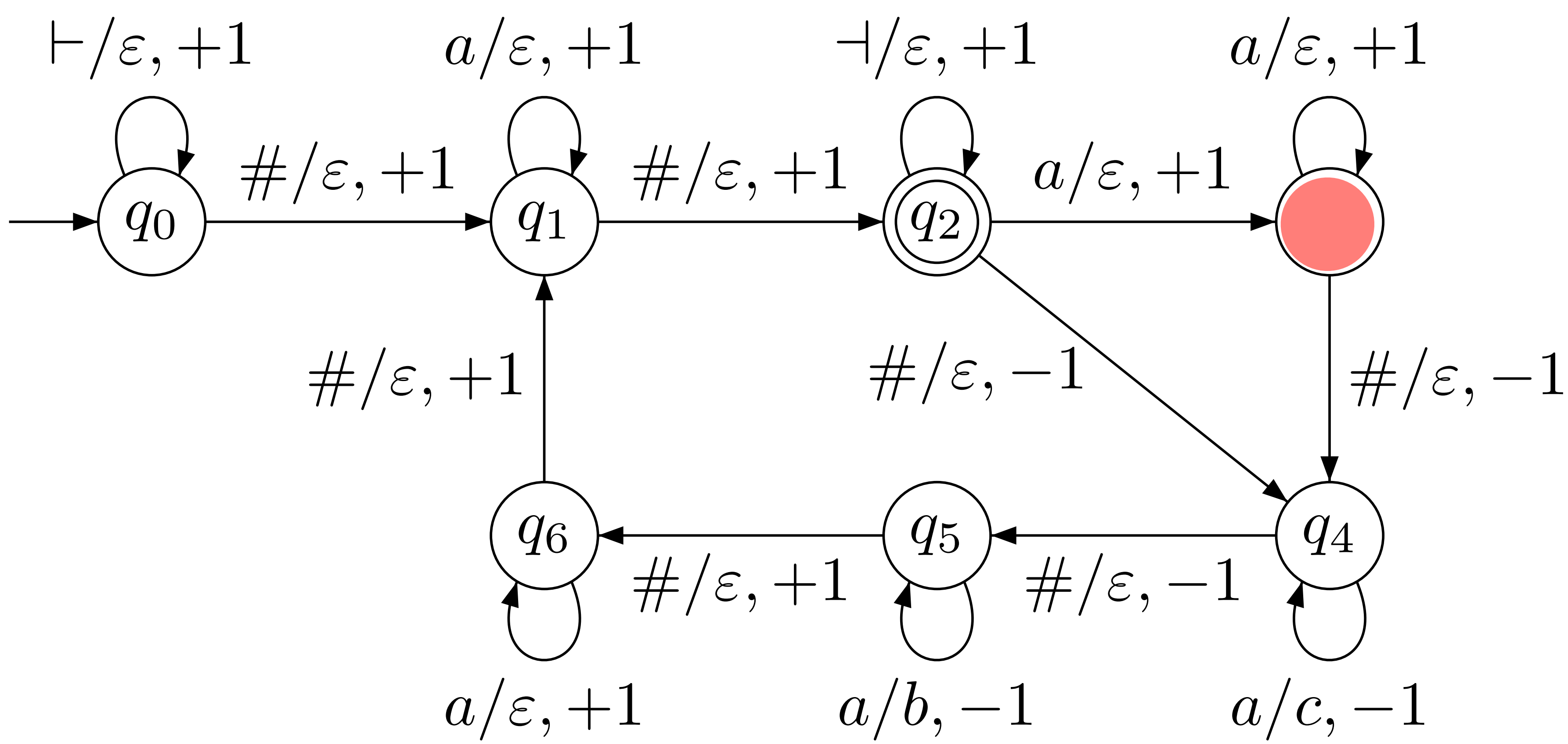


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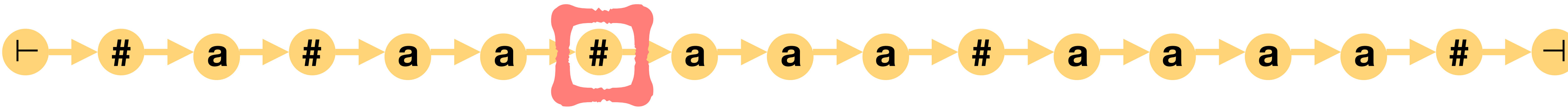


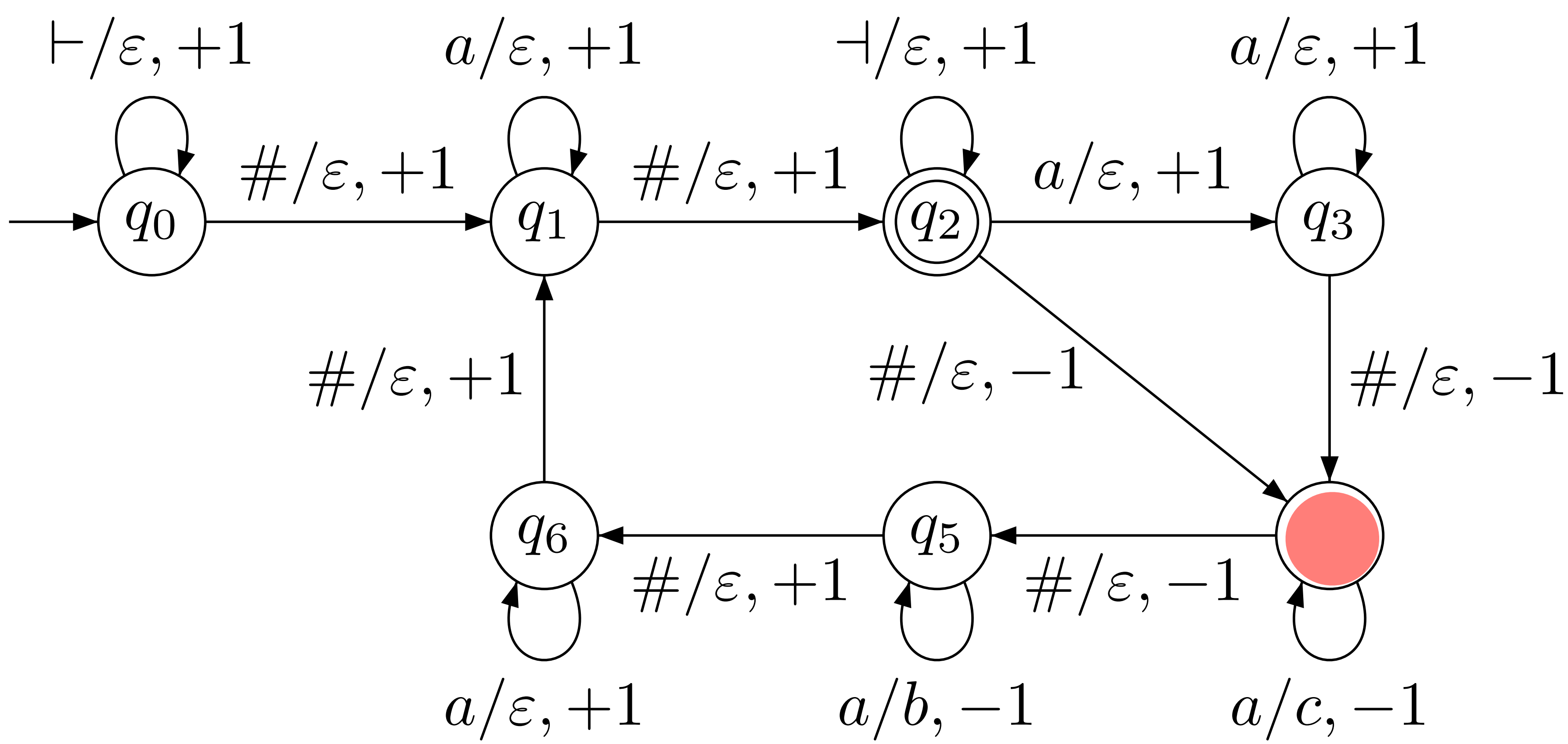


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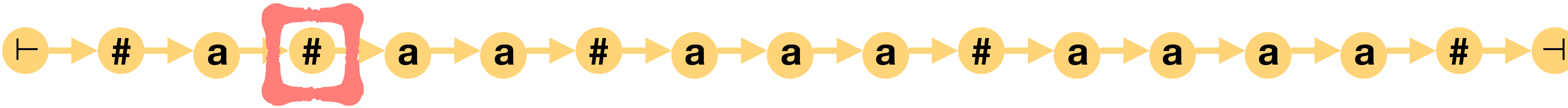


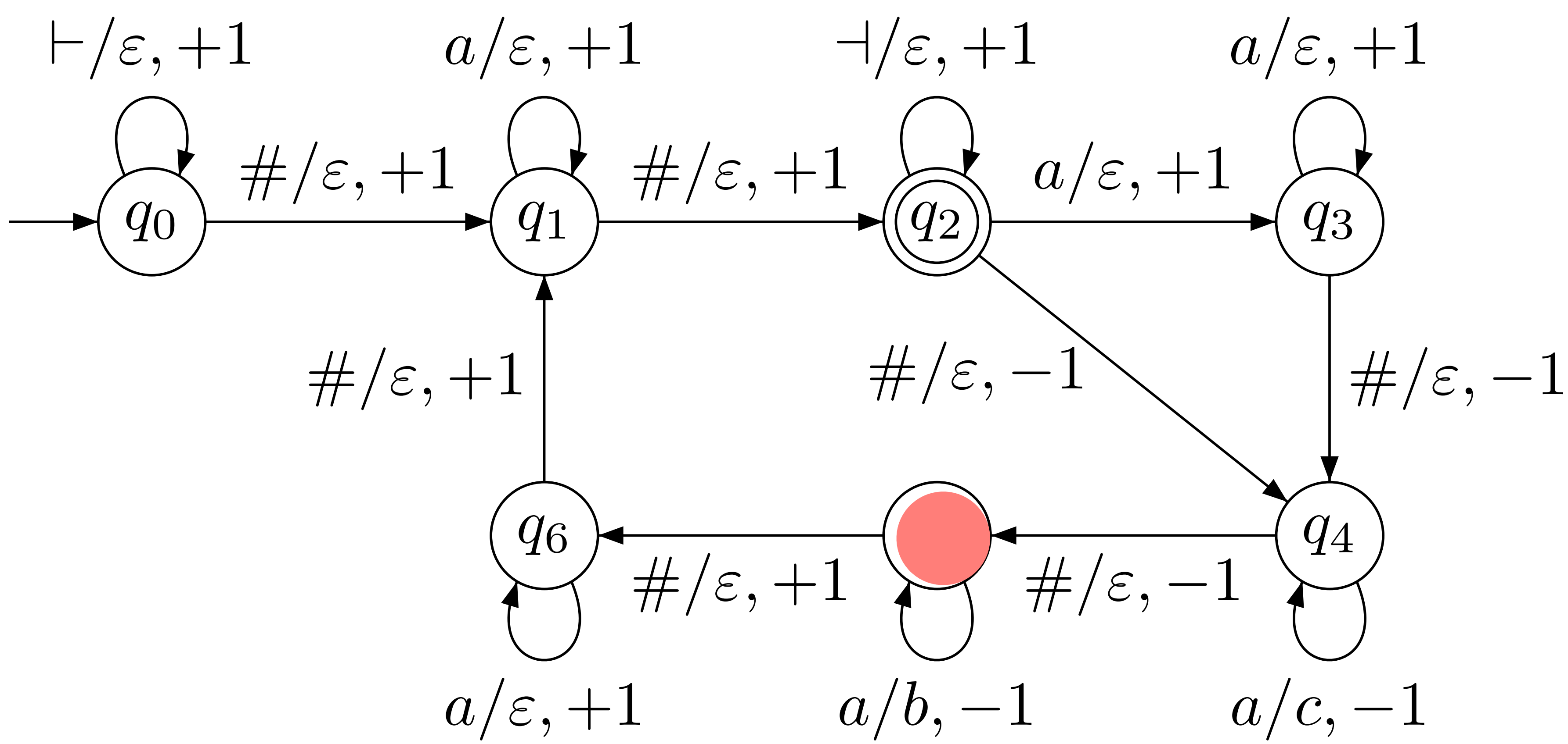


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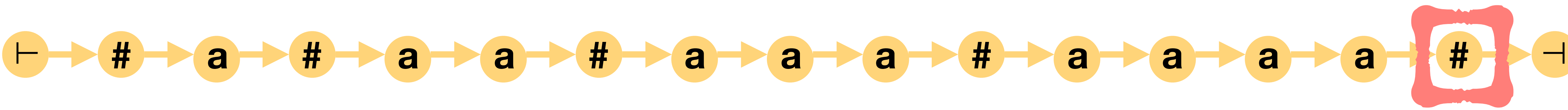


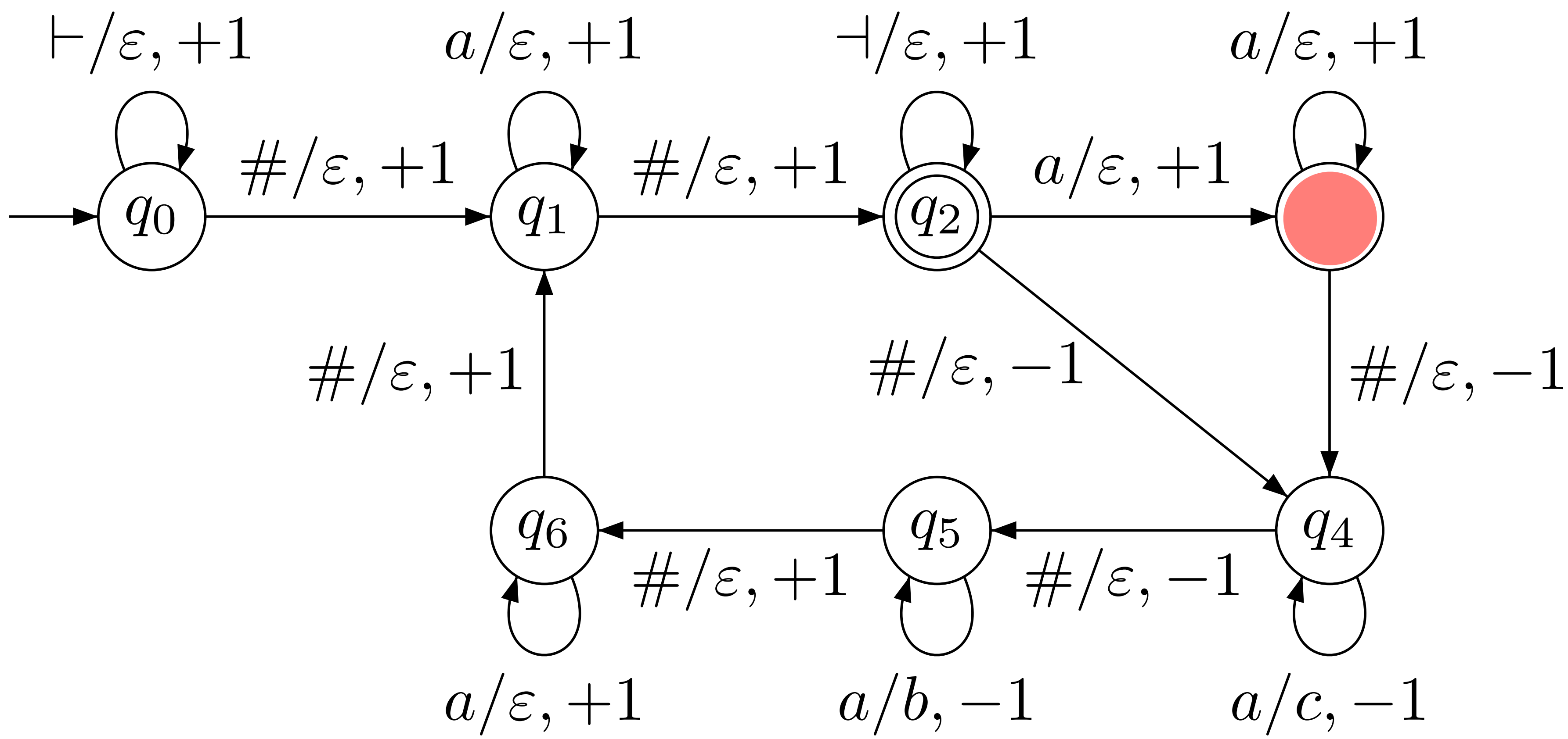


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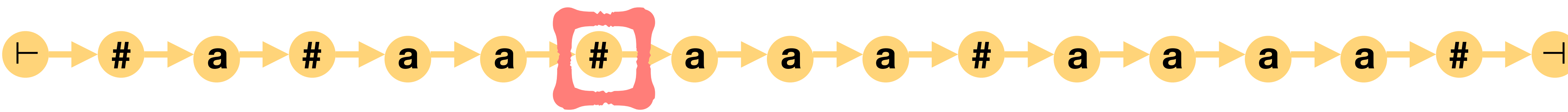


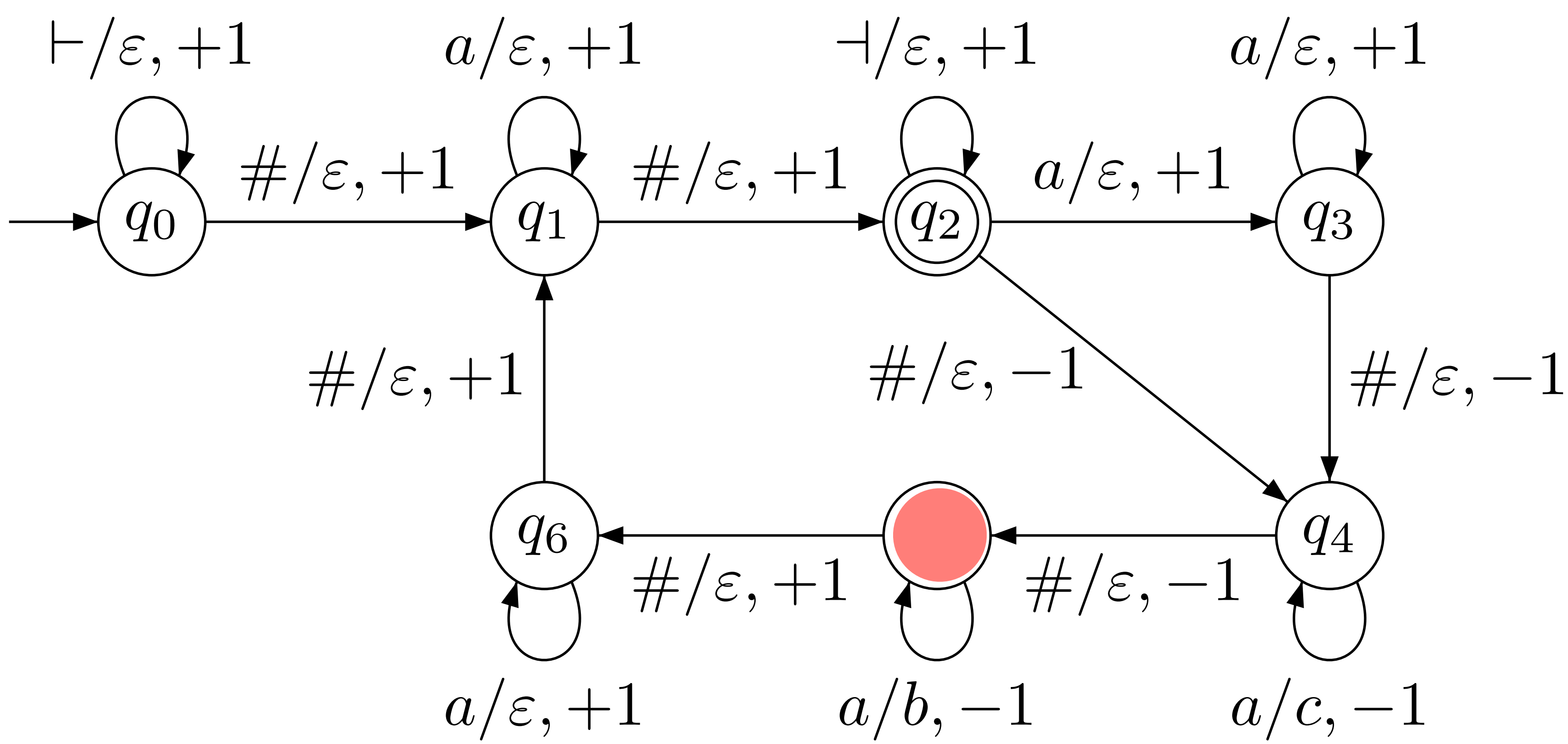


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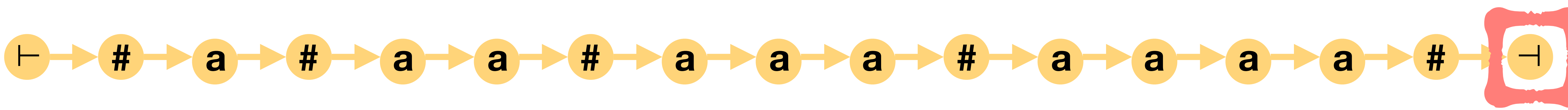


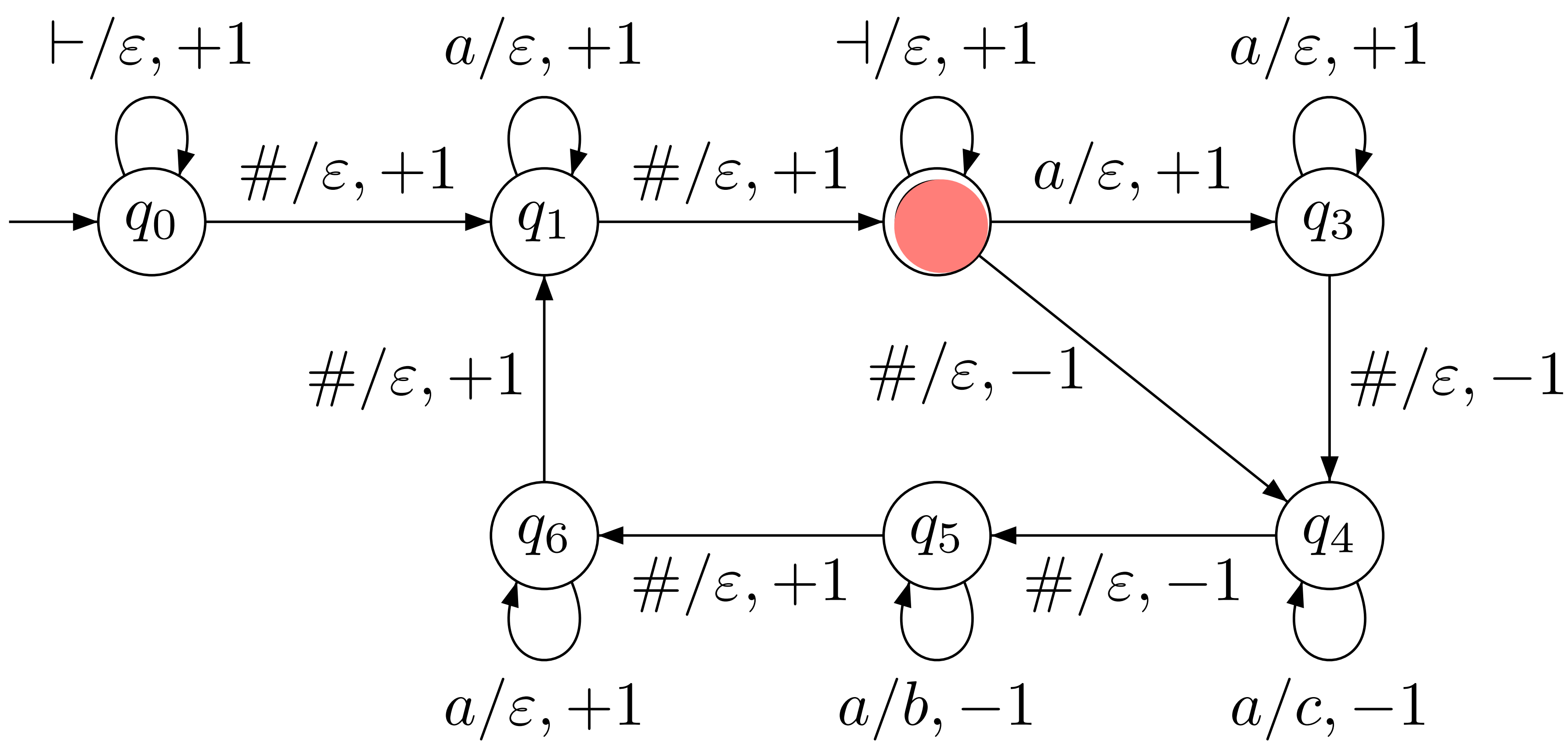


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- Unambiguous Forest Factorization
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# Regular Transducer Expressions

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

$$[\![C]\!] \colon \Sigma^* \to \Gamma^* \text{ partial function}$$

# Regular Transducer Expressions

$d \in \Gamma^* \uplus \{\perp\}$

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# Regular Transducer Expressions

$d \in \Gamma^* \uplus \{\bot\}$

$K \subseteq \Sigma^*$ regular

$$C ::= d \mid K \,?\, C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

$[\![C]\!] \colon \Sigma^* \to \Gamma^*$ partial function

$C_a = a \,?\, a : (b \,?\, \varepsilon : \bot)$

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$\mathsf{dom}(C_a) = \mathsf{dom}(C_b) = \{a, b\}$

**If then else**

$$(K \,?\, f : g)(u) = \begin{cases} f(u) & \text{if } u \in K \\ g(u) & \text{otherwise} \end{cases}$$

# Regular Transducer Expressions

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$\mathsf{dom}([\![{C_a}^{\boxplus}]\!]) = \{a, b\}^+$

$[\![{C_a}^{\boxplus}]\!](baabbbab) = a^3$

**Unambiguous Kleene-plus**

$$f^{\boxplus}(w) = f(u_1)f(u_2)\cdots f(u_n)$$

If $w = u_1u_2\cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in \mathsf{dom}(f)$

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$[\![{C_a}^{\boxplus} \odot {C_b}^{\boxplus}]\!](baabbbab) = a^3 b^5$

**Hadamard product**

$$f \odot g(u) = f(u) g(u)$$

# Regular Transducer Expressions

$d \in \Gamma^* \uplus \{\bot\}$

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**b a a b b b a b # a b a b**

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b b b b b #

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b a a b b b a b # a b a b

b b b b b # a a

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$\mathsf{copy} = (a\,?\,a : (b\,?\,b : \perp))^{\boxplus}$

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$\mathsf{dom}([\![C]\!]) = (a+b)^+ \# (a+b)^+$

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**b a a b b b a b # a b a b**

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b a a b b b a b # a b a b

a b a b # b a a b b b a b

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$$f(\#a^{m_1}\#a^{m_2}\cdots\#a^{m_{k-1}}\#a^{m_k}) = c^{m_2}b^{m_1}c^{m_3}b^{m_2}\cdots c^{m_k}b^{m_{k-1}}$$

$$C_b = (a\,?\,b : \perp)^{\boxplus} \qquad C_c = (a\,?\,c : \perp)^{\boxplus}$$

$$C = [\#a^+, ((\#\,?\,\varepsilon : \perp) \boxdot C_b) \overleftarrow{\boxdot} ((\#\,?\,\varepsilon : \perp) \boxdot C_c)]^{2\boxplus}$$

Unambiguous 2-chained Kleene-plus

$$[K, f]^{2\boxplus}(w) = f(u_1u_2)f(u_2u_3)\cdots f(u_{n-1}u_n)$$

If $w = u_1u_2\cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in K$

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**# a** **# a a** **# a a a** **# a a a a**

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**# a # a a # a a a # a a a a**

**c c b**

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$d \in \Gamma^* \uplus \{\perp\}$

$K \subseteq \Sigma^*$ regular

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$$

$$[\![C]\!](\#a\#a^2\#a^3\#a^4) = c^2b^1c^3b^2c^4b^3$$

\# a \# a a \# a a a \# a a a a

c c b c c c b b

$C_b = (a\,?\,b : \perp)^{\boxplus} \qquad C_c = (a\,?\,c : \perp)^{\boxplus}$

$$C = [\#a^+, ((\#\,?\,\varepsilon : \perp) \boxdot C_b) \overleftarrow{\boxdot} ((\#\,?\,\varepsilon : \perp) \boxdot C_c)]^{2\boxplus}$$

Unambiguous 2-chained Kleene-plus

$$[K, f]^{2\boxplus}(w) = f(u_1u_2)f(u_2u_3)\cdots f(u_{n-1}u_n)$$

If $w = u_1u_2\cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in K$

# Regular Transducer Expressions

$d \in \Gamma^* \uplus \{\perp\}$

$K \subseteq \Sigma^*$ regular

$$C ::= d \mid K \,?\, C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

$$[\![C]\!](\#a\#a^2\#a^3\#a^4) = c^2 b^1 c^3 b^2 c^4 b^3$$

# a # a a # a a a # a a a a

c c b c c c b b c c c c b b b

$C_b = (a \,?\, b : \perp)^{\boxplus}$ $\quad$ $C_c = (a \,?\, c : \perp)^{\boxplus}$

$$C = [\#a^+, ((\# \,?\, \varepsilon : \perp) \boxdot C_b) \overleftarrow{\boxdot} ((\# \,?\, \varepsilon : \perp) \boxdot C_c)]^{2\boxplus}$$

Unambiguous 2-chained **Kleene-plus**

$$[K, f]^{2\boxplus}(w) = f(u_1 u_2) f(u_2 u_3) \cdots f(u_{n-1} u_n)$$

If $w = u_1 u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in K$

# RTE and 2DFT

**Main Theorem:**

2DFTs and RTEs define the same class of functions. More precisely,

1. given an RTE $C$, we can construct a 2DFT $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = [\![C]\!]$,
2. given a 2DFT $\mathcal{A}$, we can construct an RTE C such that $[\![\mathcal{A}]\!] = [\![C]\!]$.

# Summary

- MSO Transductions
- 2-way Deterministic Transducers (2DFT)
- Regular Transducer Expressions (RTE)
- From RTE to 2DFT
- Transition Monoid
- Unambiguous Forest Factorization
- From 2DFT to RTE
- Conclusion

# RTE to 2DFT

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

**Main Lemma:**

Let $K \subseteq \Sigma^*$ be regular, and let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$ respectively. Then, one can construct

1. a 2DFT $\mathcal{A}$ such that $[\![K\,?\,f : g]\!] = [\![\mathcal{A}]\!]$.
2. a 2DFT $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = [\![f \odot g]\!]$.
3. 2DFTs $\mathcal{A}$, $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![f \boxdot g]\!]$ and $[\![\mathcal{B}]\!] = [\![f \overleftarrow{\boxdot} g]\!]$.
4. 2DFTs $\mathcal{A}$, $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![f^{\boxplus}]\!]$ and $[\![\mathcal{B}]\!] = [\![f^{\overleftarrow{\boxplus}}]\!]$.
5. 2DFTs $\mathcal{A}$, $\mathcal{B}$ such that $[\![\mathcal{A}]\!] = [\![[K, f]^{2\boxplus}]\!]$ and $[\![\mathcal{B}]\!] = [\![[K, f]^{\overleftarrow{2\boxplus}}]\!]$.

# RTE to 2DFT: If then else

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

| | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vdash$ | **a** | **a** | **b** | **a** | **b** | **b** | **a** | **a** | **a** | **b** | **a** | **a** | **b** | **b** | **b** | **b** | **a** | **b** | **a** | **a** | **a** | **b** | **b** | **b** | **b** | $\dashv$ |

**Main Lemma:**

Let $K \subseteq \Sigma^*$ be regular, and let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$ respectively. Then, one can construct

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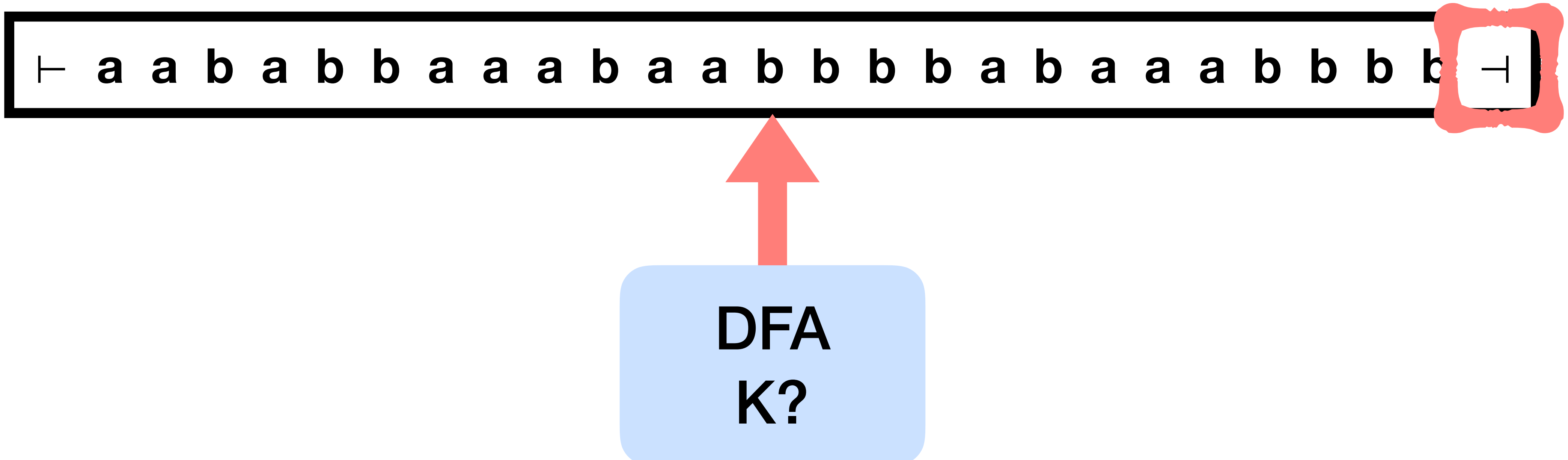


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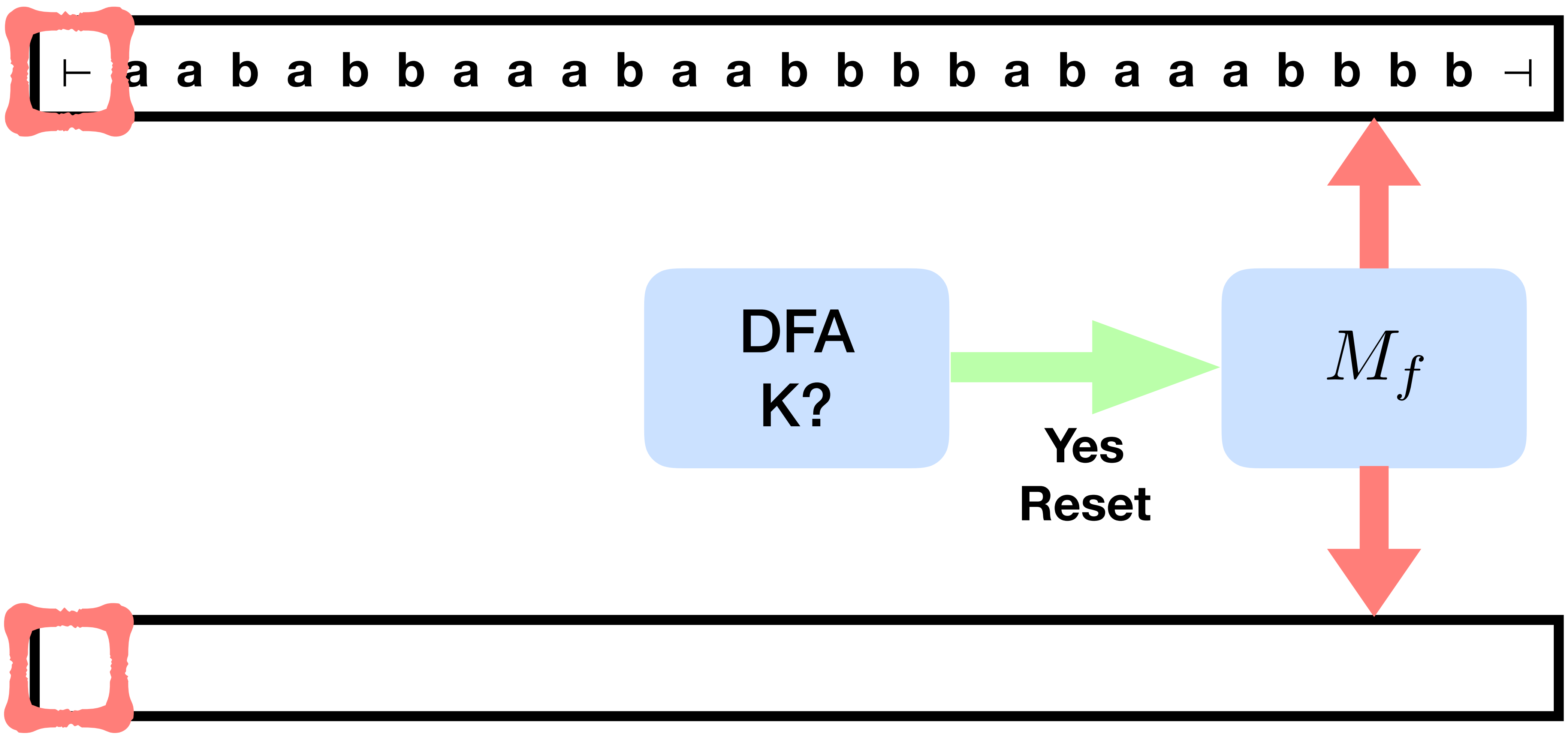


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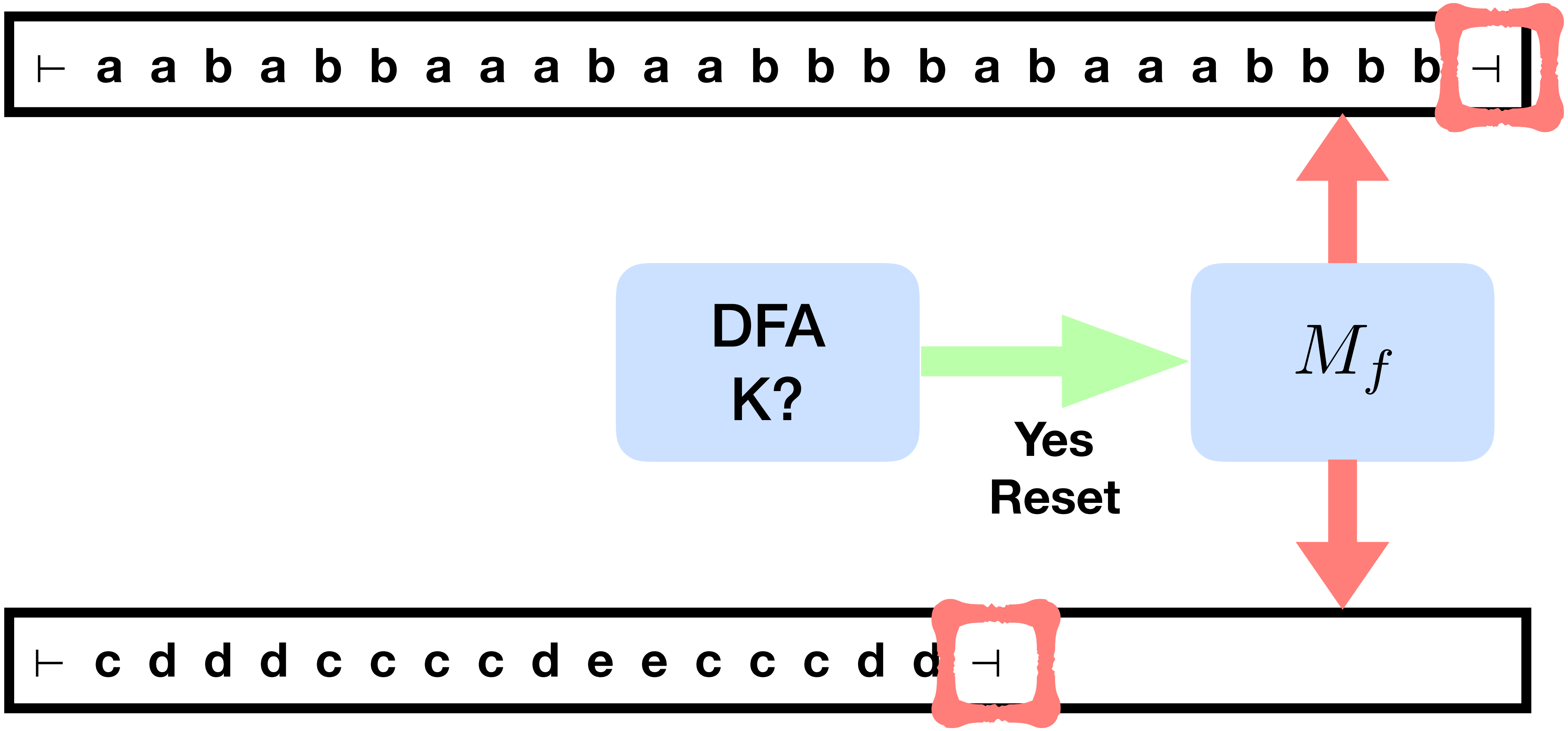


**Main Lemma:**

Let $K \subseteq \Sigma^*$ be regular, and let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$ respectively. Then, one can construct

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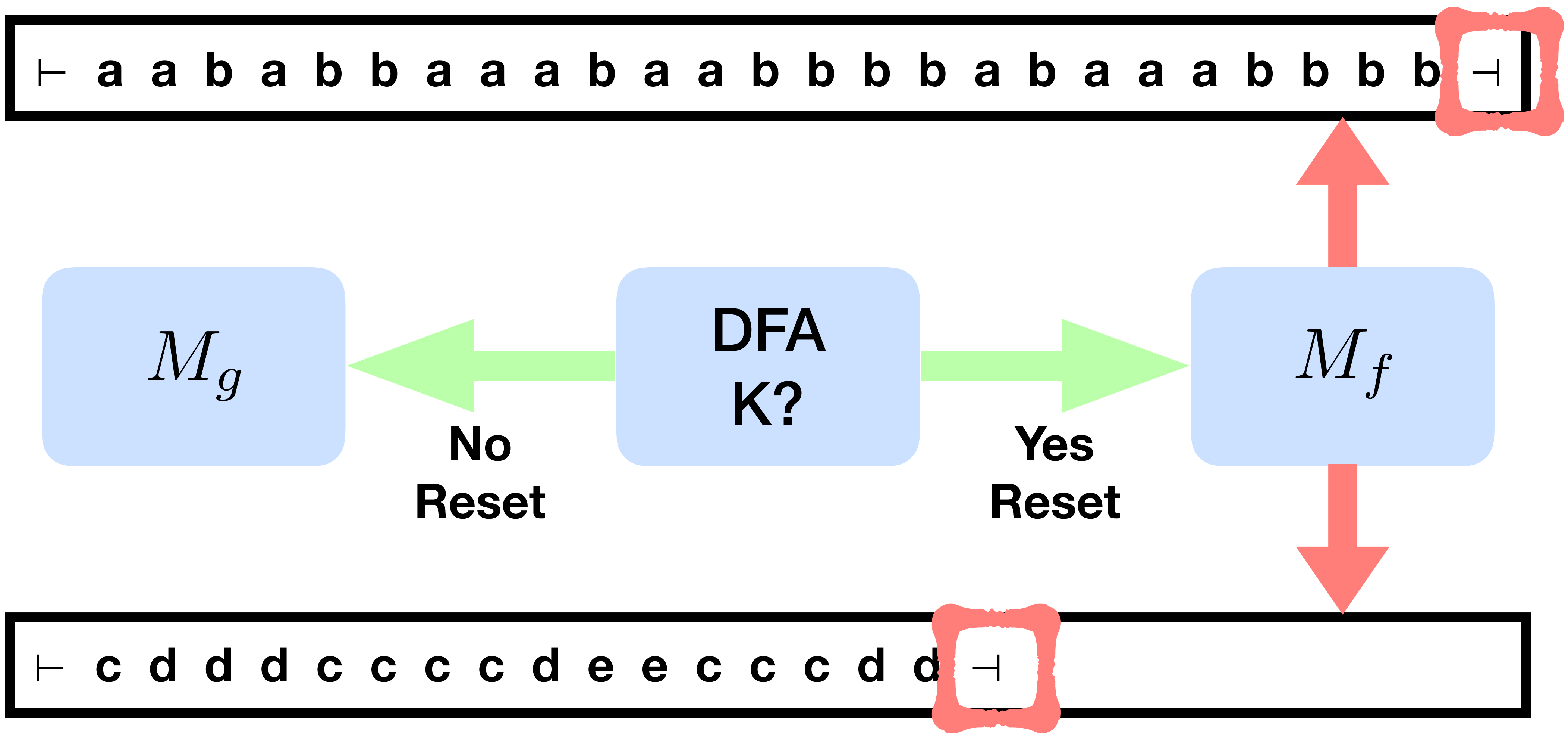


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# RTE to 2DFT: Hadamard product

$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

$\vdash$ **a a b a b b a a a b a a b b b b a b a a a b b b b** $\dashv$

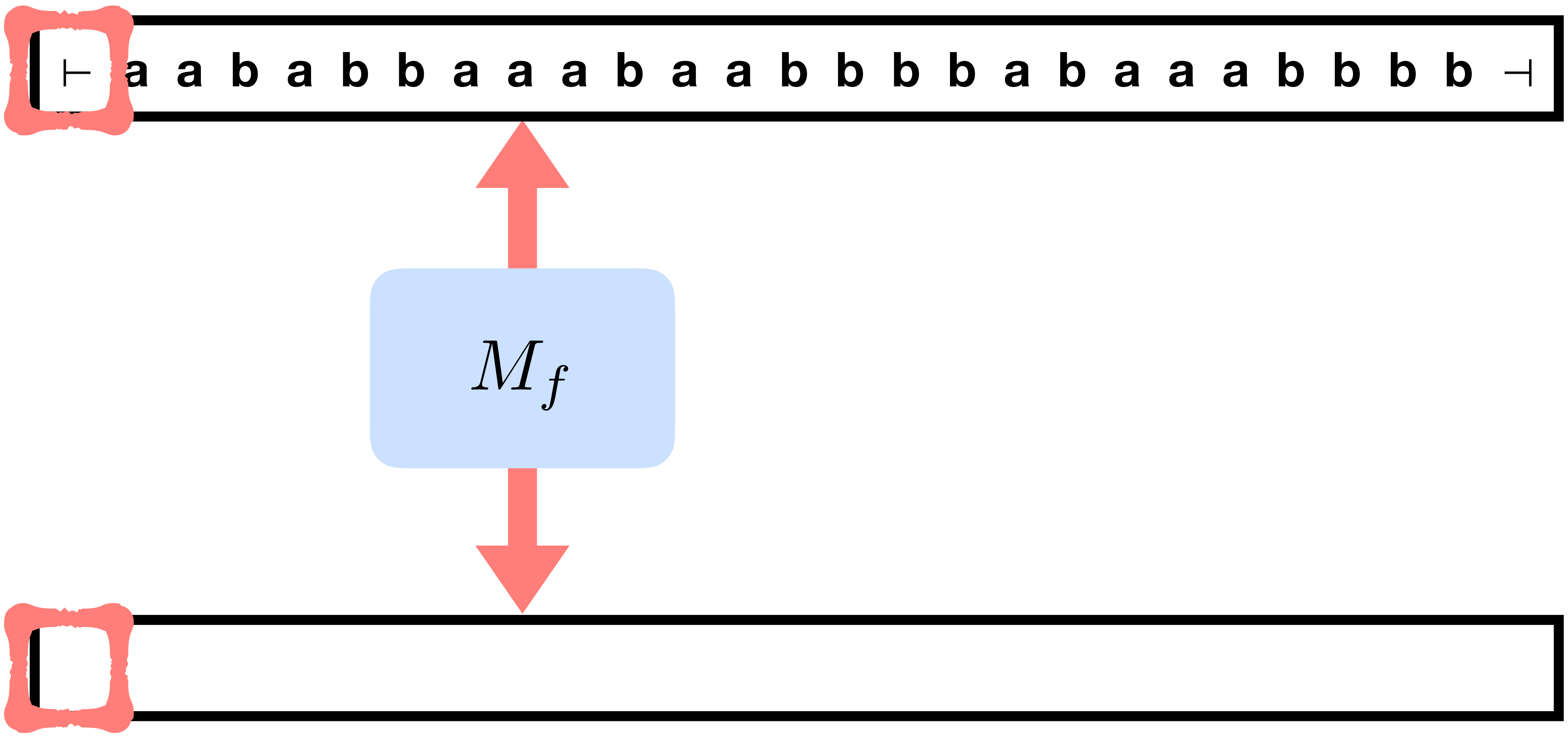
**Main Lemma:**

Let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$. Then, one can construct

2. a 2DFT $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = [\![f \odot g]\!]$.

# RTE to 2DFT: Hadamard product

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$$

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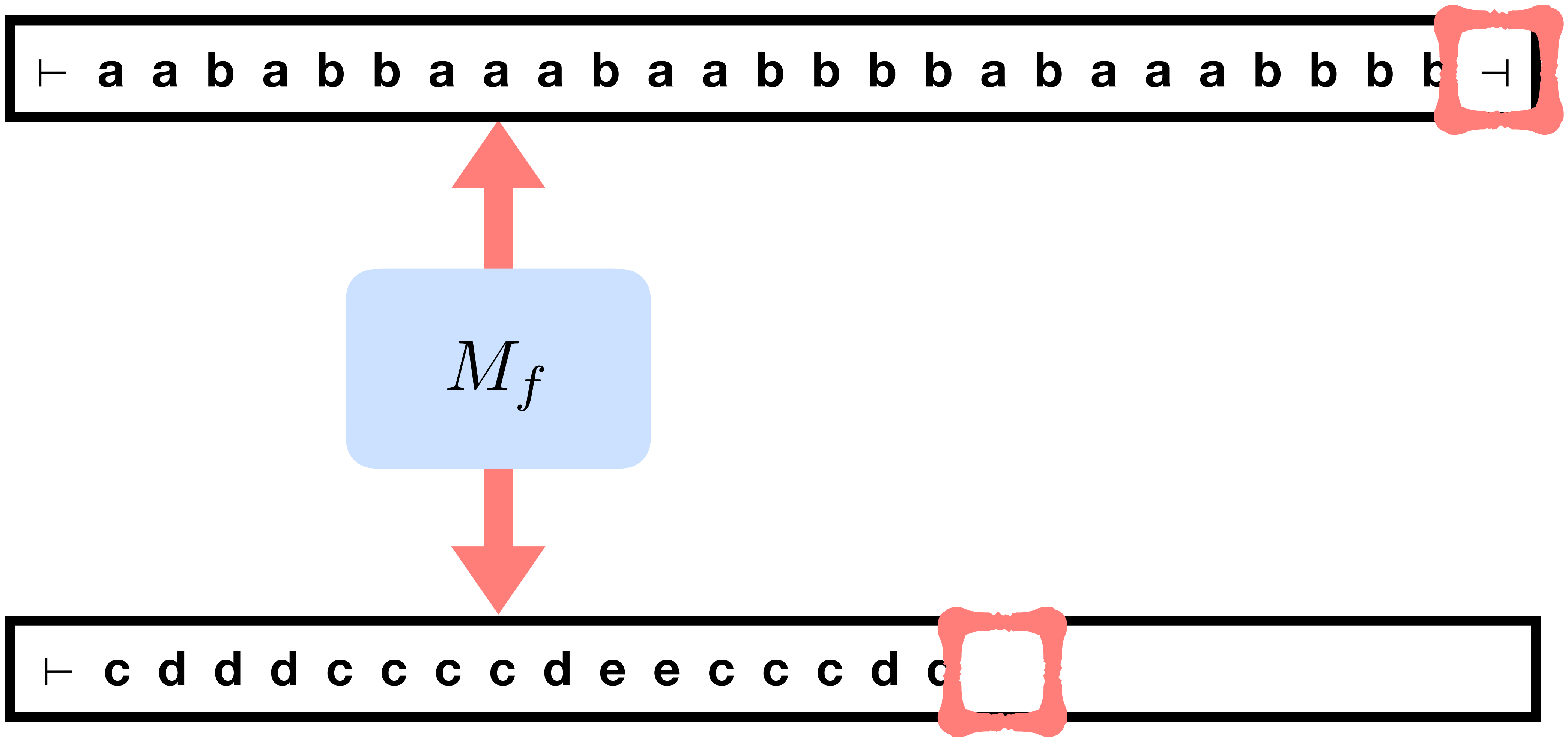


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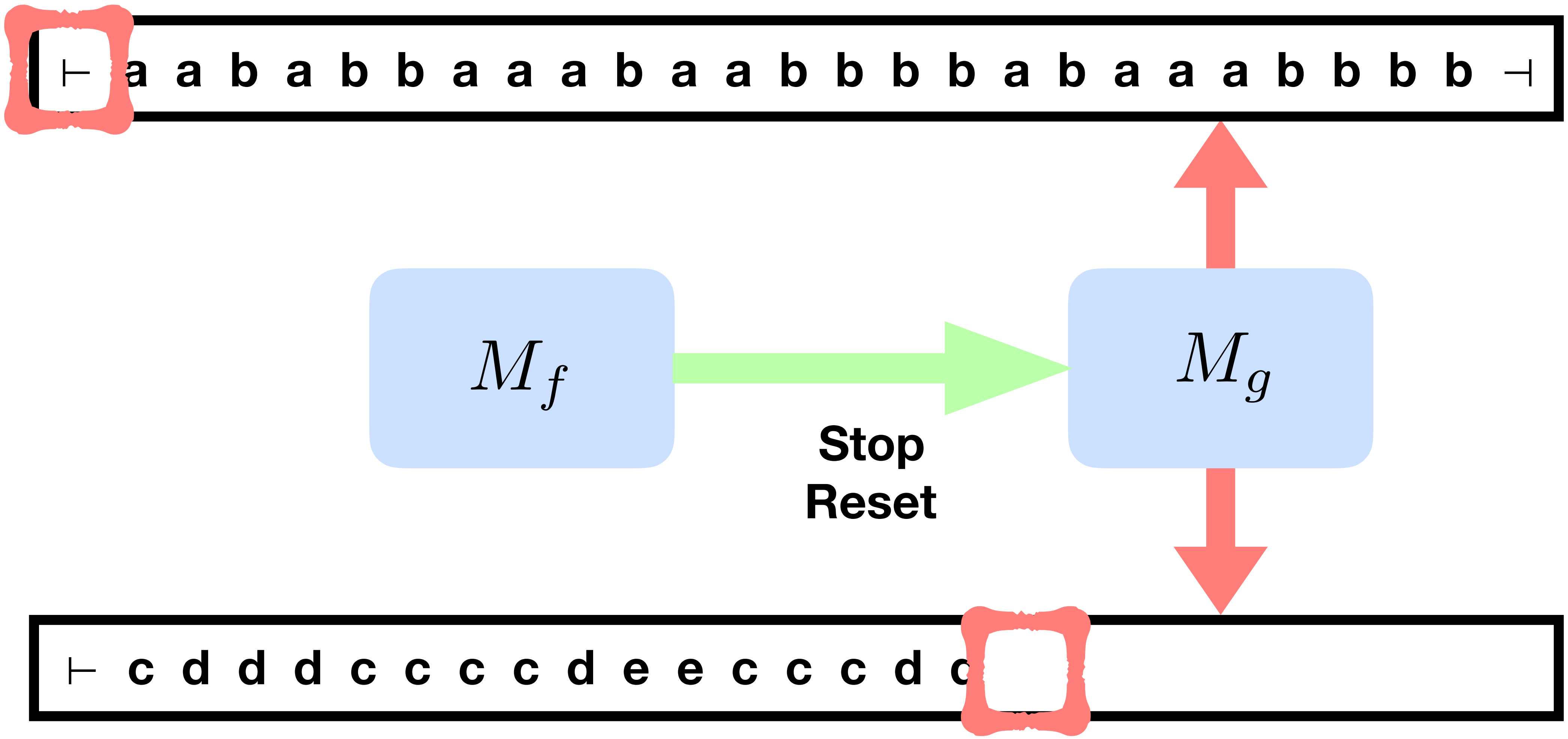


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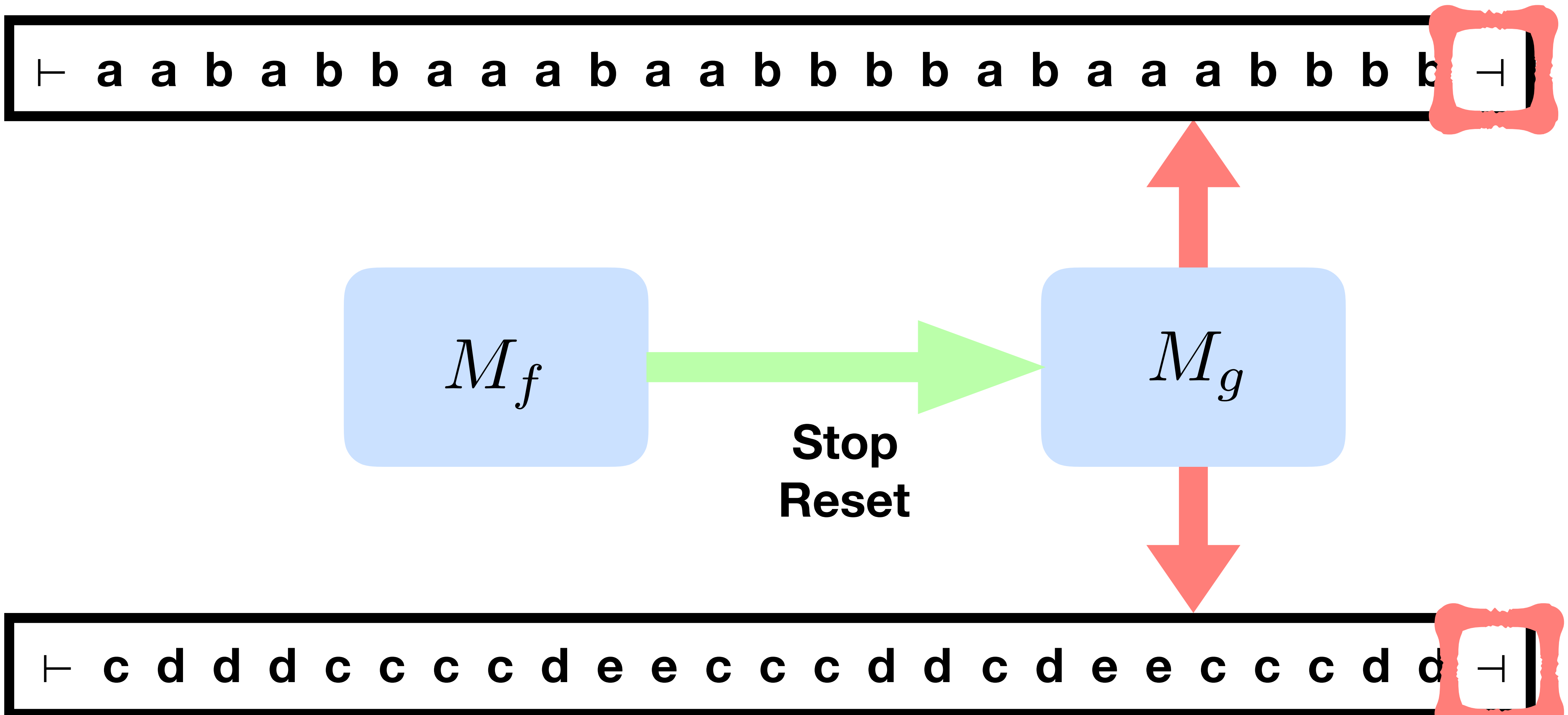


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# RTE to 2DFT: Cauchy product

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

| $\vdash$ a a b a b b a a a b a a b b b a b a a a b b b b $\dashv$ |
|---|

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$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$$

$\vdash$ a a b a b b a a a b a a b b b a b a a a b b b b $\dashv$

**DFA**
$\mathsf{dom}(f) \cdot \mathsf{dom}(g)$

**Unambiguous**

**Main Lemma:**

Let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$. Then, one can construct

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$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$$

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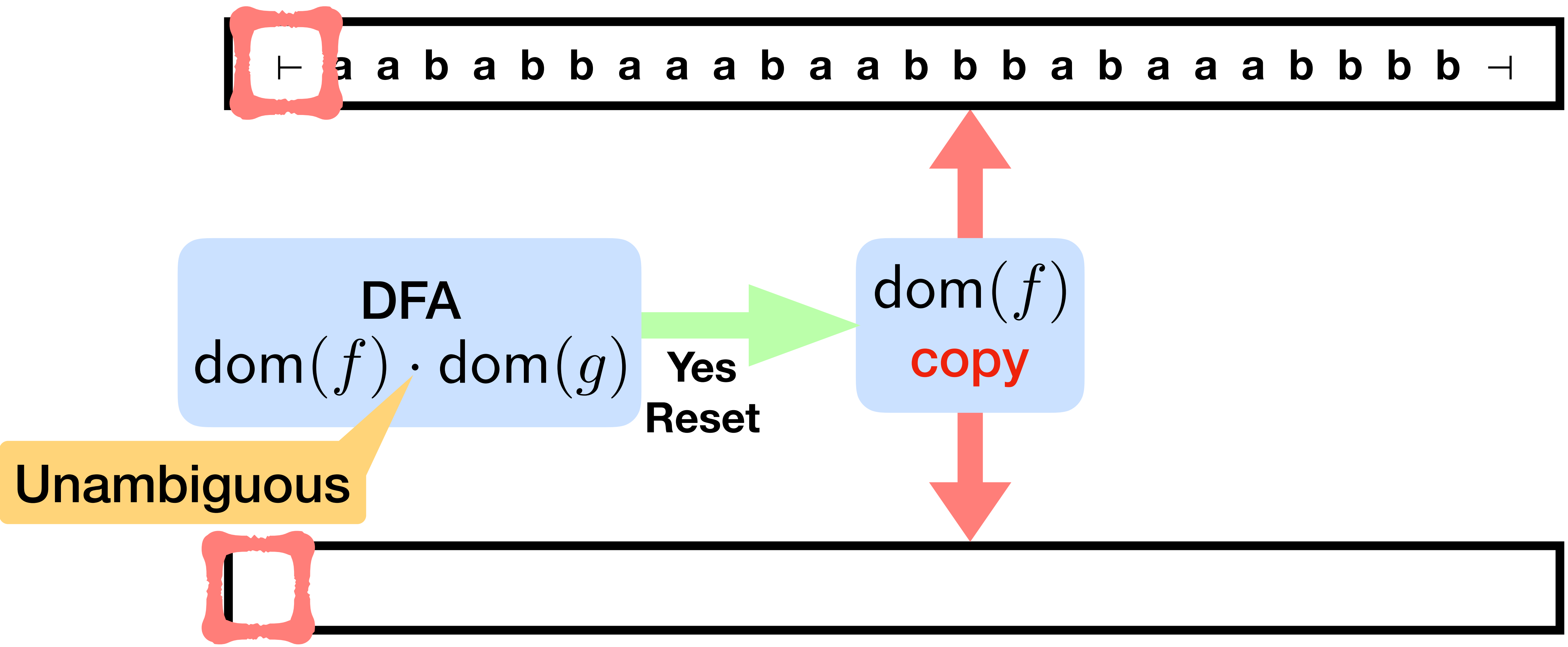
$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$

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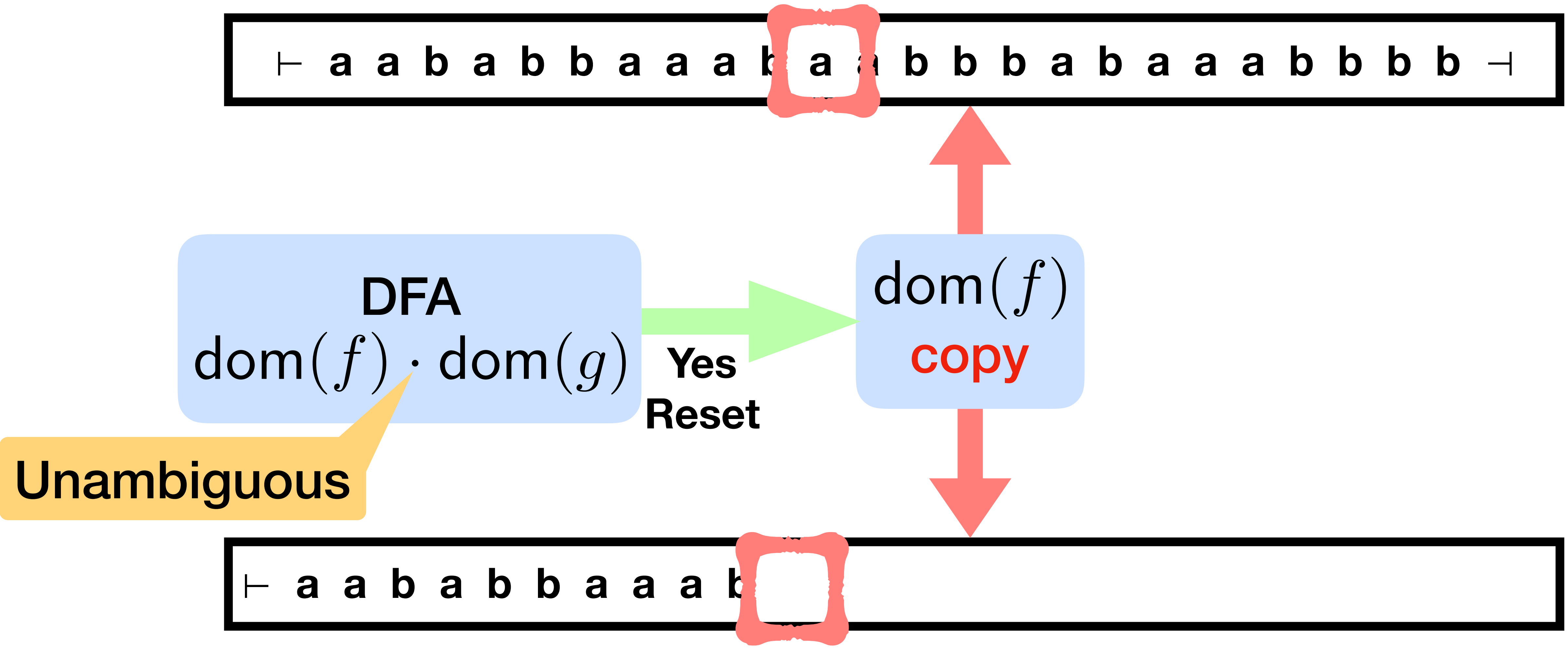
# RTE to 2DFT: Cauchy product

$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

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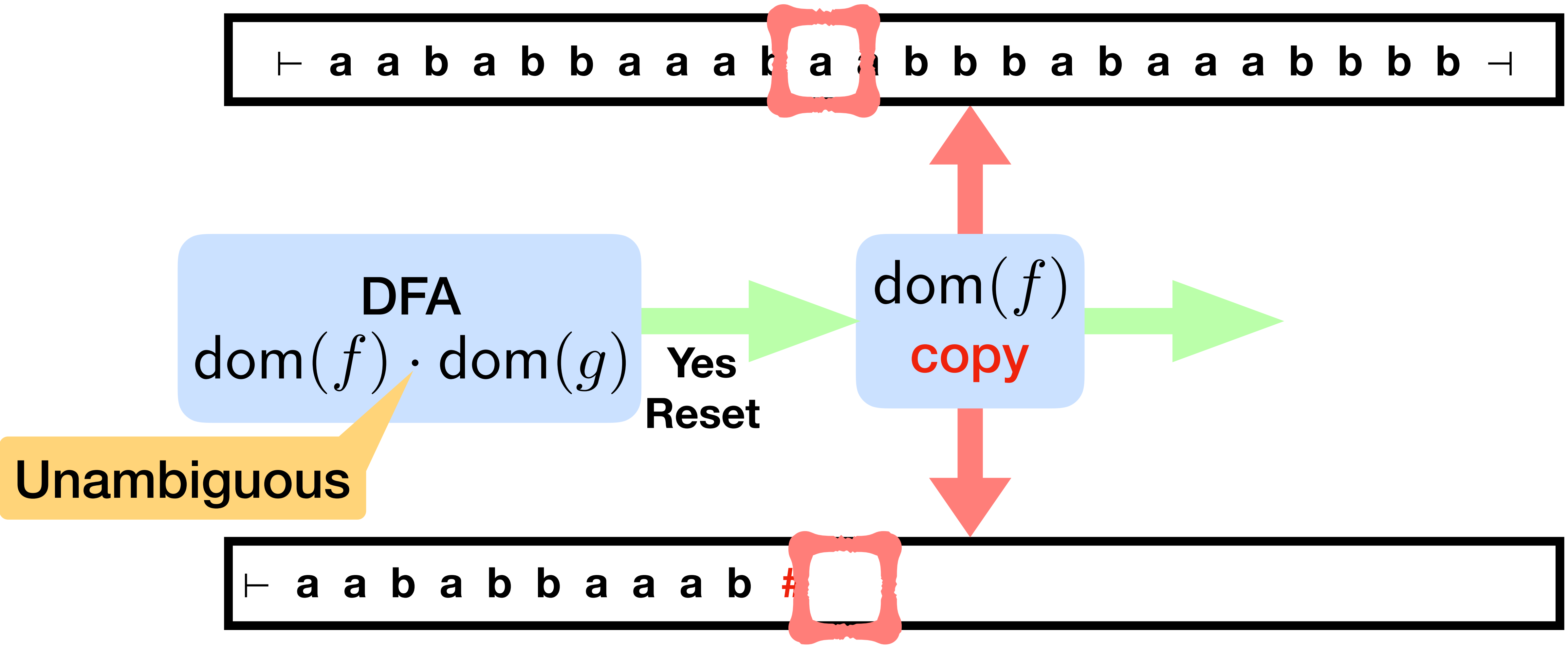
# RTE to 2DFT: Cauchy product

$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

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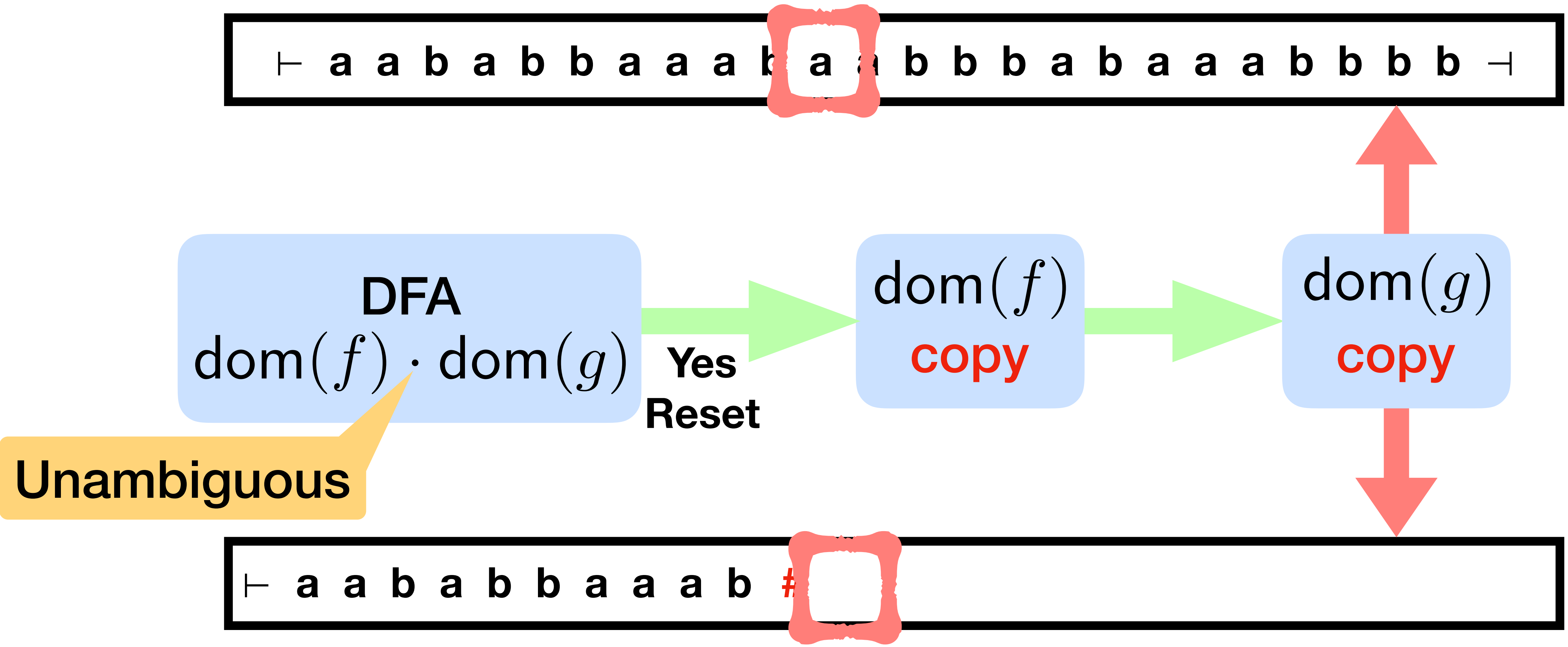
# RTE to 2DFT: Cauchy product

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid \boxed{C \boxdot C} \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$$

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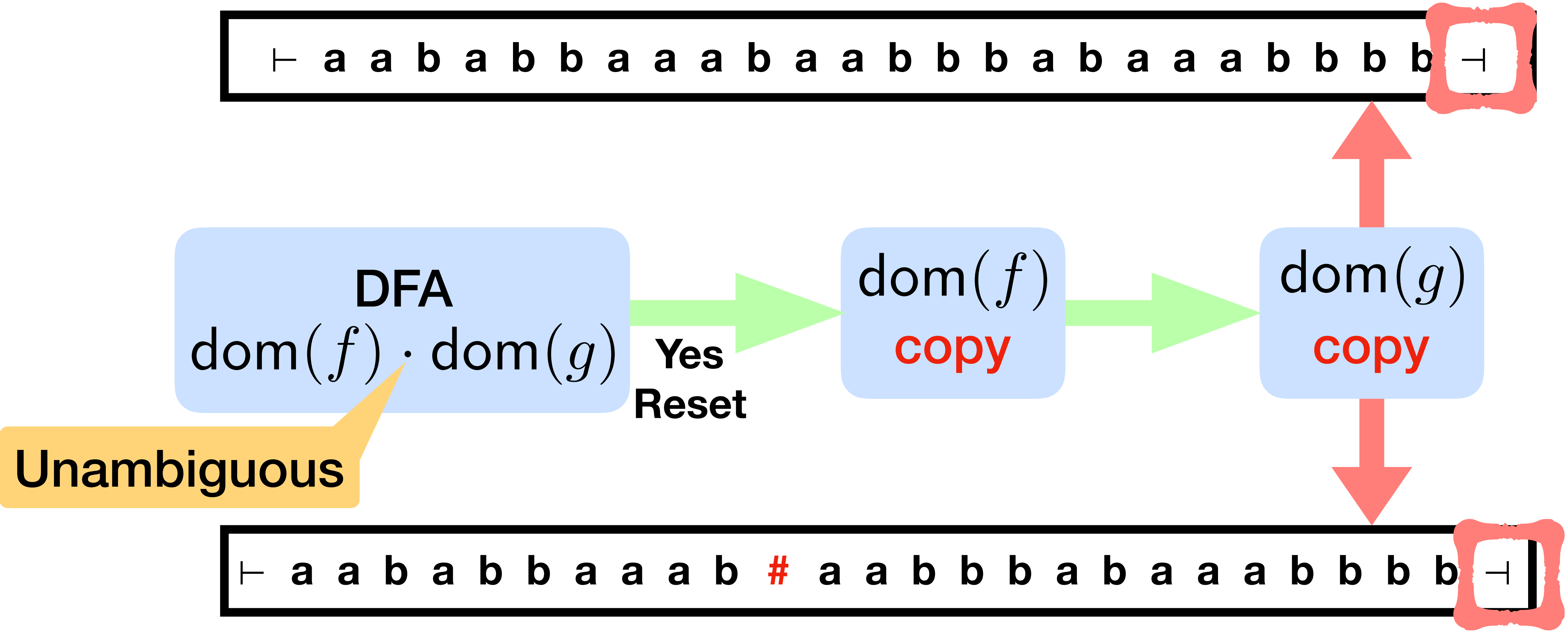


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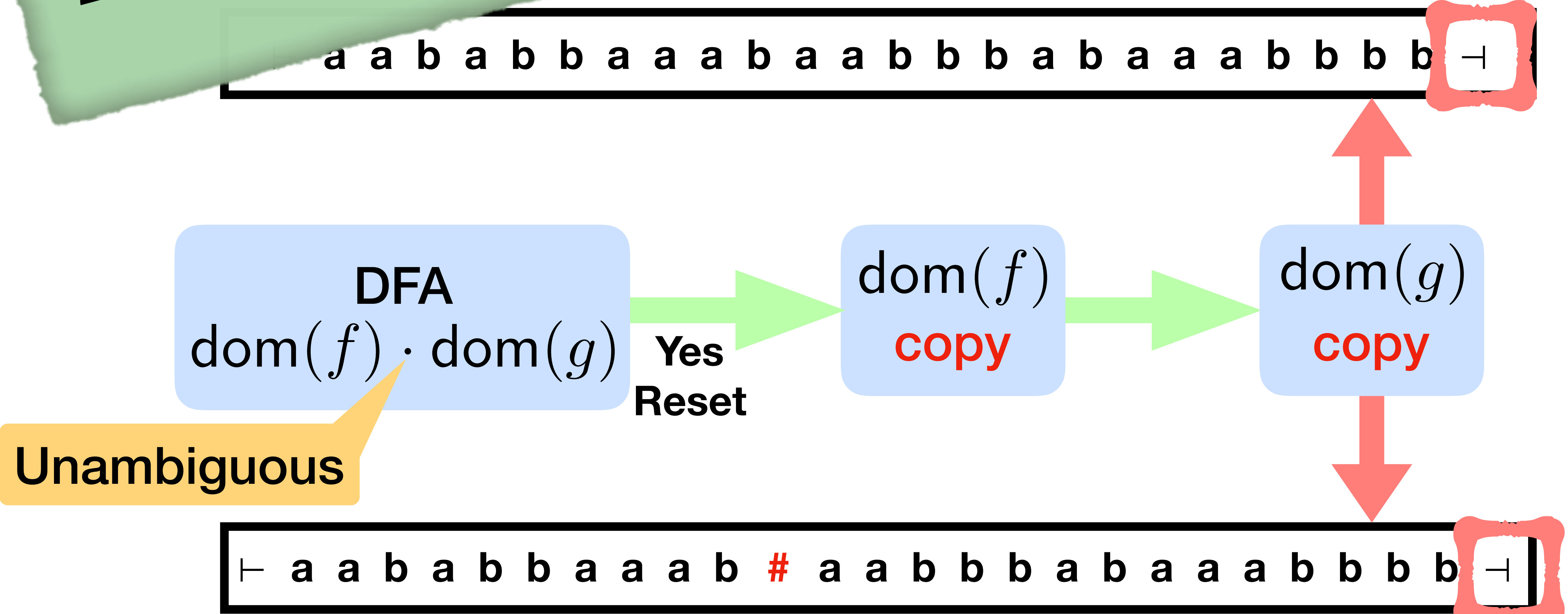
# RTE to 2D Cauchy product

2-way Unambigous Transducer
(2UFT)

$C ::= \ldots \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

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Let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$. Then, one can construct

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# RTE to 2DFT: Cauchy product

$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$

$\vdash$ **a a b a b b a a a b** **#** **a a b b b a b a a a b b b b** $\dashv$

**Main Lemma:**

Let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$. Then, one can construct

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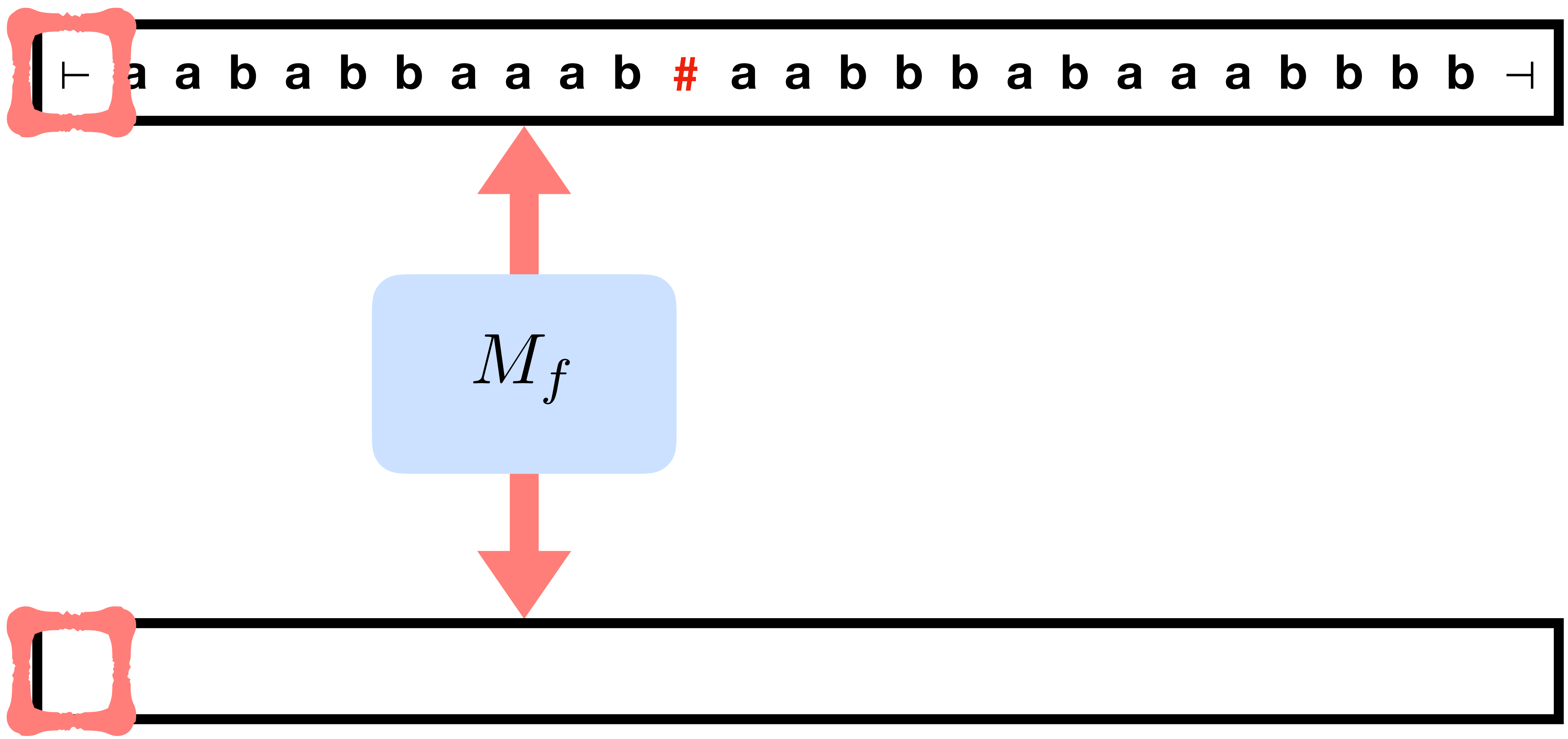
# RTE to 2DFT: Cauchy product

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$$

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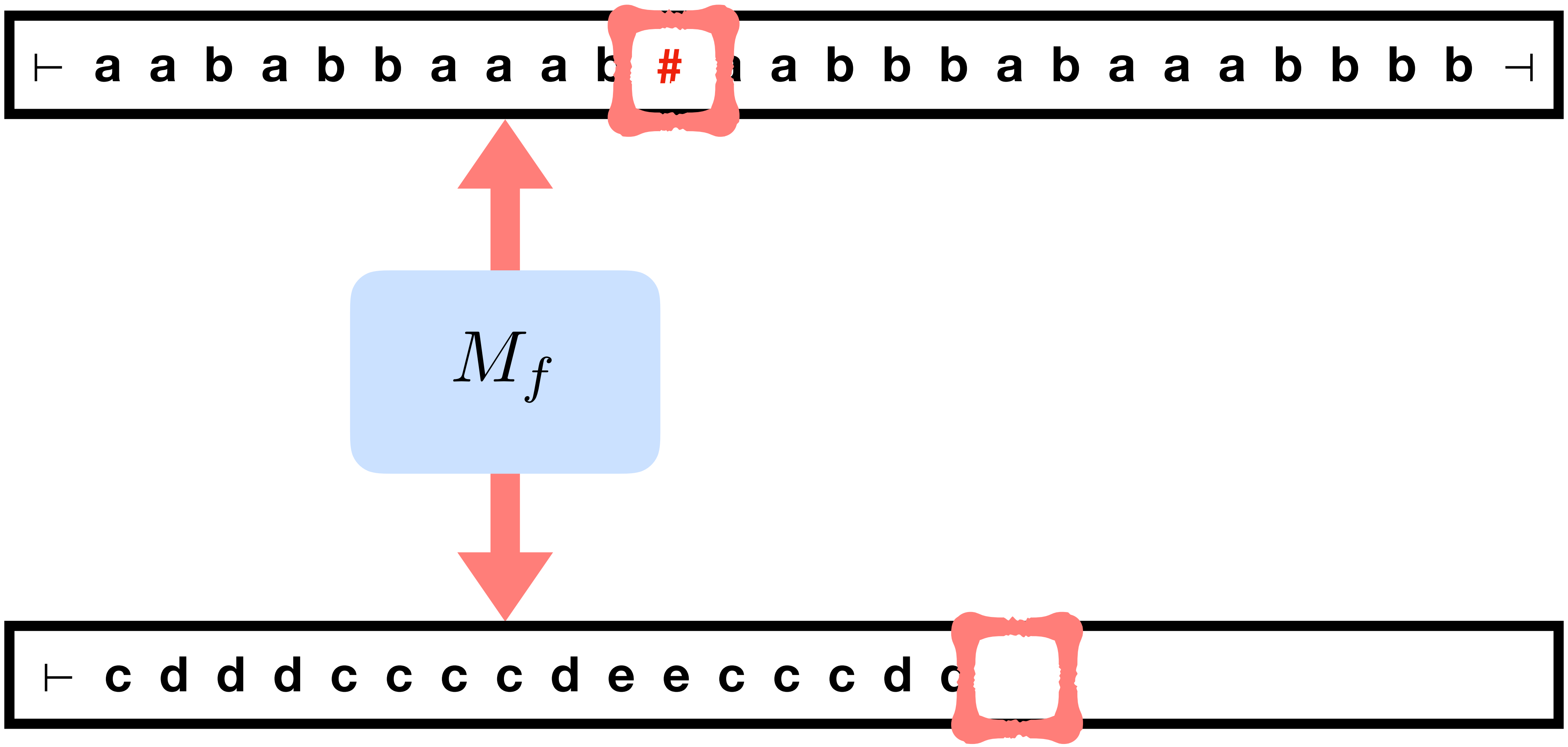
# RTE to 2DFT: Cauchy product

$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K,C]^{2\boxplus} \mid [K,C]^{\overleftarrow{2\boxplus}}$$




**Main Lemma:**

Let $f$ and $g$ be RTEs with $[\![f]\!] = [\![M_f]\!]$ and $[\![g]\!] = [\![M_g]\!]$ for 2DFTs $M_f$ and $M_g$. Then, one can construct

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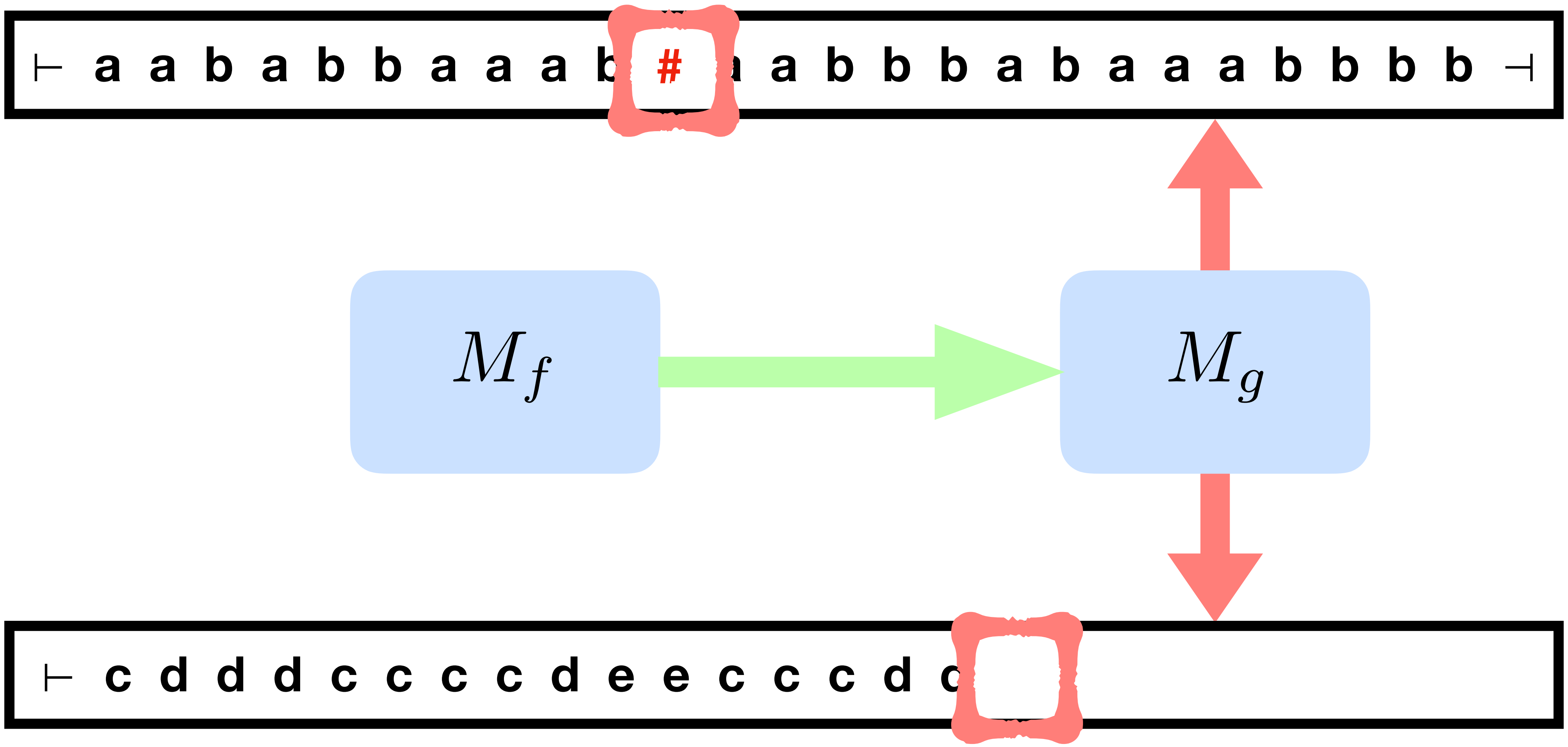


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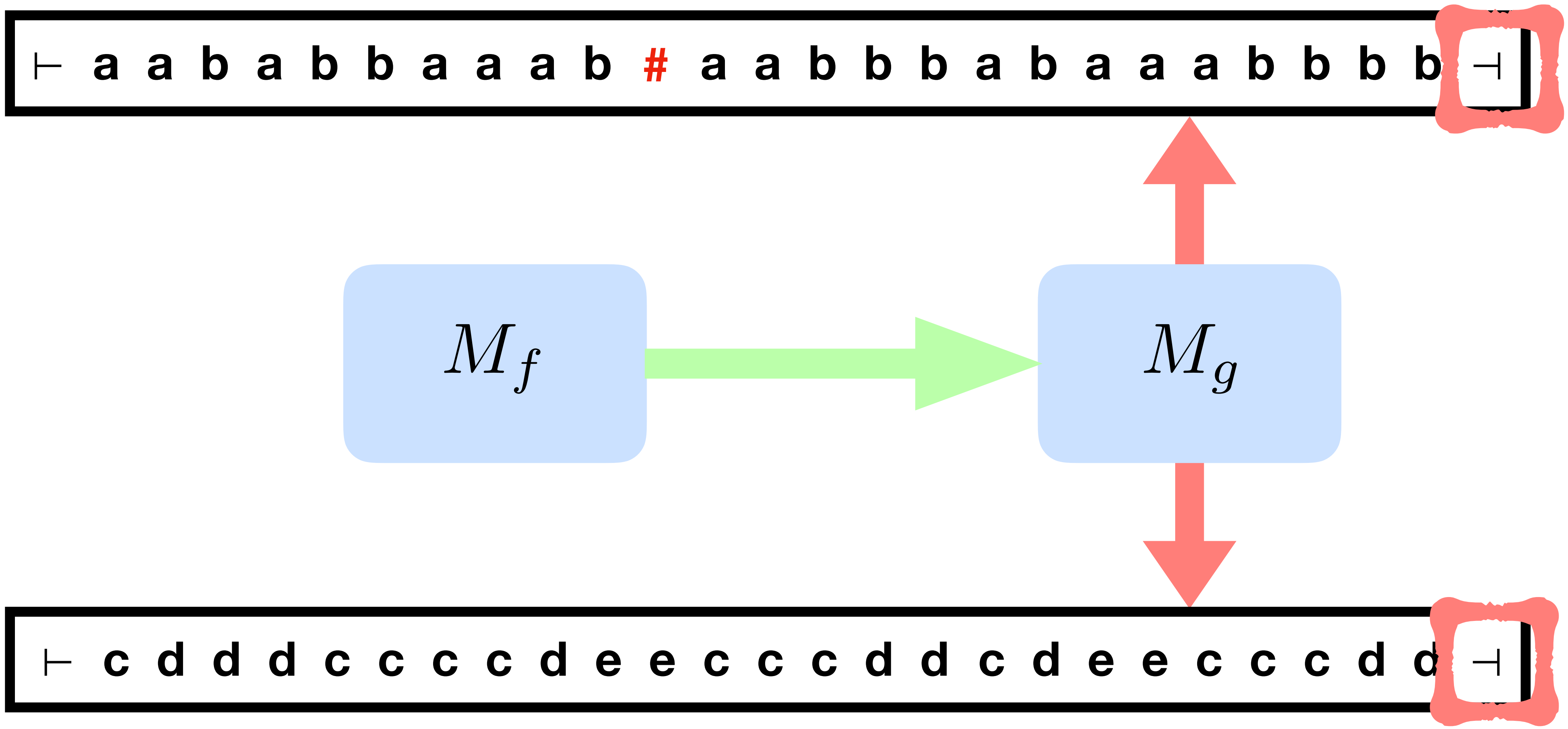


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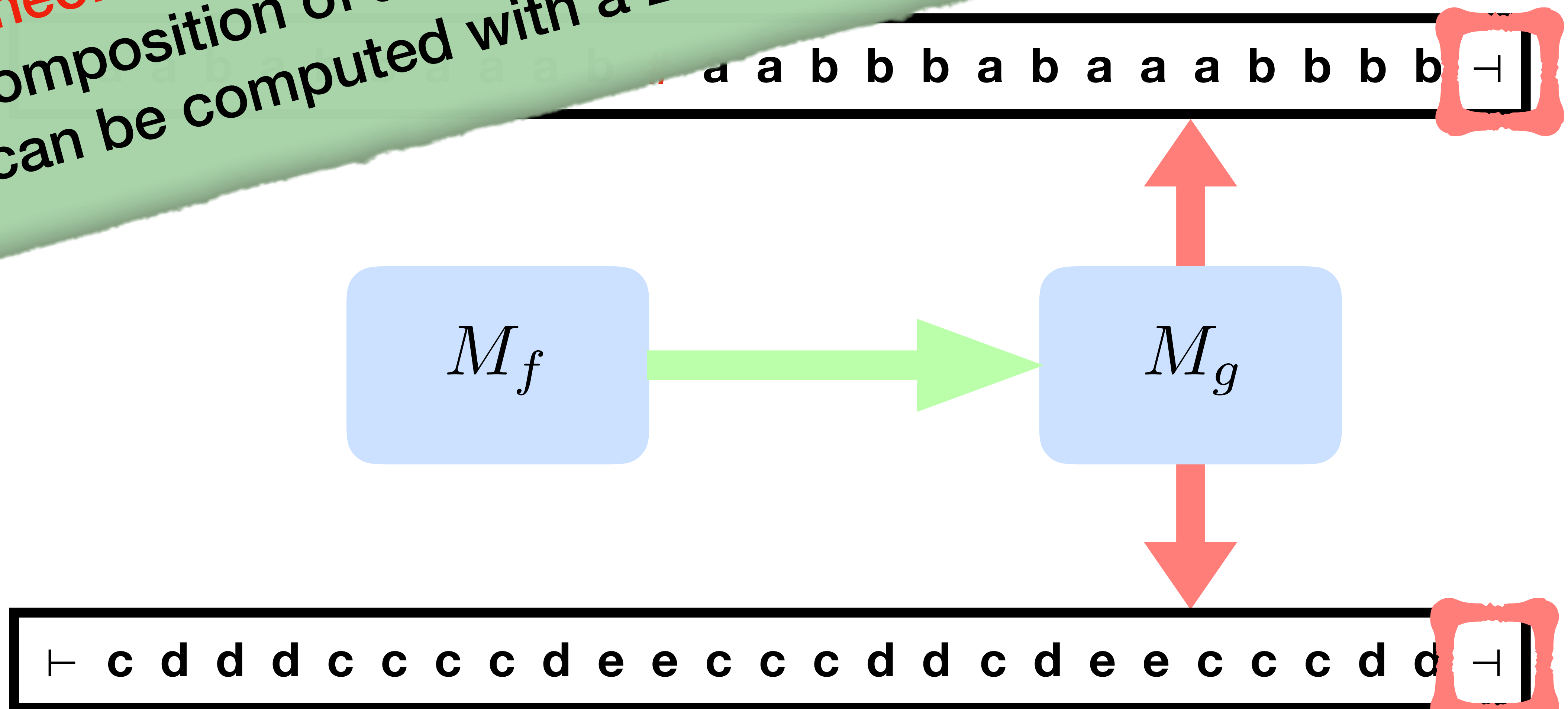
# RTE to 2DFT: Cauchy product

$C ::= d \mid$ … $C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

**Theorem: (Chytil & Jákl, 1977)**
The composition of a 2UFT with a 2DFT can be computed with a 2DFT

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# Summary

- MSO Transductions
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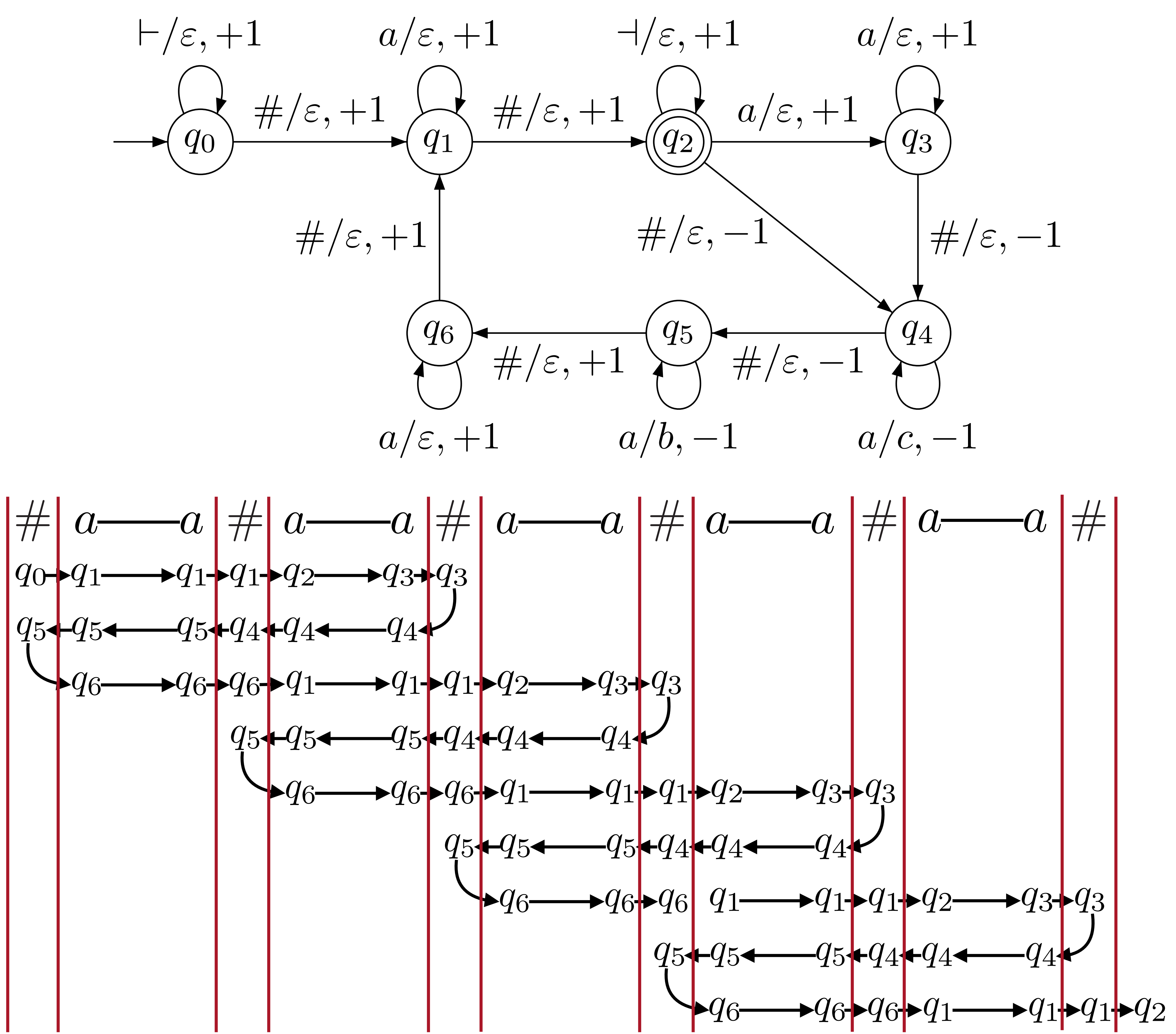
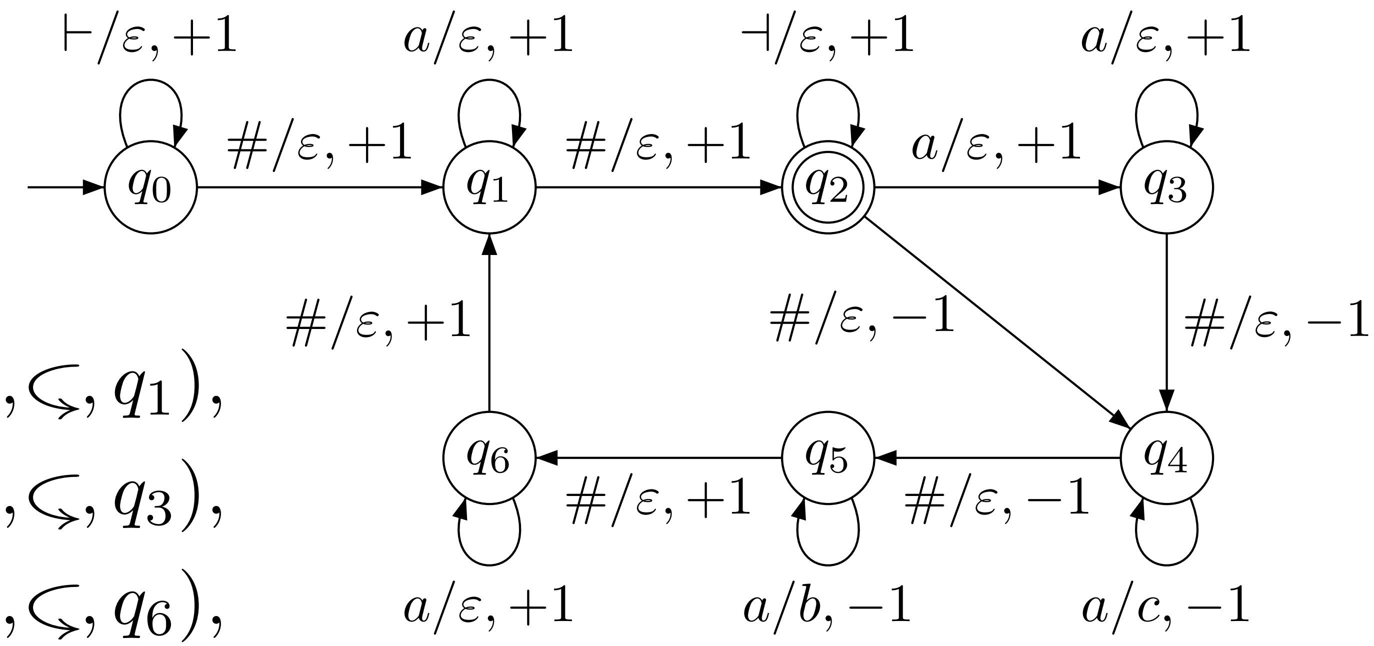
# Transition Monoid

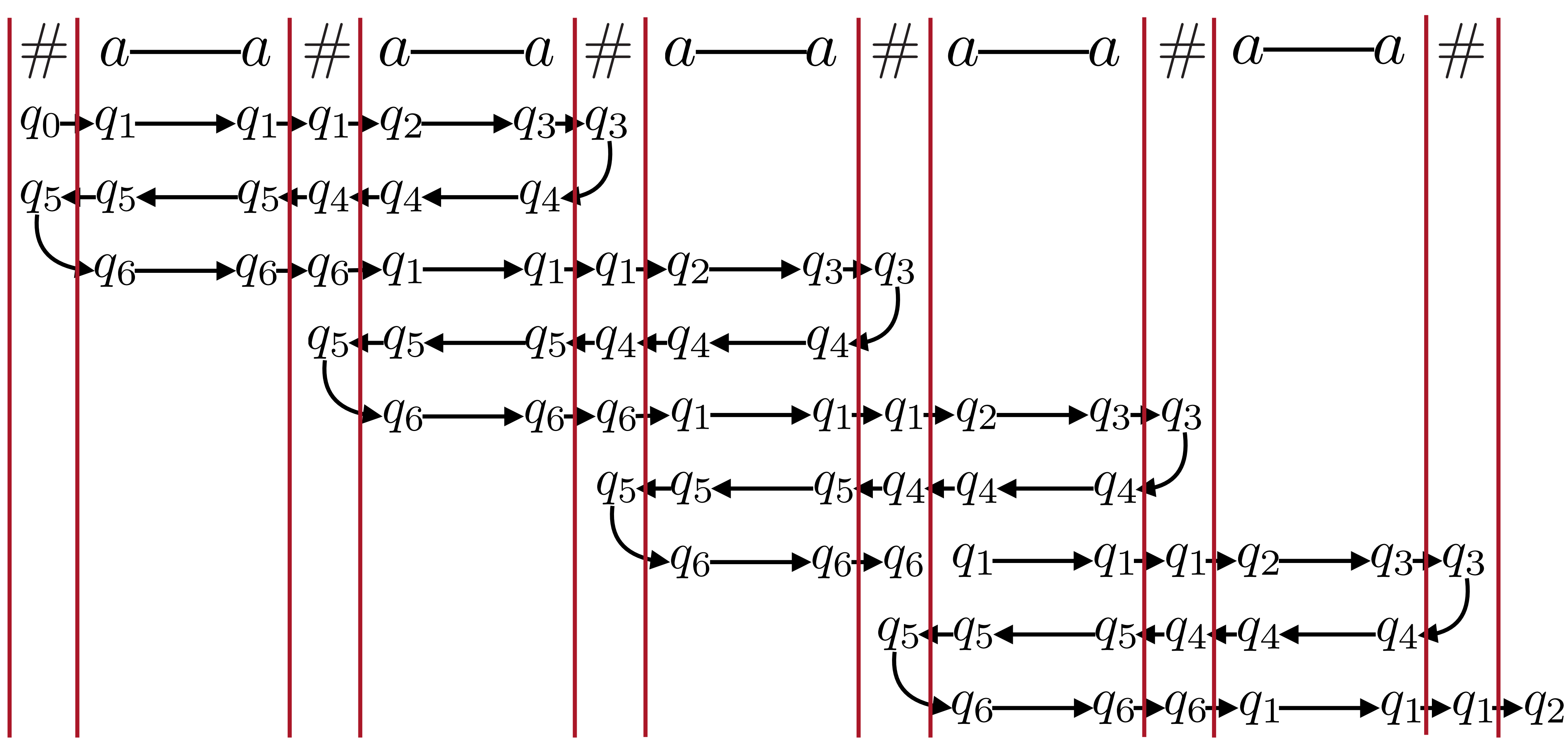


# Transition Monoid

$$\mathrm{Tr}\colon \Sigma^* \to \mathrm{TrM}$$




$$\begin{aligned}\mathrm{Tr}(\#a^+\#a^+) = \{&(q_0,\rightarrow,q_3),(q_1,\curvearrowleft,q_5),(q_1,\curvearrowright,q_1),\\ &(q_2,\curvearrowleft,q_4),(q_2,\curvearrowright,q_3),(q_3,\curvearrowleft,q_4),(q_3,\curvearrowright,q_3),\\ &(q_4,\curvearrowleft,q_5),(q_4,\curvearrowright,q_1),(q_5,\rightarrow,q_1),(q_5,\curvearrowright,q_6),\\ &(q_6,\rightarrow,q_3),(q_6,\curvearrowright,q_6)\}\end{aligned}$$



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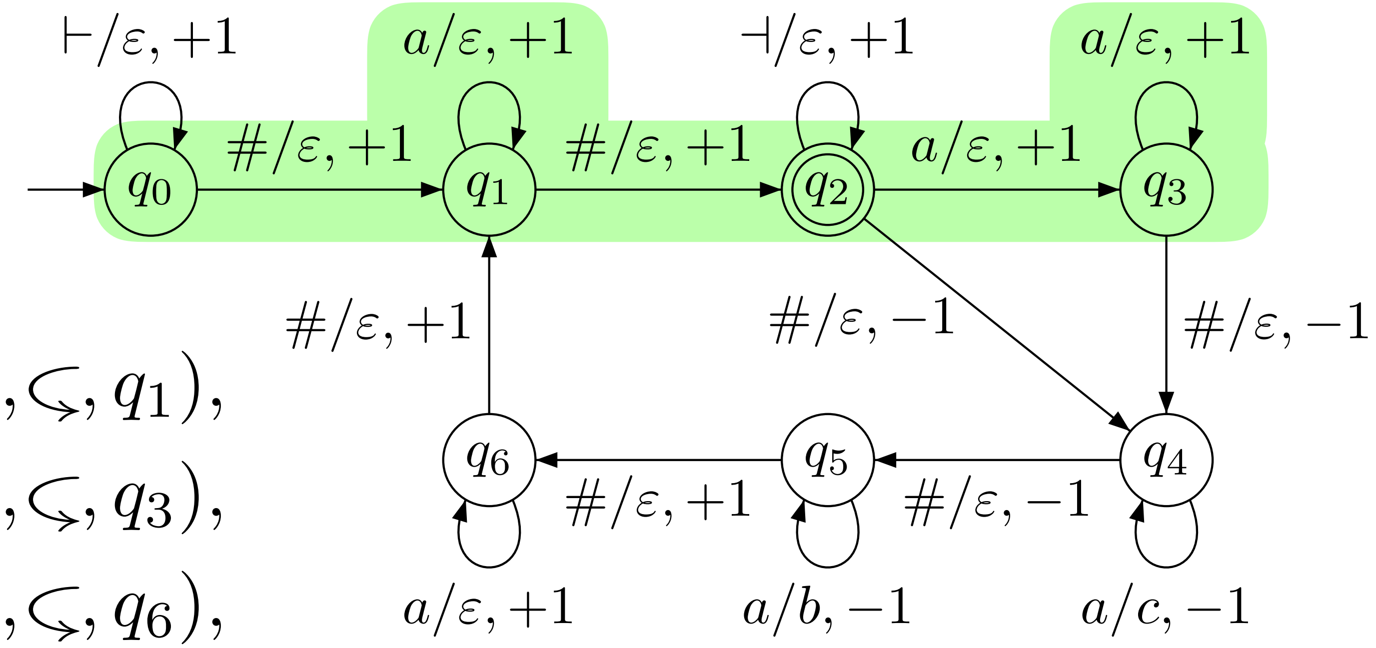


$$\mathrm{Tr}(\#a^+\#a^+) = \{(q_0,\rightarrow,q_3),(q_1,\supsetneq,q_5),(q_1,\subsetneq,q_1),$$
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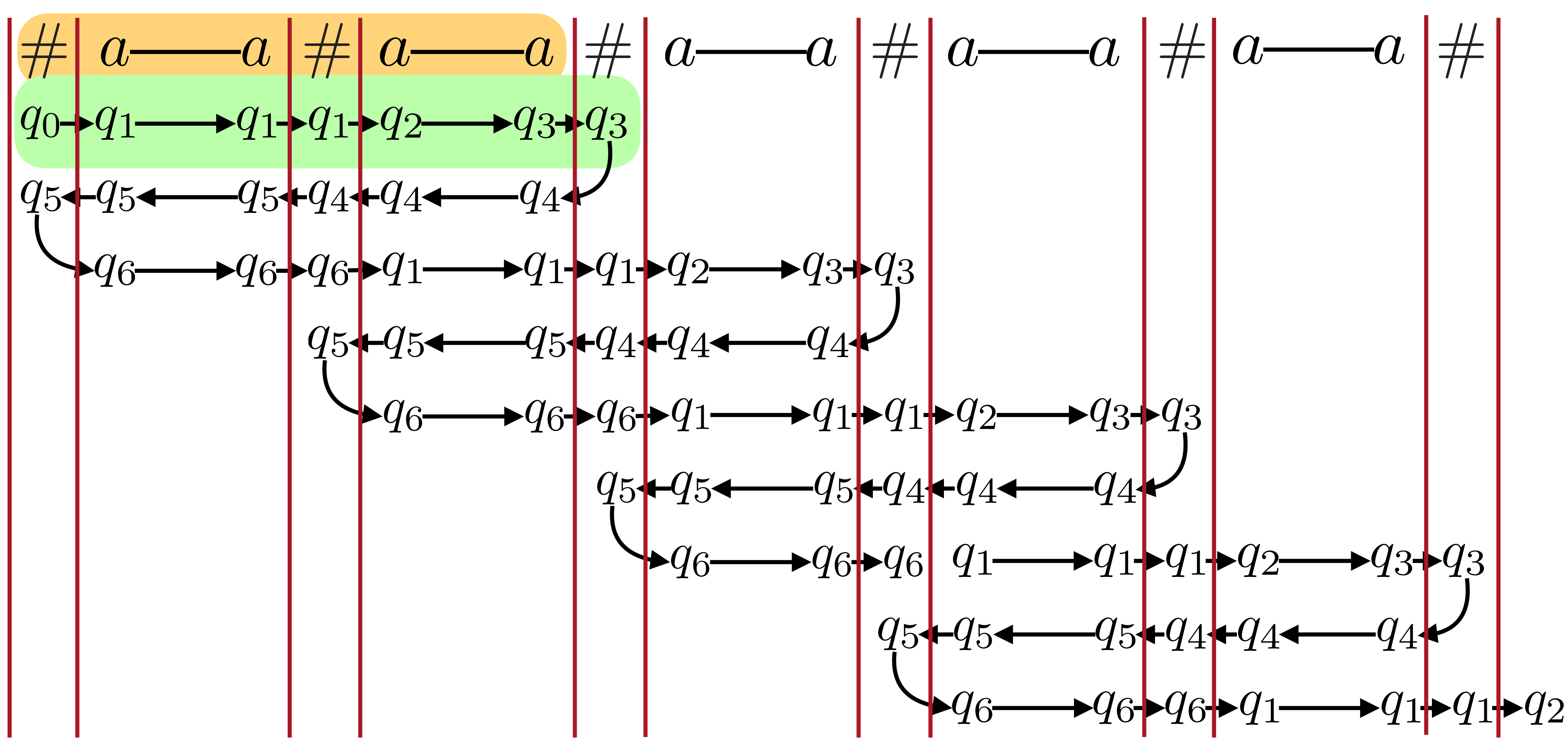

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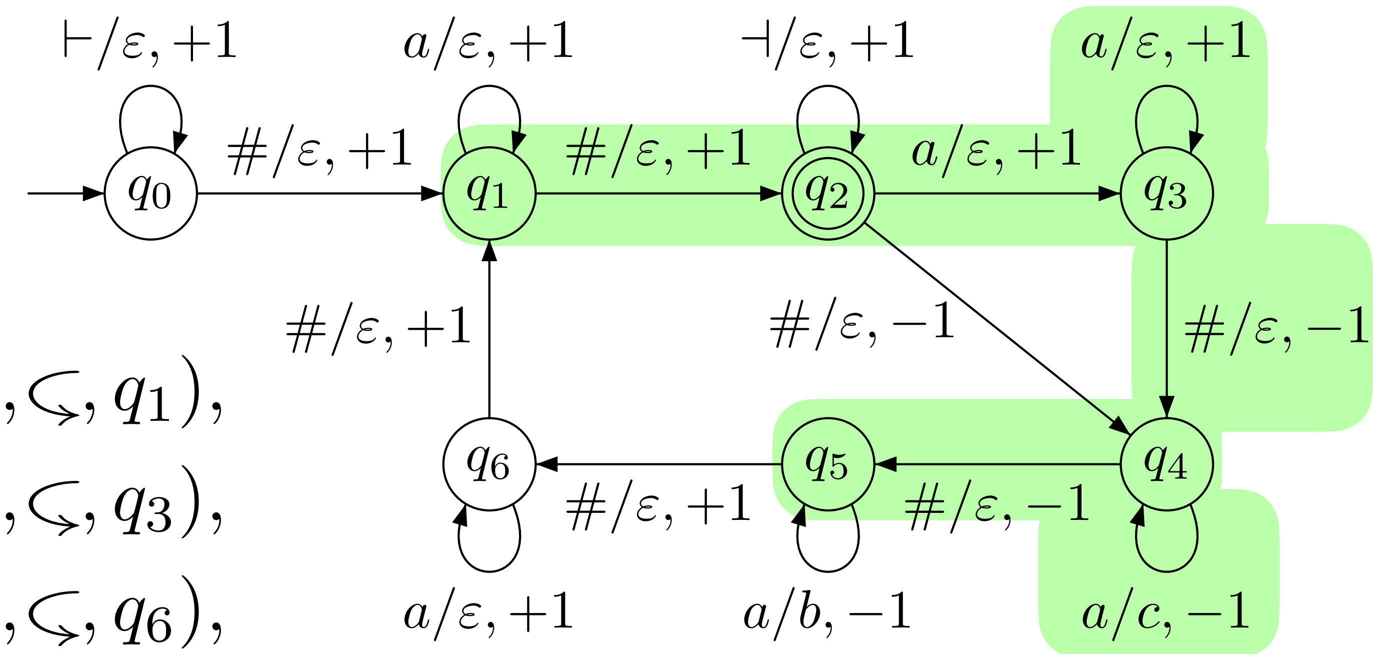


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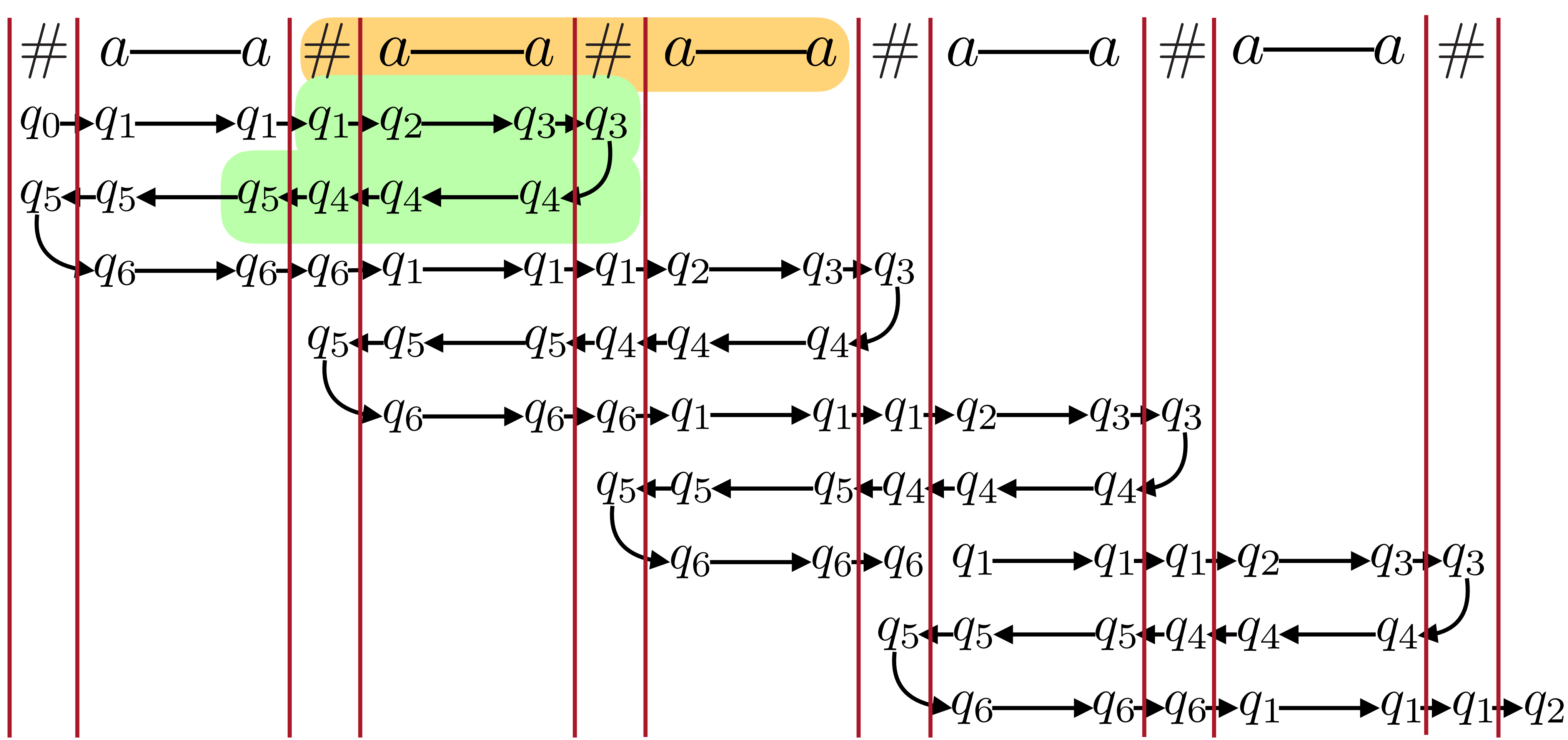

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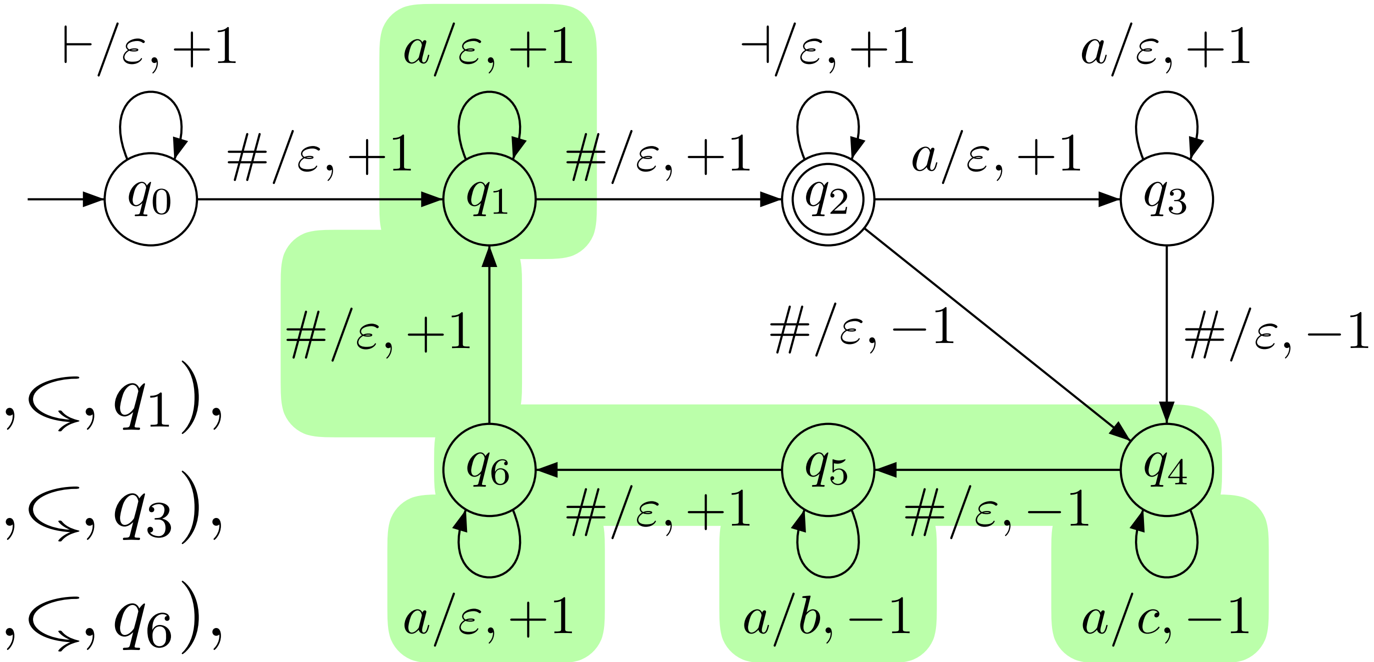


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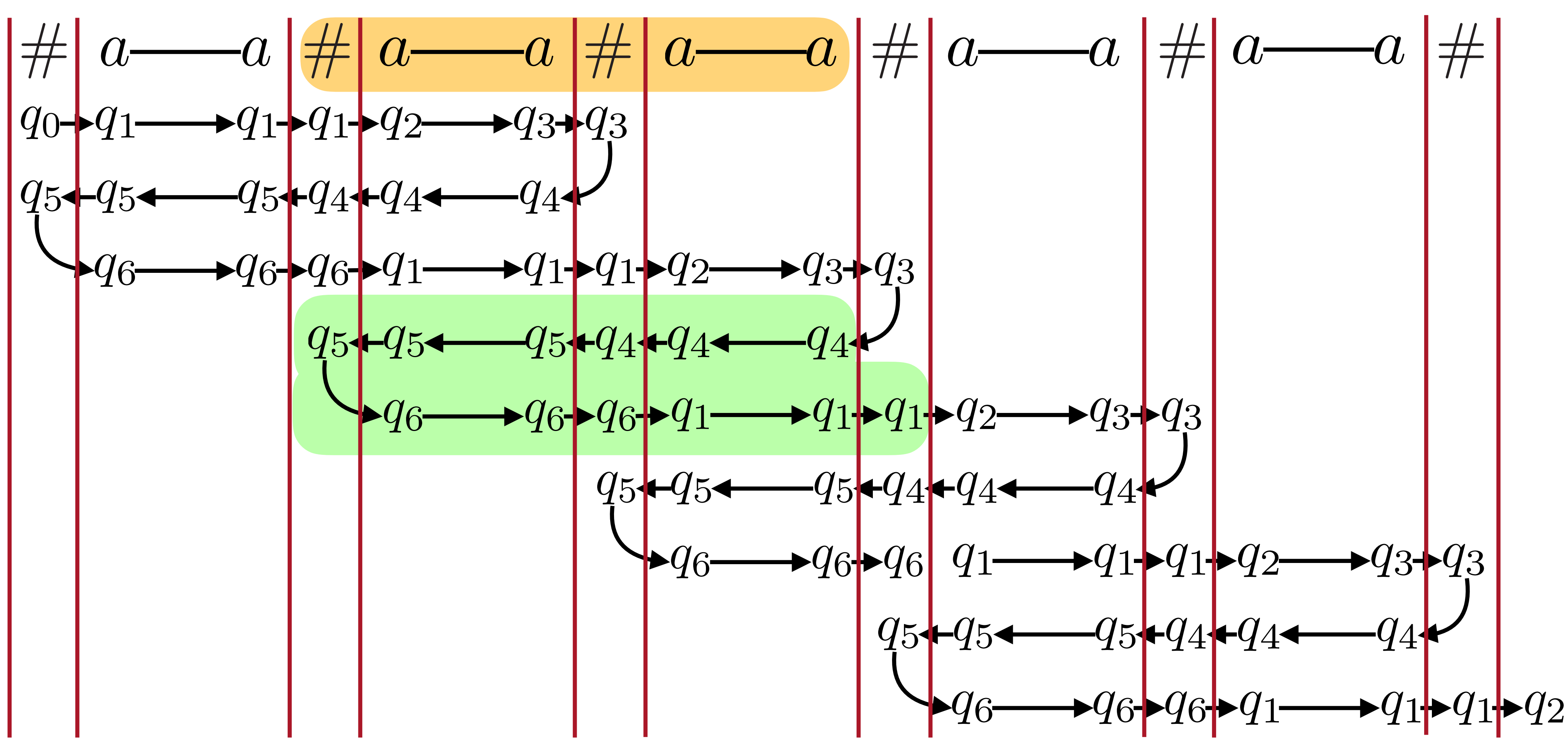

# Summary

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# Forest Factorization

$\varphi\colon \Sigma^* \to S$ morphism

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$

$\# \; a \; a \; a \; a \; \# \; a \; a \; a \; a \; \# \; a \; a \; a \; a$

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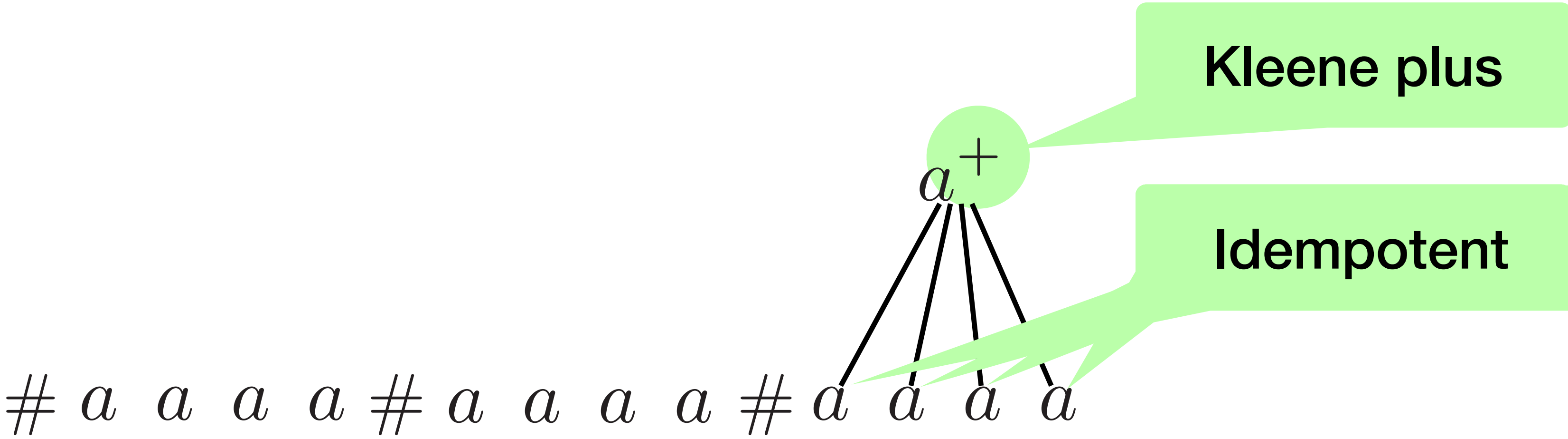

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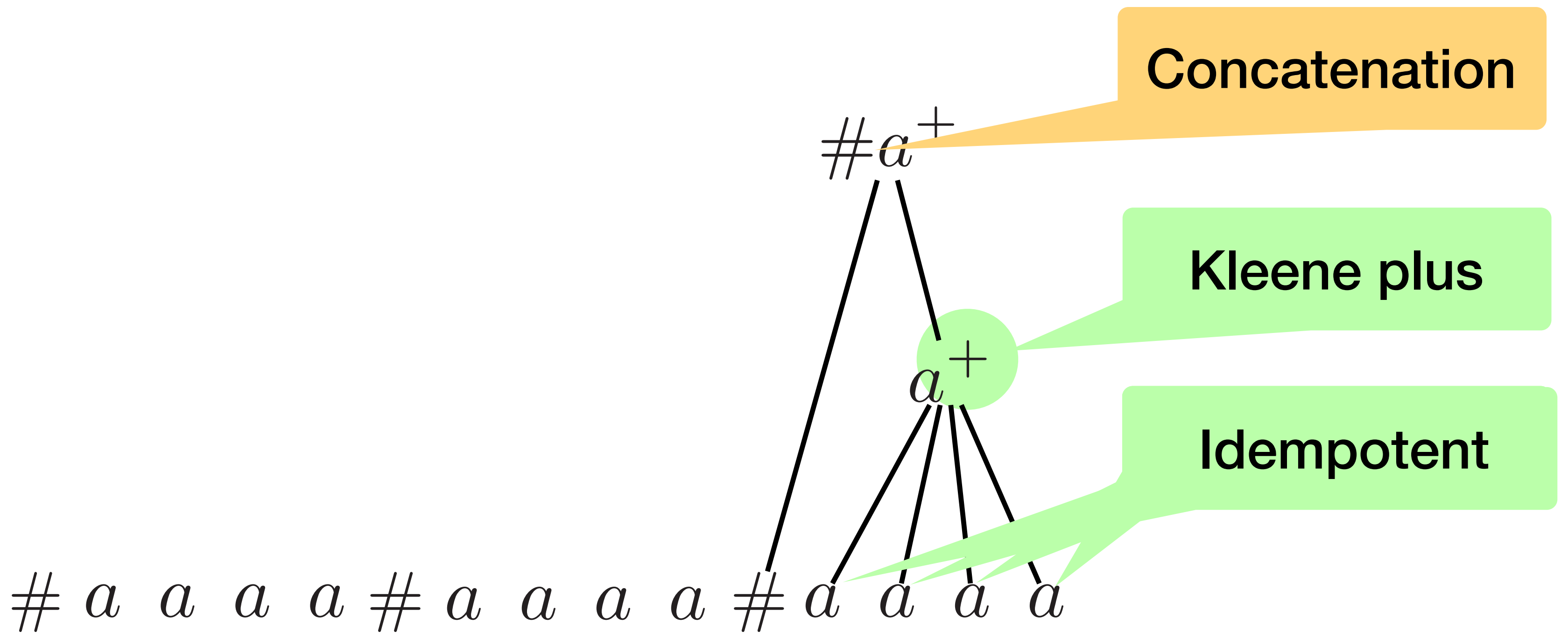

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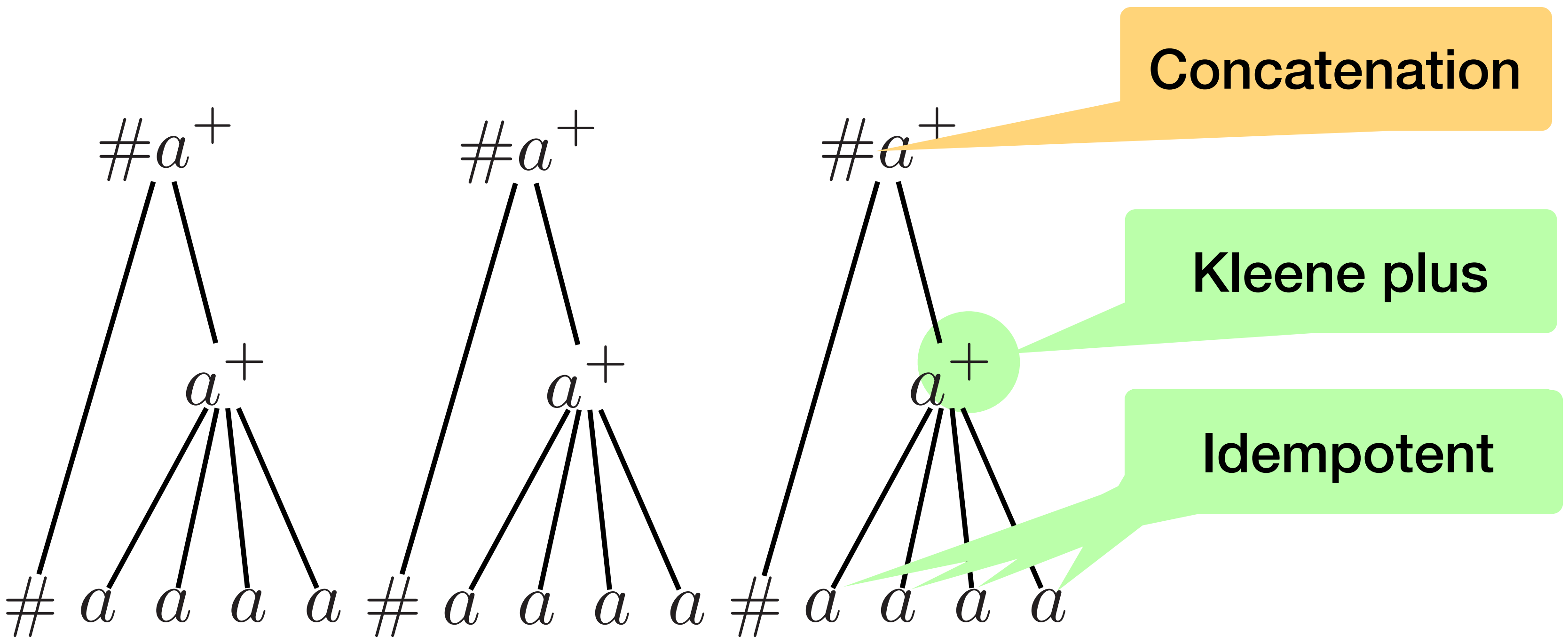

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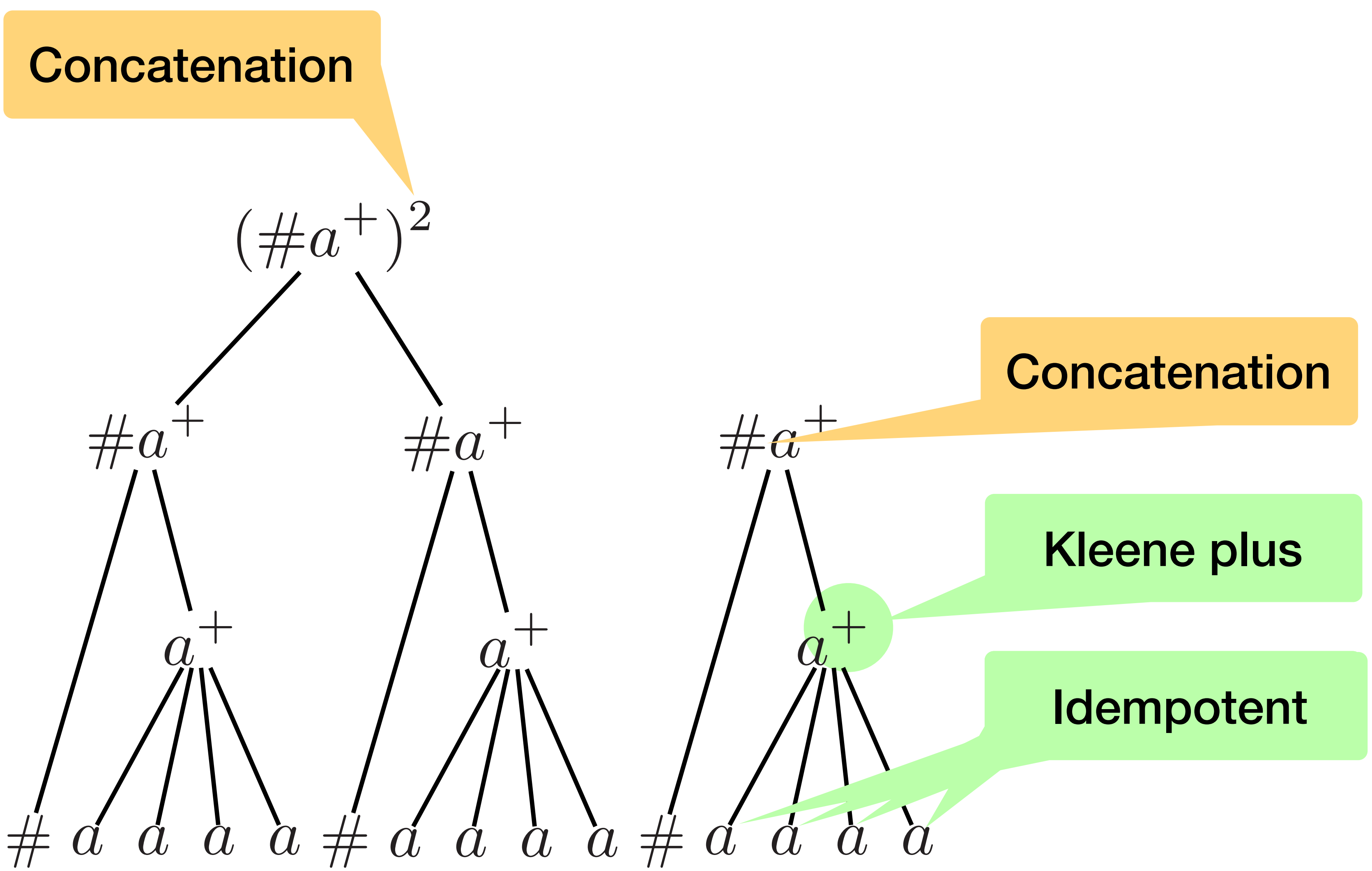

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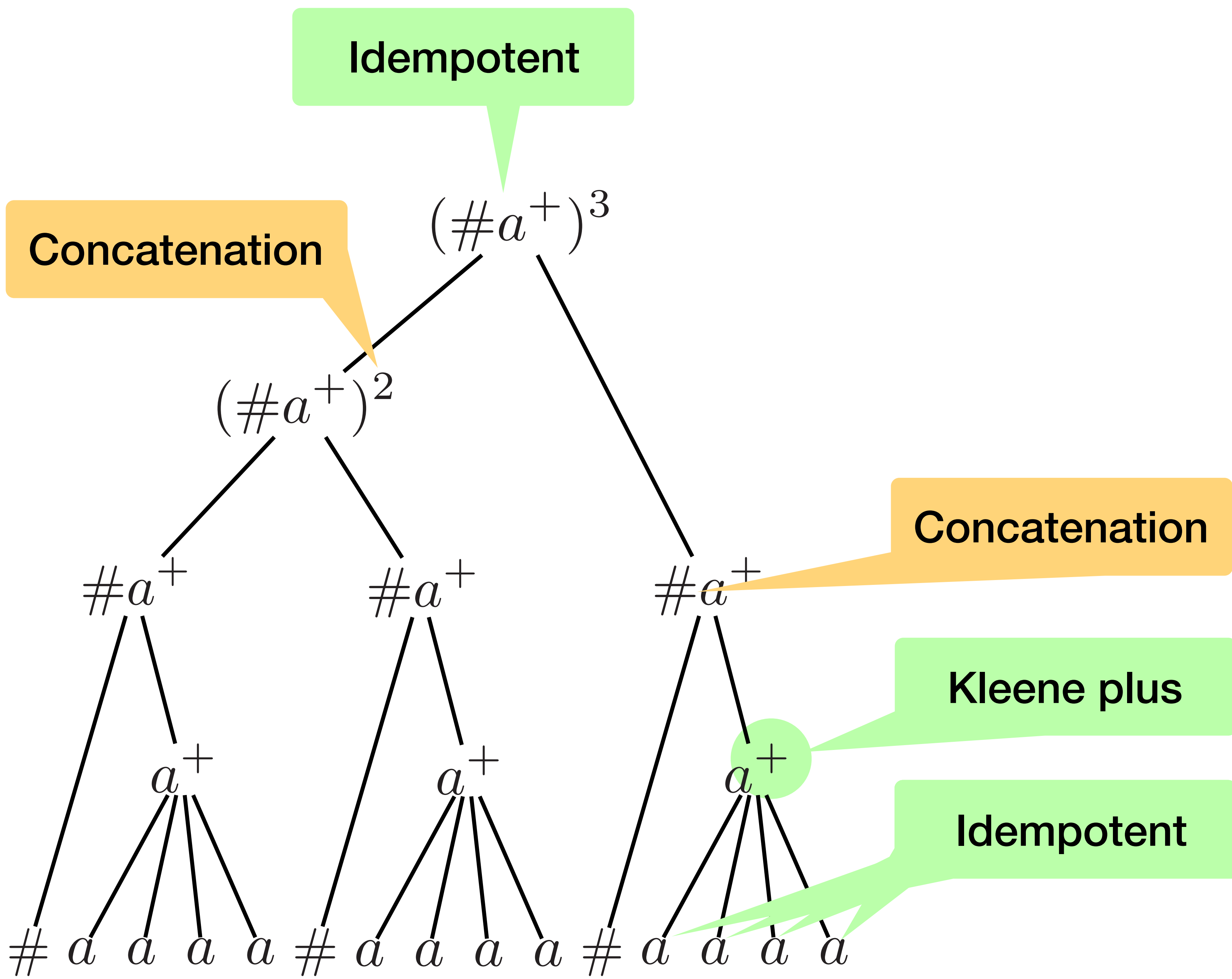

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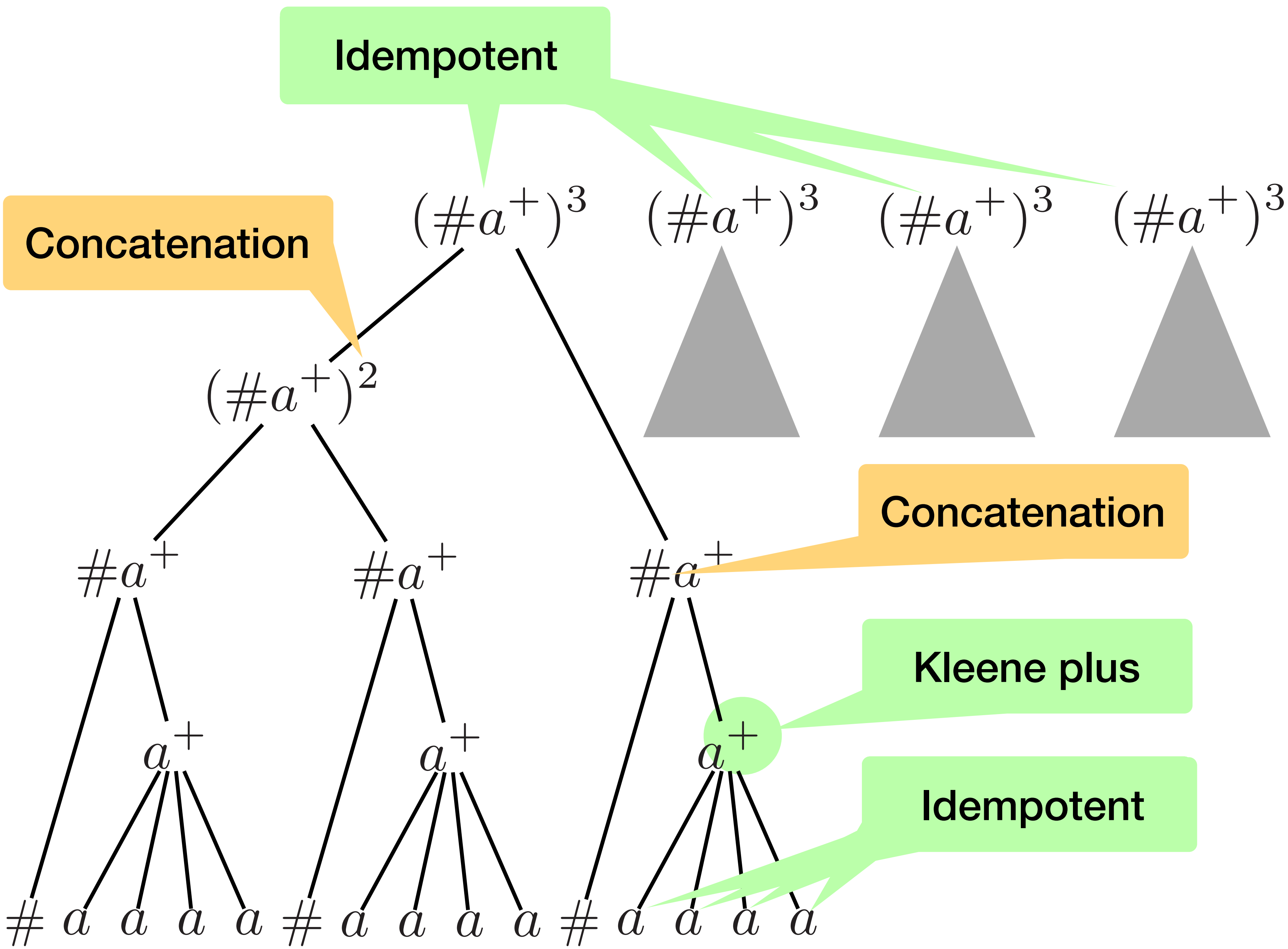

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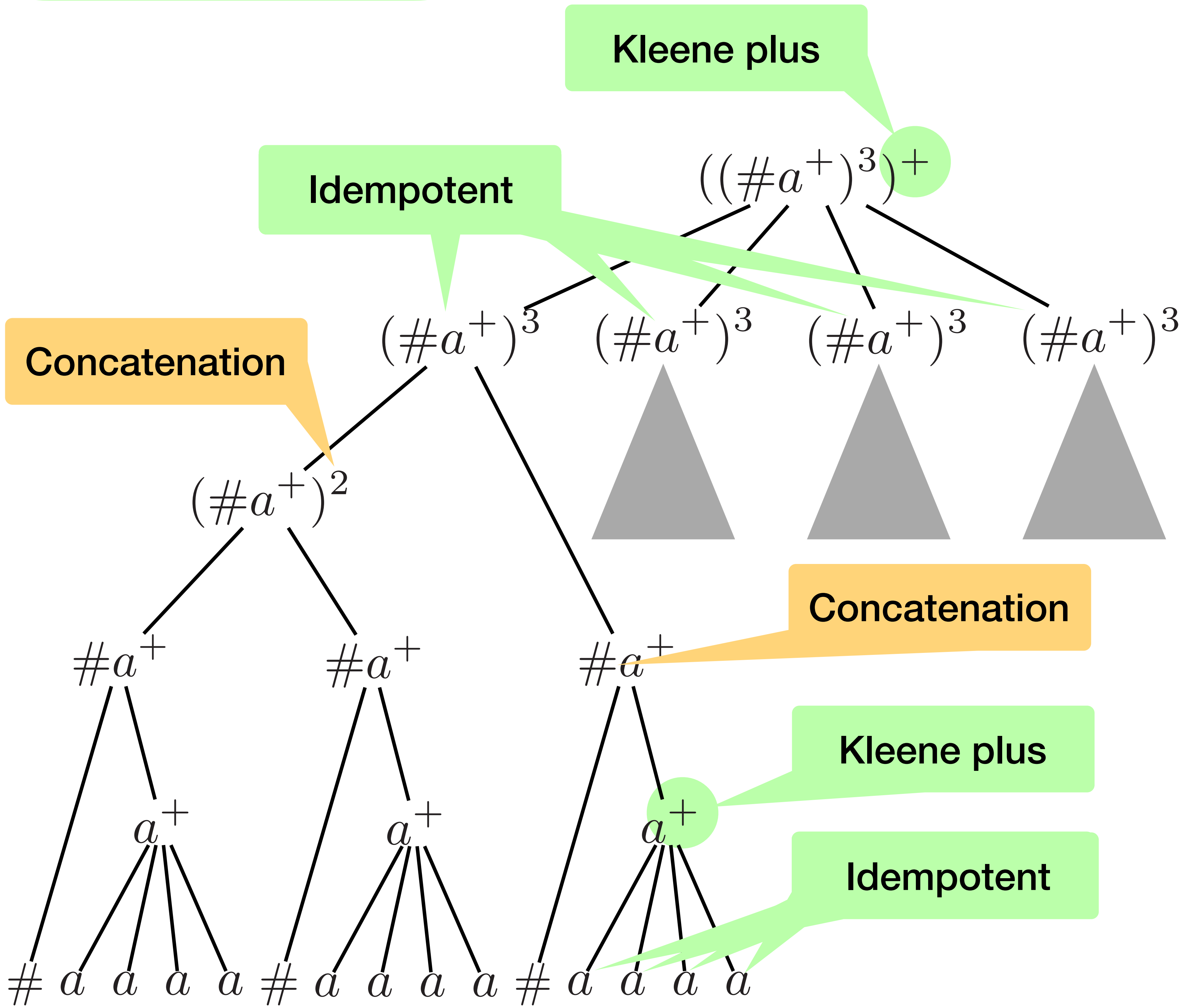

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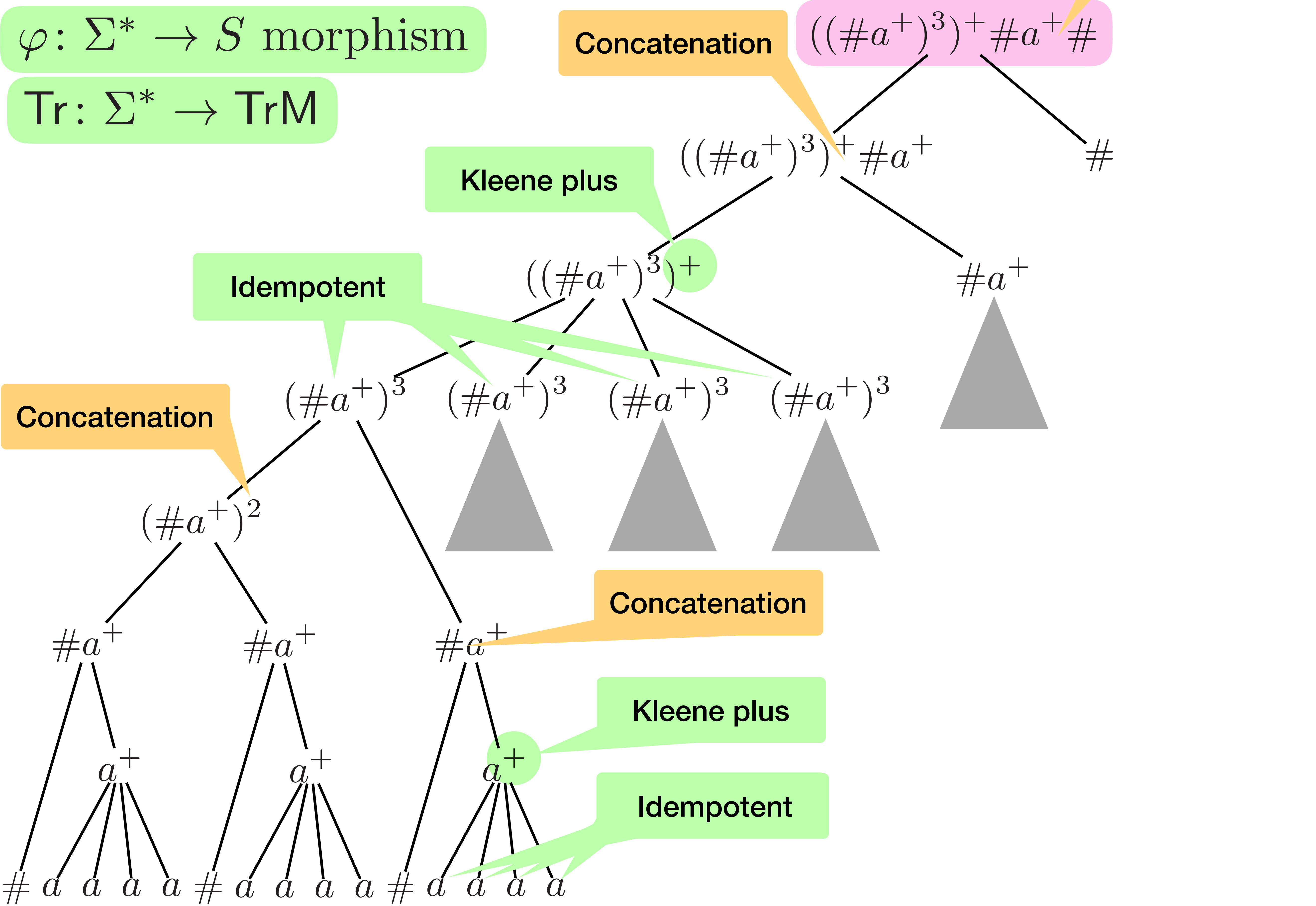

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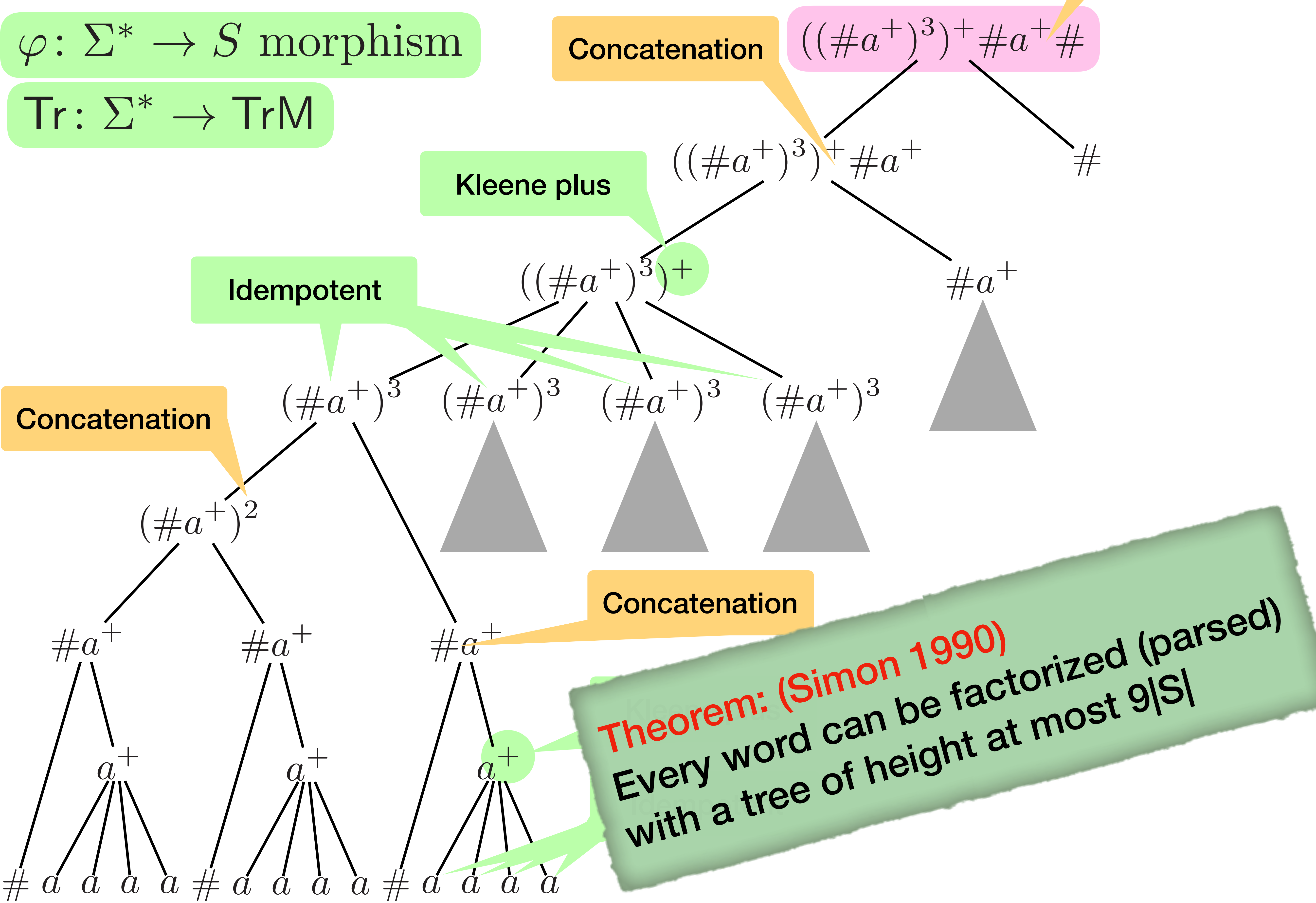


**Theorem: (Simon 1990)**
Every word can be factorized (parsed) with a tree of height at most 9|S|

# Good Rational Expressions

$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

$F$ is *good* wrt. $\varphi\colon \Sigma^* \to S$ morphism to a finite monoid $S$ if

1. $F$ is unambiguous
2. If $E$ is a subexpression of $F$ then $\varphi(\mathcal{L}(E)) = \{s_E\}$ is a singleton
3. If $E^+$ is a subexpression of $F$ then $s_E$ is an idempotent.

**Theorem:** (PG, S.Krishna)

For each $s \in S$, there is an $\varepsilon$-free *good* rational expression $F_s$ such that

$$\mathcal{L}(F_s) = \varphi^{-1}(s) \setminus \{\varepsilon\} \subseteq \Sigma^+$$

Therefore, $G = \varepsilon \cup \bigcup_{s \in S} F_s$ is an *unambiguous* rational expression over $\Sigma$ such that $\mathcal{L}(G) = \Sigma^*$.

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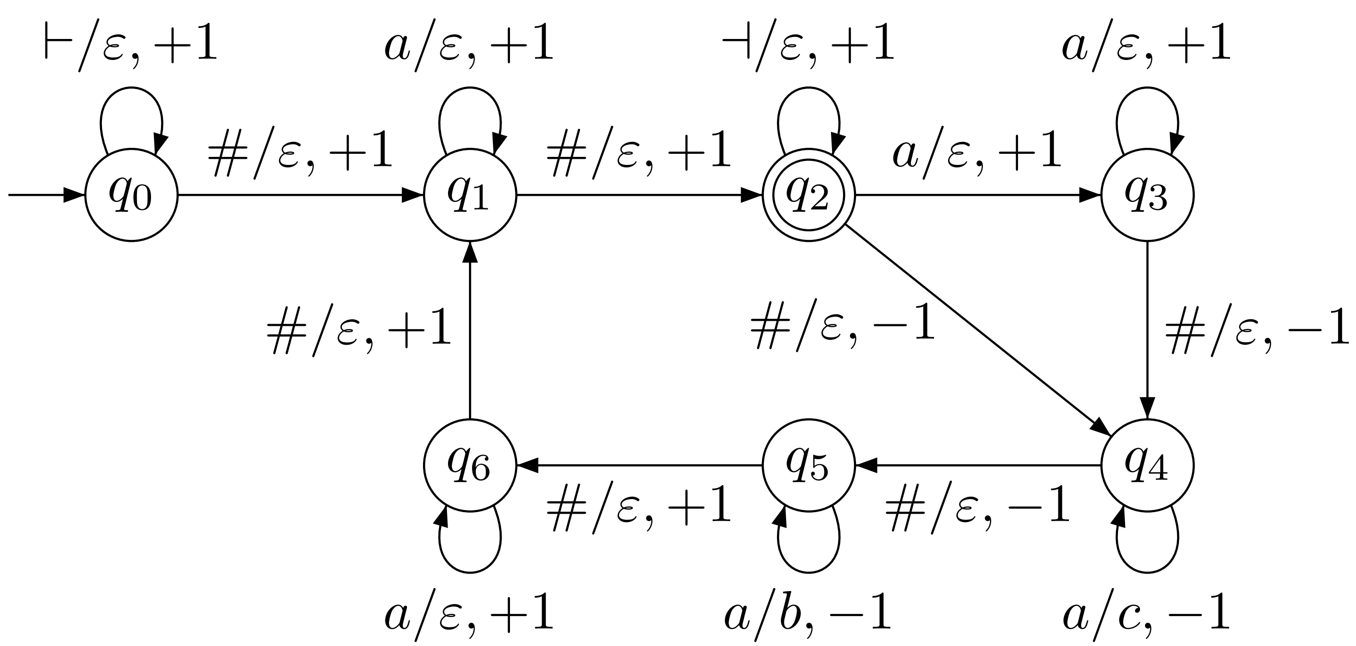


$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

$\mathsf{Tr} \colon \Sigma^* \to \mathsf{TrM}$

$$F_1 = \#a^+\#$$
$$F_2 = \#a^+\#a^+\#$$
$$F_3 = ((\#a^+)^3)^+\#$$
$$F_4 = ((\#a^+)^3)^+\#a^+\#$$
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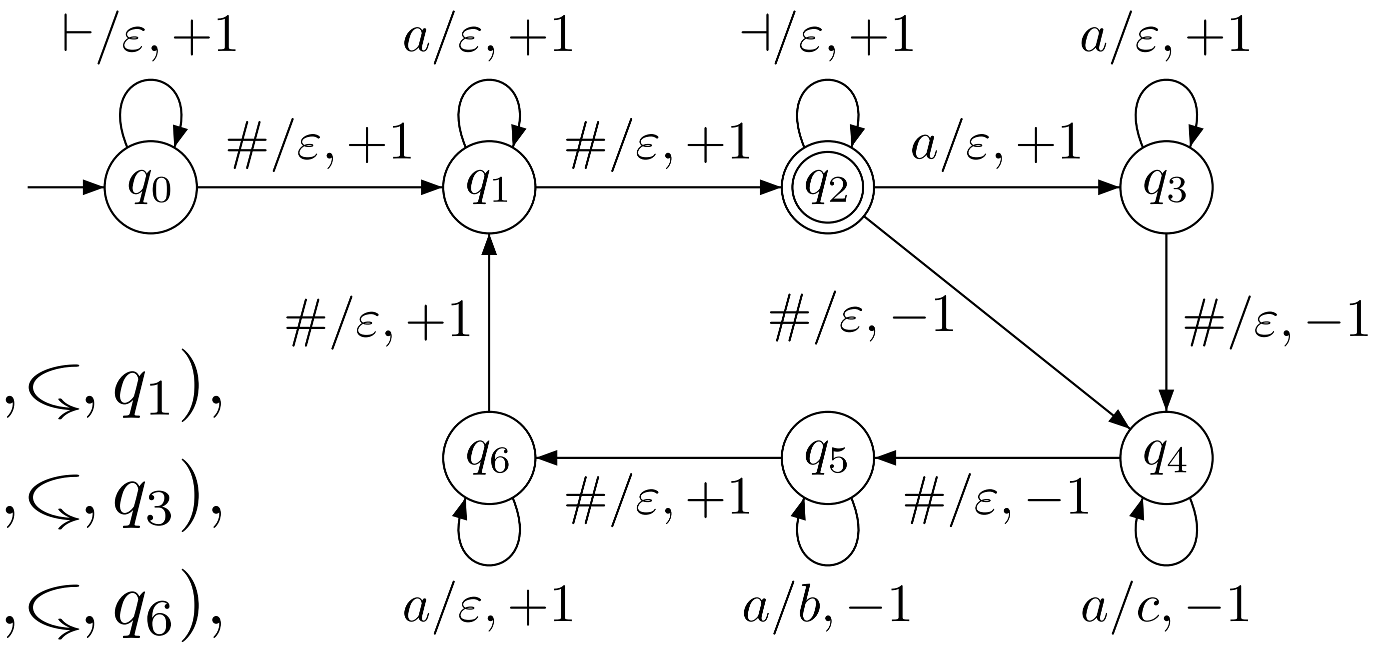
# 2DFA to RTE

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$




$$\mathsf{Tr}(\#a^+\#a^+) = \{(q_0, \rightarrow, q_3), (q_1, \curvearrowleft, q_5), (q_1, \curvearrowright, q_1), (q_2, \curvearrowleft, q_4), (q_2, \curvearrowright, q_3), (q_3, \curvearrowleft, q_4), (q_3, \curvearrowright, q_3), (q_4, \curvearrowleft, q_5), (q_4, \curvearrowright, q_1), (q_5, \rightarrow, q_1), (q_5, \curvearrowright, q_6), (q_6, \rightarrow, q_3), (q_6, \curvearrowright, q_6)\}$$

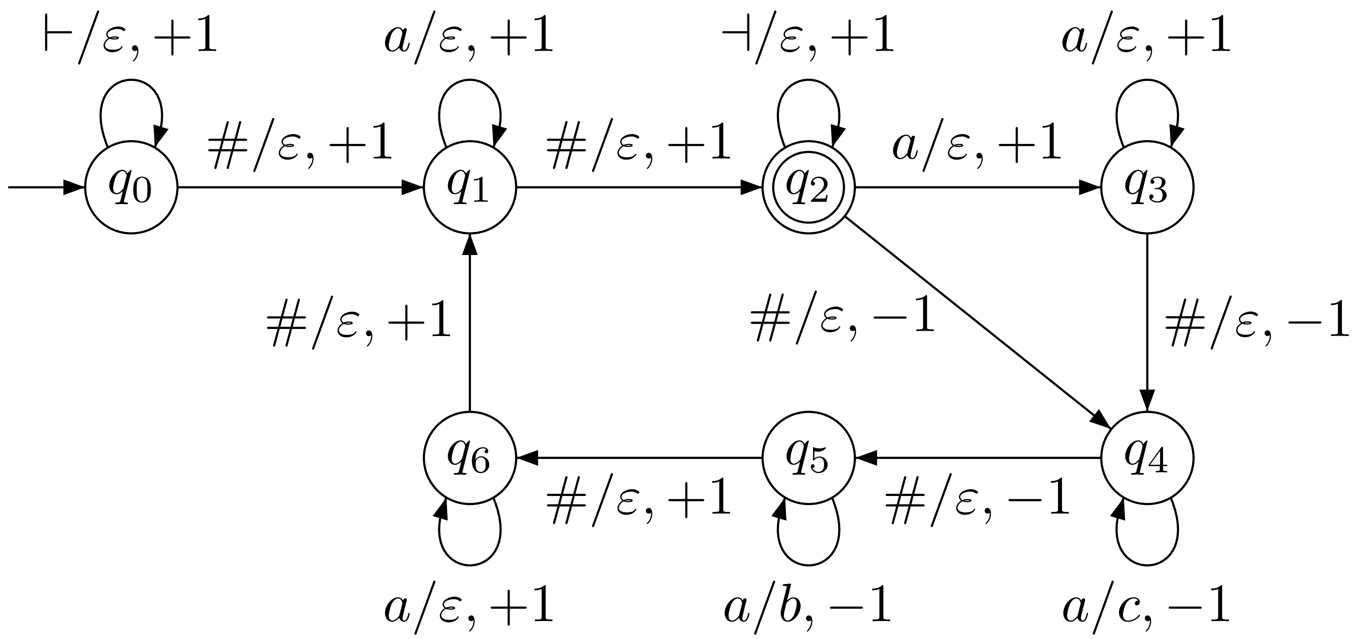
**Main Lemma:** $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

Let $F$ be an $\varepsilon$-free $\mathsf{Tr}$-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F\colon s_F \to \mathsf{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step $x$ on $u$ (i.e., running $\mathcal{A}$ on $u$ from $p$ to $q$ following direction $d$).

# 2DFA to RTE: atomic

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$




$$\mathsf{Tr}(a) = \{(q_1, \rightarrow, q_1), \ldots, (q_4, \leftarrow, q_4), \ldots, (q_5, \leftarrow, q_5), \ldots\}$$

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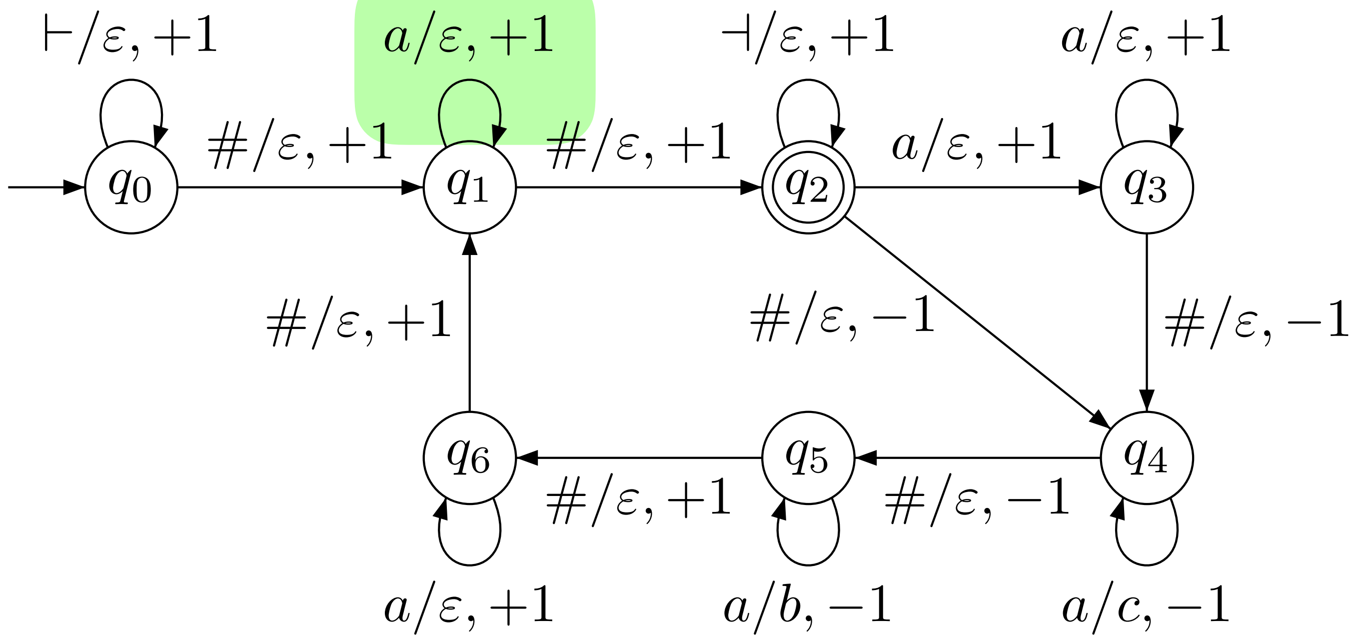
# 2DFA to RTE: atomic

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$

$C_a(q_1, \rightarrow, q_1) = (a\,?\,\varepsilon : \bot)$

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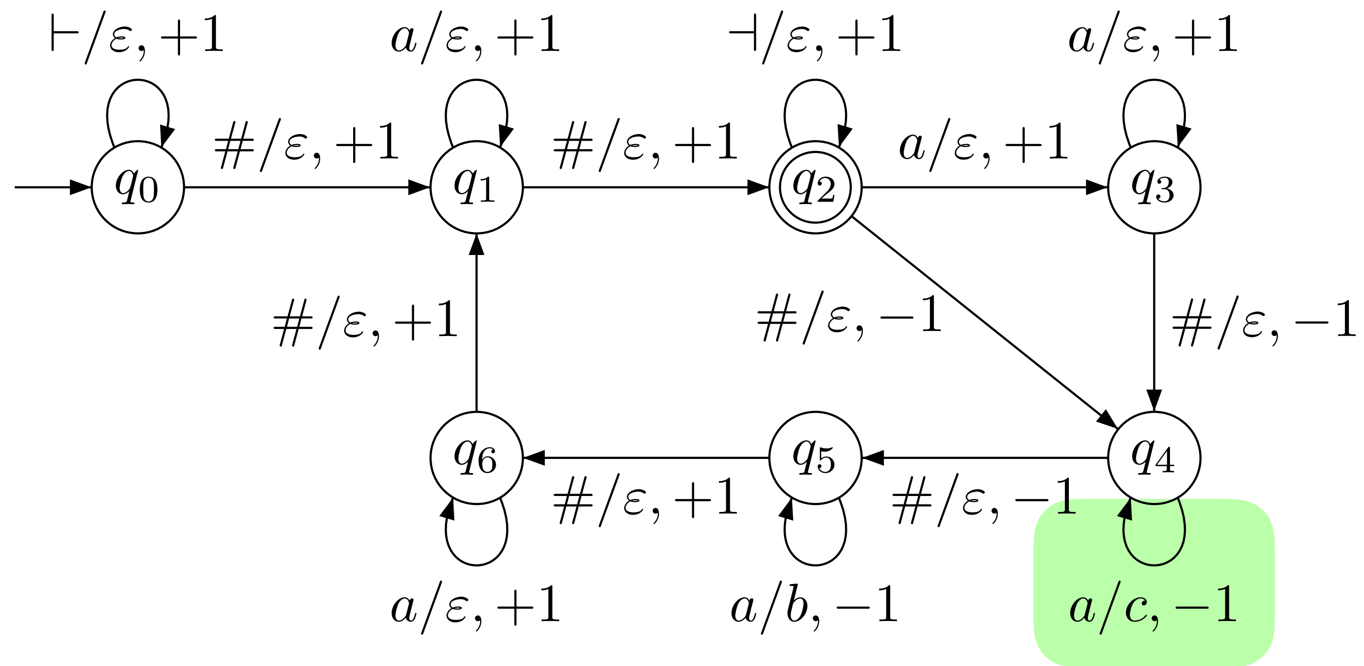
# 2DFA to RTE: atomic

$$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$$

$$C_a(q_1, \rightarrow, q_1) = (a\,?\,\varepsilon : \bot)$$

$$C_a(q_4, \leftarrow, q_4) = (a\,?\,c : \bot)$$




$$\mathsf{Tr}(a) = \{(q_1, \rightarrow, q_1), \ldots, (q_4, \leftarrow, q_4), \ldots, (q_5, \leftarrow, q_5), \ldots\}$$

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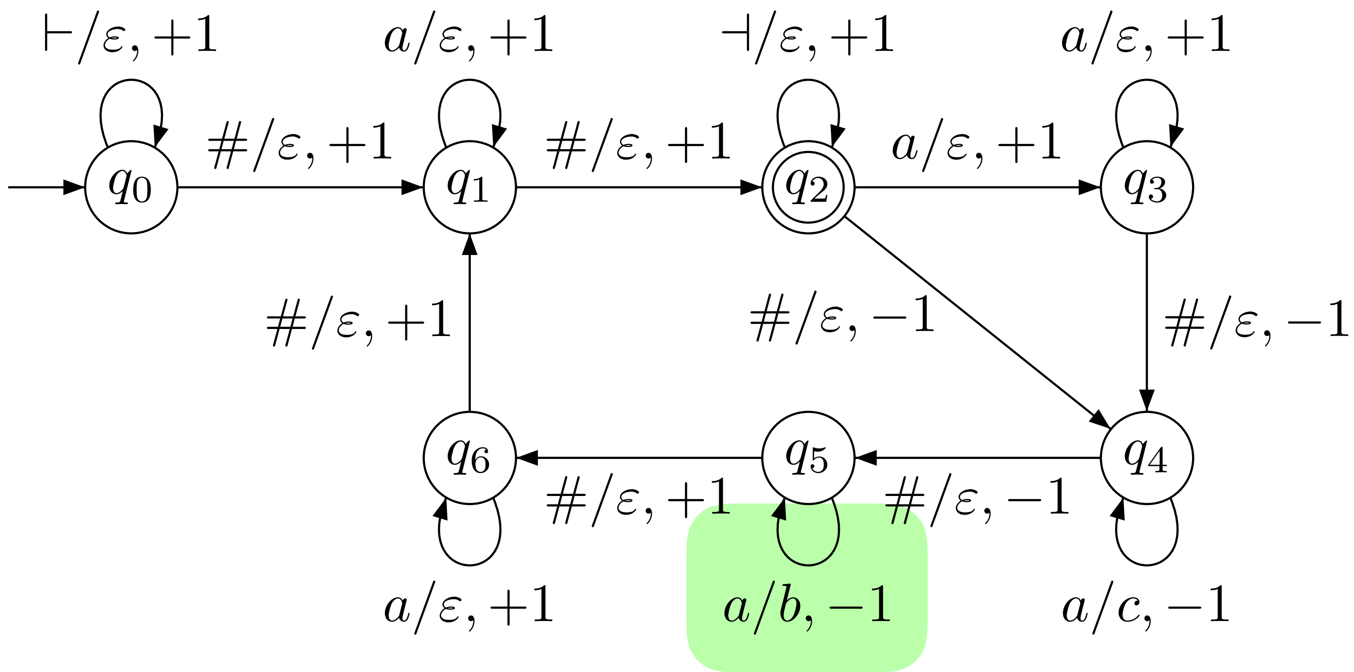
# 2DFA to RTE: atomic

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$

$C_a(q_1, \rightarrow, q_1) = (a\,?\,\varepsilon : \bot)$

$C_a(q_4, \leftarrow, q_4) = (a\,?\,c : \bot)$

$C_a(q_5, \leftarrow, q_5) = (a\,?\,b : \bot)$

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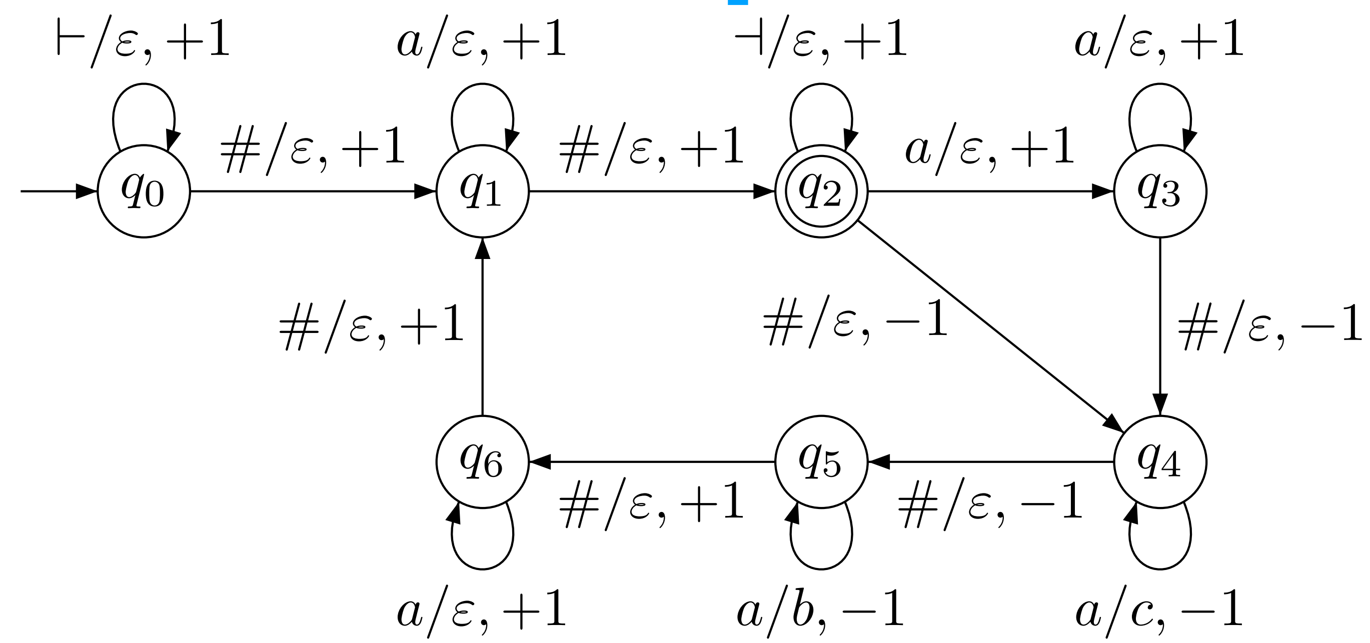
# 2DFA to RTE: Kleene-plus

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$




$$\mathsf{Tr}(a) = \{(q_1, \rightarrow, q_1), \ldots, (q_4, \leftarrow, q_4), \ldots, (q_5, \leftarrow, q_5), \ldots\} = \mathsf{Tr}(a^+)$$

**Main Lemma:** $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

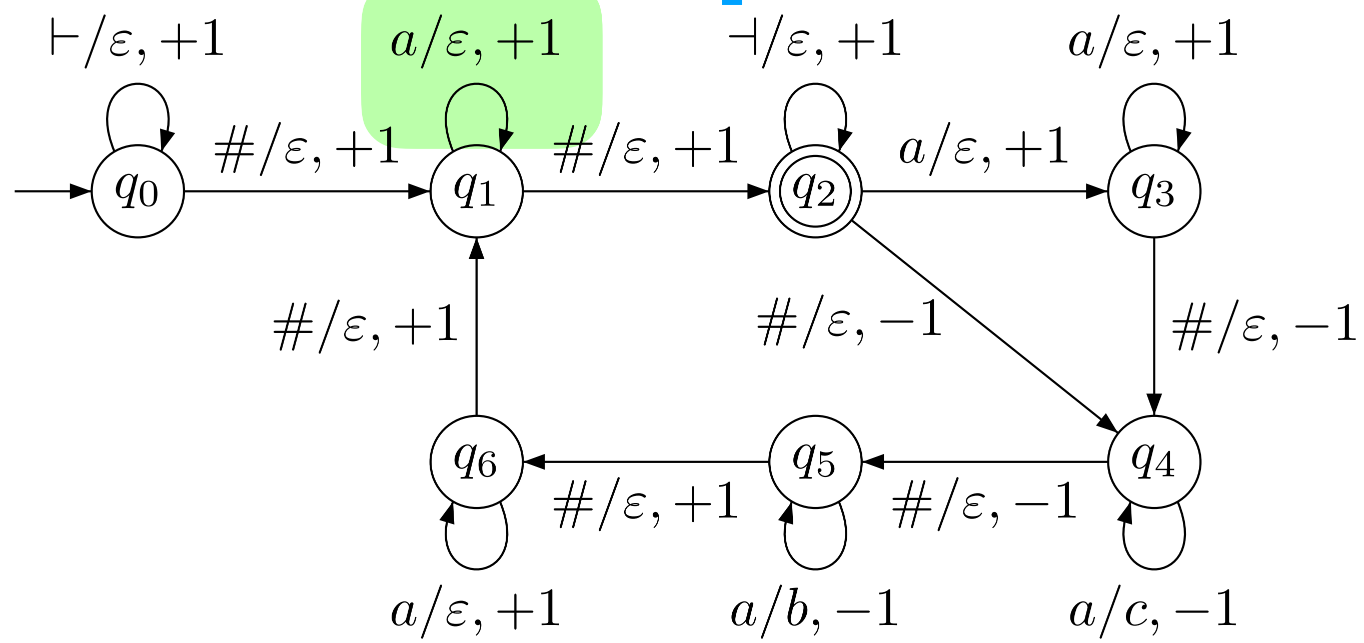
Let $F$ be an $\varepsilon$-free $\mathsf{Tr}$-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F\colon s_F \to \mathsf{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step $x$ on $u$ (i.e., running $\mathcal{A}$ on $u$ from $p$ to $q$ following direction $d$).

# 2DFA to RTE: Kleene-plus

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$

$C_{a^+}(q_1, \rightarrow, q_1) = (a\,?\,\varepsilon : \bot)^{\boxplus}$




$\mathsf{Tr}(a) = \{(q_1, \rightarrow, q_1), \ldots, (q_4, \leftarrow, q_4), \ldots, (q_5, \leftarrow, q_5), \ldots\} = \mathsf{Tr}(a^+)$

**Main Lemma:** $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

Let $F$ be an $\varepsilon$-free $\mathsf{Tr}$-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F\colon s_F \to \mathsf{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

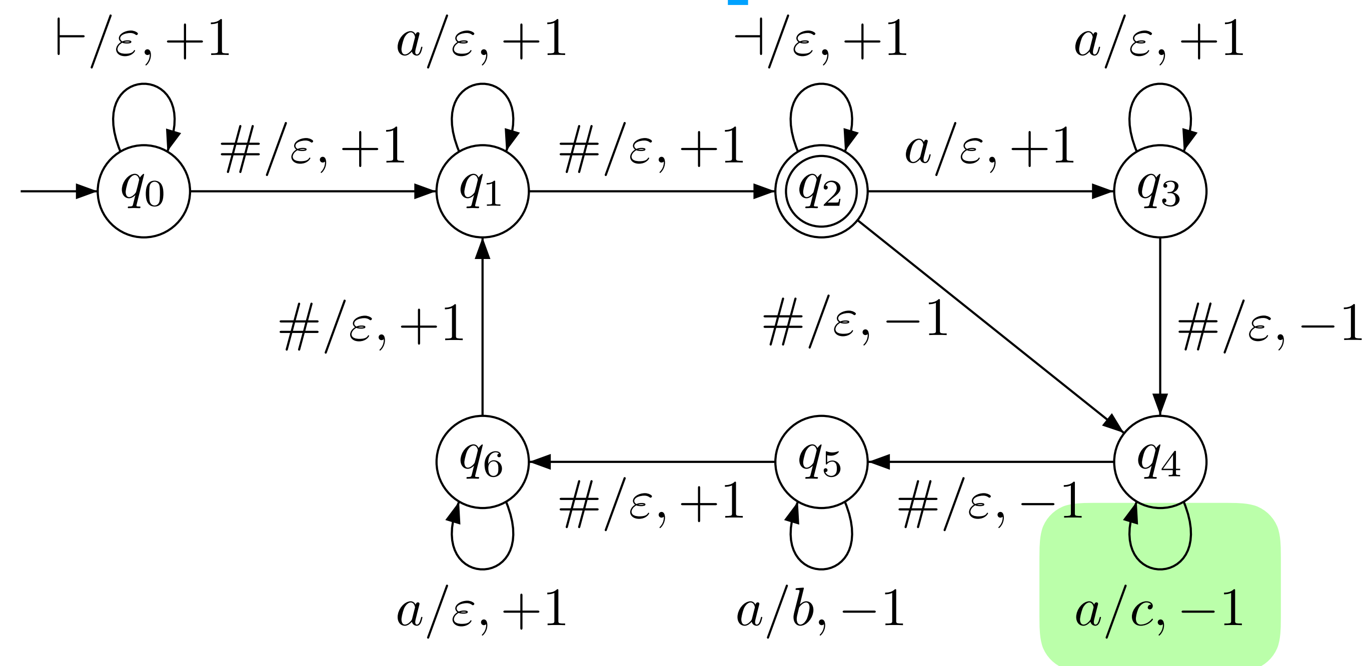
1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step $x$ on $u$ (i.e., running $\mathcal{A}$ on $u$ from $p$ to $q$ following direction $d$).

# 2DFA to RTE: Kleene-plus

$$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$$

$$C_{a^+}(q_1, \rightarrow, q_1) = (a \,?\, \varepsilon : \bot)^{\boxplus}$$

$$C_{a^+}(q_4, \leftarrow, q_4) = (a \,?\, c : \bot)^{\overleftarrow{\boxplus}}$$



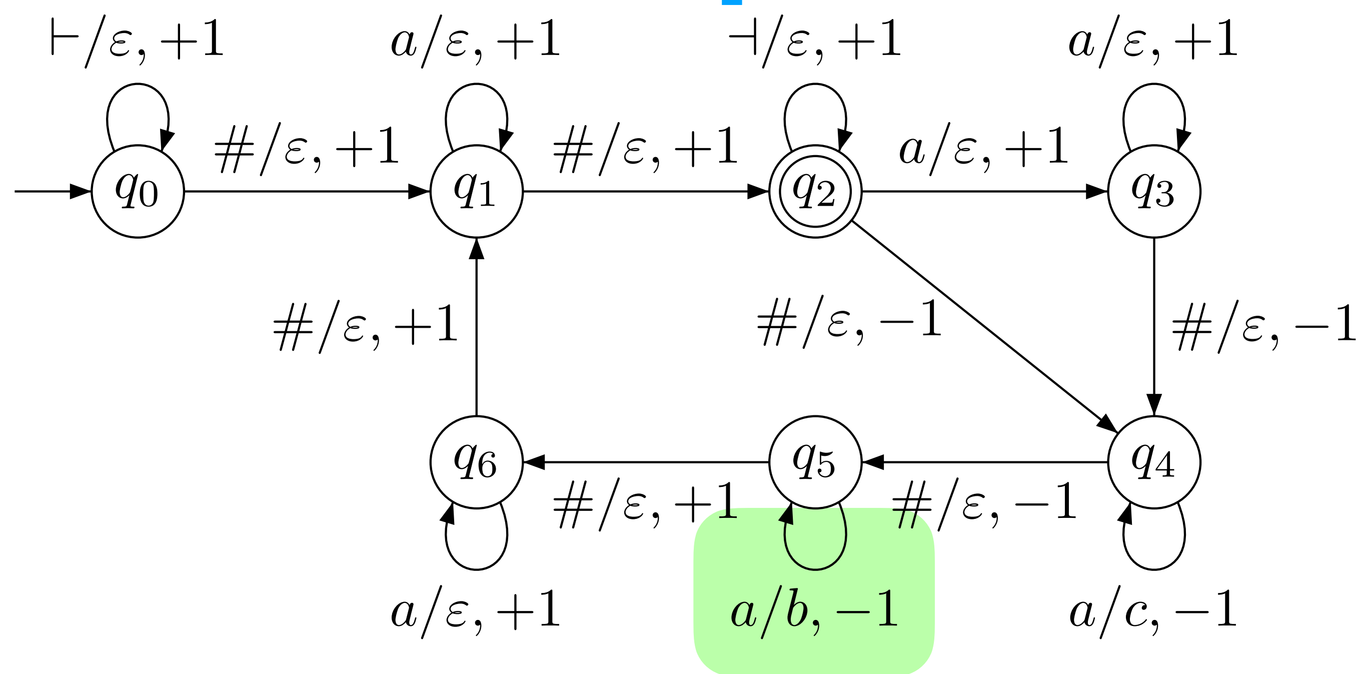

$$\mathsf{Tr}(a) = \{(q_1, \rightarrow, q_1), \ldots, (q_4, \leftarrow, q_4), \ldots, (q_5, \leftarrow, q_5), \ldots\} = \mathsf{Tr}(a^+)$$

**Main Lemma:** $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

Let $F$ be an $\varepsilon$-free $\mathsf{Tr}$-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F\colon s_F \to \mathsf{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step $x$ on $u$ (i.e., running $\mathcal{A}$ on $u$ from $p$ to $q$ following direction $d$).

# 2DFA to RTE: Kleene-plus




$$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$$

$$C_{a^+}(q_1, \rightarrow, q_1) = (a\,?\,\varepsilon : \bot)^{\boxplus}$$

$$C_{a^+}(q_4, \leftarrow, q_4) = (a\,?\,c : \bot)^{\overleftarrow{\boxplus}}$$

$$C_{a^+}(q_5, \leftarrow, q_5) = (a\,?\,b : \bot)^{\overleftarrow{\boxplus}}$$

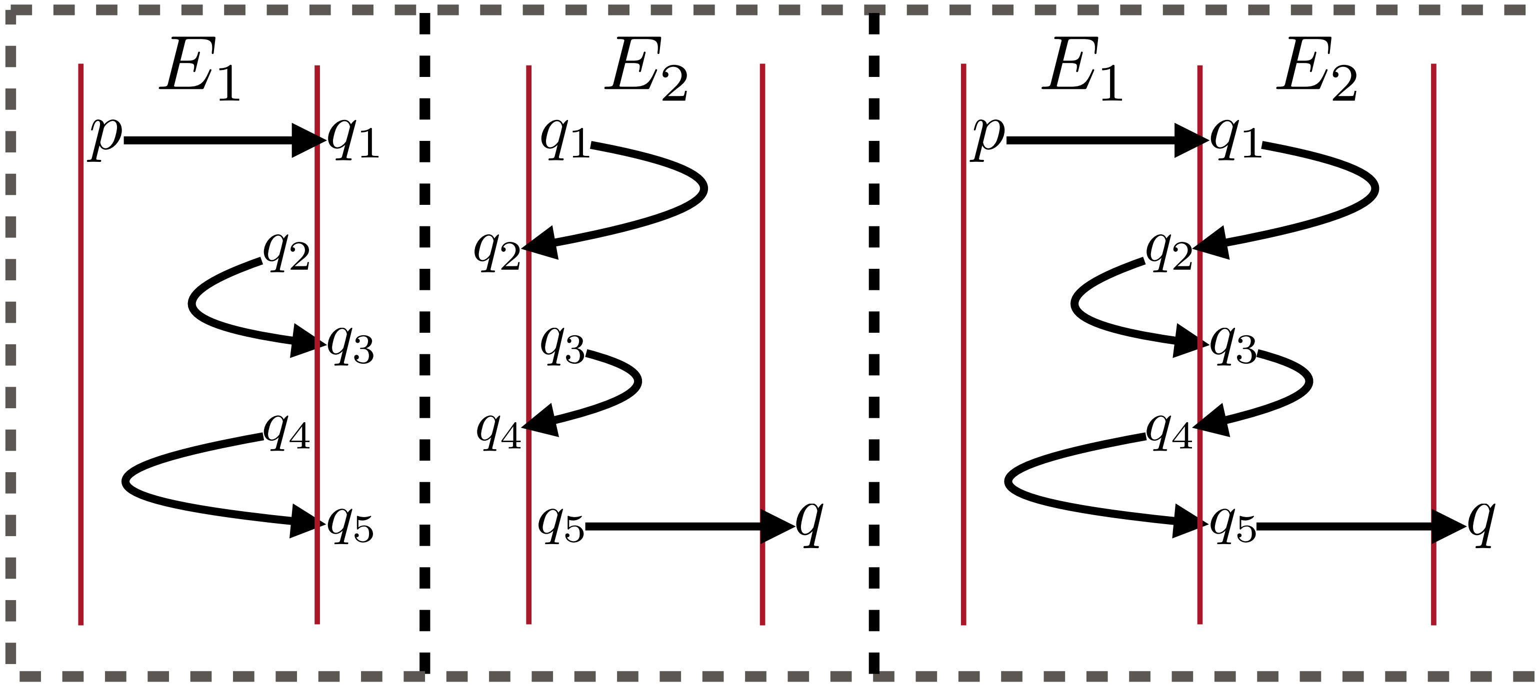
$$\mathsf{Tr}(a) = \{(q_1, \rightarrow, q_1), \ldots, (q_4, \leftarrow, q_4), \ldots, (q_5, \leftarrow, q_5), \ldots\} = \mathsf{Tr}(a^+)$$

**Main Lemma:** $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

Let $F$ be an $\varepsilon$-free $\mathsf{Tr}$-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F\colon s_F \to \mathsf{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step $x$ on $u$ (i.e., running $\mathcal{A}$ on $u$ from $p$ to $q$ following direction $d$).

# 2DFA to RTE: concatenation




$$
\begin{aligned}
x &= (p, \rightarrow, q)\\
x_1 &= (p, \rightarrow, q_1)\\
x_2 &= (q_1, \rightleftarrows, q_2)\\
x_3 &= (q_2, \leftrightarrows, q_3)\\
x_4 &= (q_3, \rightleftarrows, q_4)\\
x_5 &= (q_4, \leftrightarrows, q_5)\\
x_6 &= (q_5, \rightarrow, q)
\end{aligned}
$$

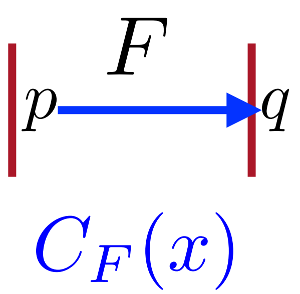
$$C_{E_1 \cdot E_2}(x) = (C_{E_1}(x_1) \boxdot C_{E_2}(x_2)) \odot (C_{E_1}(x_3) \boxdot C_{E_2}(x_4)) \odot (C_{E_1}(x_5) \boxdot C_{E_2}(x_6))$$

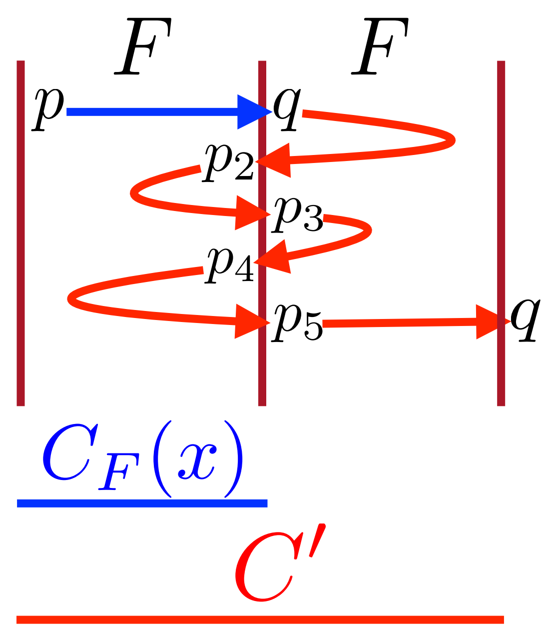
**Main Lemma:** $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

Let $F$ be an $\varepsilon$-free $\mathsf{Tr}$-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F \colon s_F \to \mathsf{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step $x$ on $u$ (i.e., running $\mathcal{A}$ on $u$ from $p$ to $q$ following direction $d$).

# 2DFA to RTE: Kleene-plus




$$x = (p, \rightarrow, q)$$
$$x_2 = (q, \curvearrowleft, p_2)$$
$$x_3 = (p_2, \curvearrowright, p_3)$$
$$x_4 = (p_3, \curvearrowleft, p_4)$$
$$x_5 = (p_4, \curvearrowright, p_5)$$
$$x_6 = (p_5, \rightarrow, q)$$

$$C_{F^+}(x) = \big(C_F(x) \boxdot (F^* \,?\, \varepsilon : \bot)\big) \odot [F, C']^{2\boxplus}$$

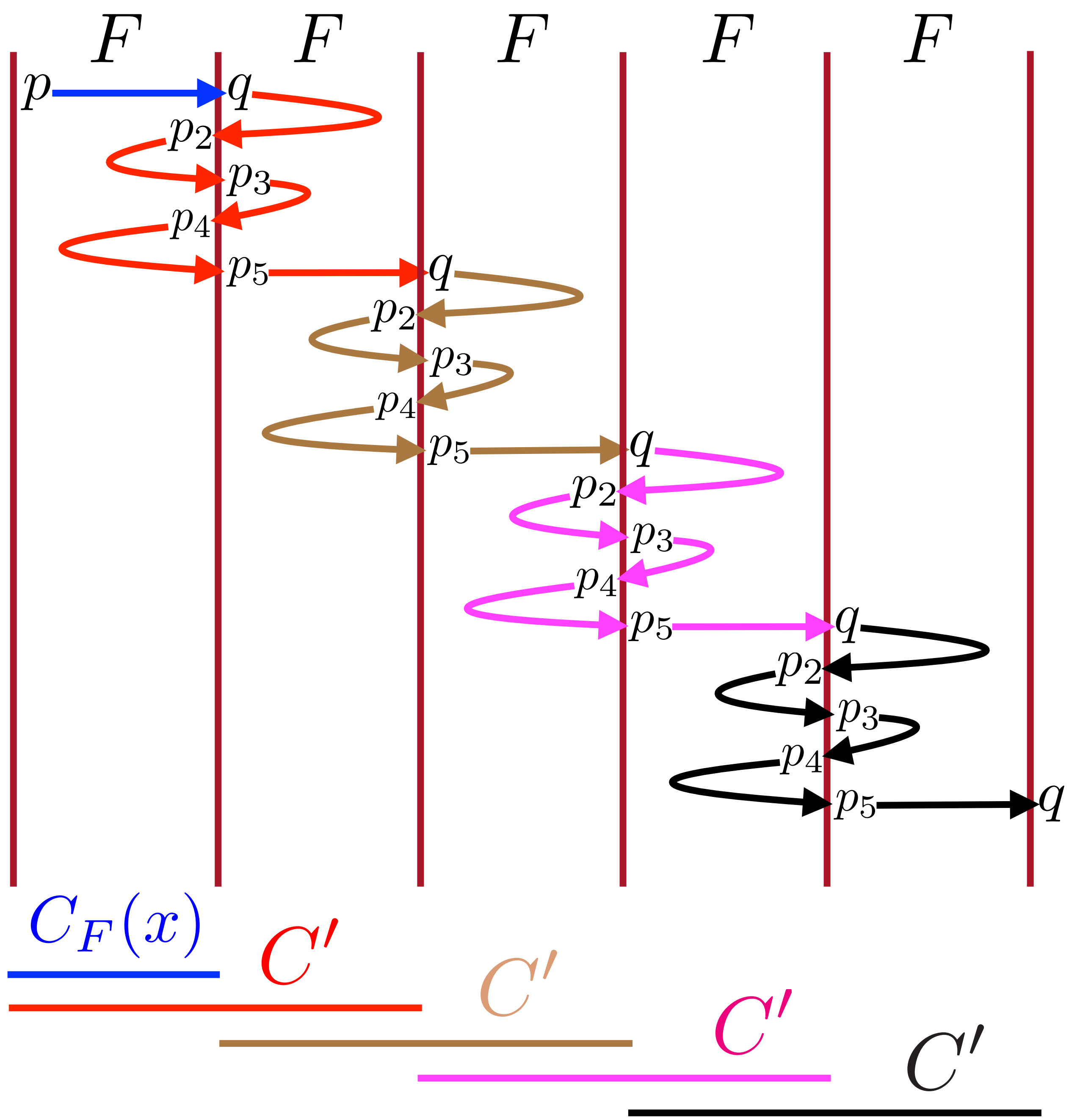
$$C' = \big((F \,?\, \varepsilon : \bot) \boxdot C_F(x_2)\big) \odot \big(C_F(x_3) \boxdot C_F(x_4)\big) \odot \big(C_F(x_5) \boxdot C_F(x_6)\big)$$

**Main Lemma:** $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

Let $F$ be an $\varepsilon$-free Tr-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F \colon s_F \to$ RTE such that for each step $x = (p, d, q) \in s_F$:

1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step $x$ on $u$ (i.e., running $\mathcal{A}$ on $u$ from $p$ to $q$ following direction $d$).

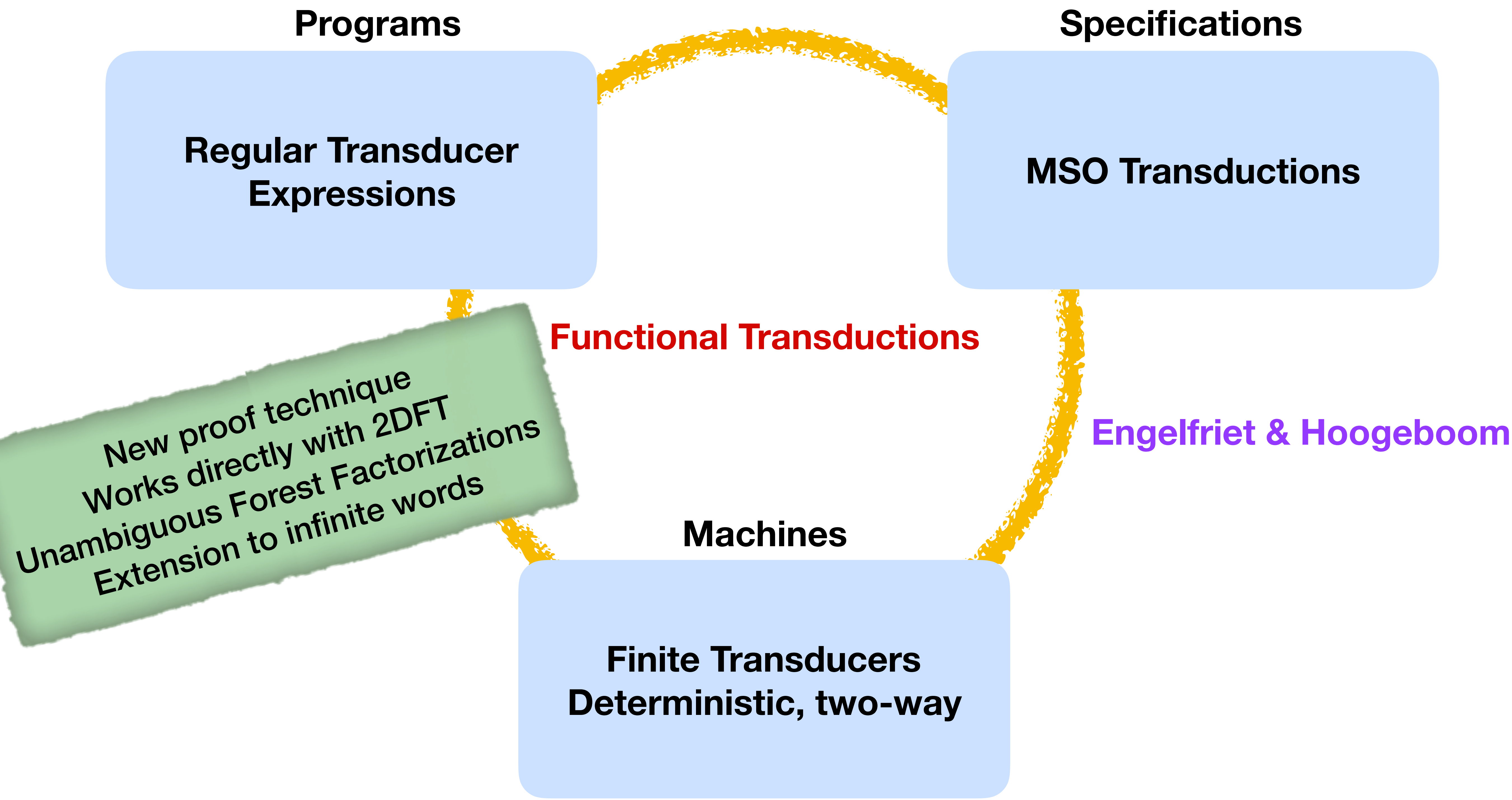
# 2DFA to RTE: Kleene-plus




$$x = (p, \rightarrow, q)$$
$$x_2 = (q, \curvearrowleft, p_2)$$
$$x_3 = (p_2, \curvearrowright, p_3)$$
$$x_4 = (p_3, \curvearrowleft, p_4)$$
$$x_5 = (p_4, \curvearrowright, p_5)$$
$$x_6 = (p_5, \rightarrow, q)$$

$$C_{F^+}(x) = \big(C_F(x) \boxdot (F^* \,?\, \varepsilon : \bot)\big) \odot [F, C']^{2\boxplus}$$
$$C' = \big((F \,?\, \varepsilon : \bot) \boxdot C_F(x_2)\big) \odot (C_F(x_3) \boxdot C_F(x_4)) \odot (C_F(x_5) \boxdot C_F(x_6))$$

# Conclusion


Programs
Regular Transducer Expressions
Specifications
MSO Transductions
Functional Transductions
Engelfriet & Hoogeboom
Machines
Finite Transducers
Deterministic, two-way
New proof technique
Works directly with 2DFT
Unambiguous Forest Factorizations
Extension to infinite words