

Regular transducer expressions for two-way deterministic transducers

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Joint work with Vrunda Dave and S. Krishna
IIT Bombay



LICS'18
Also on ArXiv

The fundamental Triptych

Programs

Specifications

```
Optimize
class JVC {
    public static void main(String args[]) throws IOException {
        String hostName = "www.javacode.com";
        ServerSocket serverSocket = new ServerSocket(8080);

        while (true) {
            Socket clientSocket = serverSocket.accept();
            BufferedReader in = new BufferedReader(new InputStreamReader(clientSocket.getInputStream()));
            String line = in.readLine();
            while (line != null) {
                System.out.println("Received: " + line);
                line = in.readLine();
            }
        }
    }
}
```

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Machines



The fundamental Triptych

Programs

Structured, Compositional
Blocks, Sequences
Loops, Conditionals

Specifications

Logical: Is n prime?
Functional: encrypt, decrypt
Relational: French \rightarrow English

Machines

Low level
Executable
Often sequential

The fundamental Triptych

Programs

Specifications

Regular Expressions

Monadic Second-order Logic
Linear Dynamic Logic

Sequential Systems

Kleene

Büchi
Elgot
Trakhtenbrot

Machines

Finite Automata
Deterministic
One-way



The fundamental Triptych

Programs

Star-Free Expressions

McNaughton & Papert

Specifications

**First-order Logic
Linear Temporal Logic**

Sequential Systems

Kamp

Machines

**Finite Automata
Deterministic
Counter-free**



The fundamental Triptych

Programs

Specifications

Context-Free Grammars

**Exists matching relation
+ MSO**

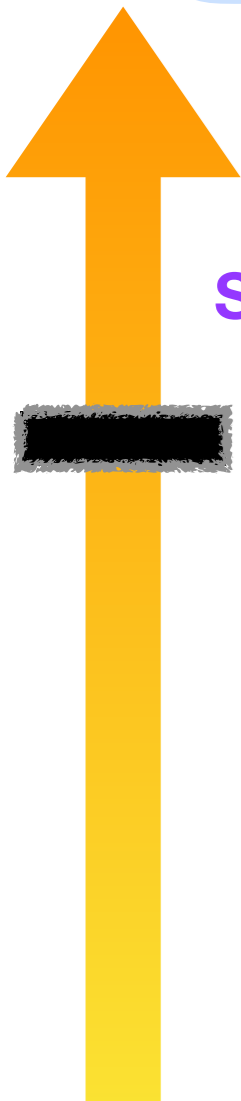
Sequential Systems

**Chomsky,
Schützenberger**

**Lautemann &
Schwentick &
Thérien**

Machines

**Finite Automata
non-deterministic
with a stack**



The fundamental Triptych

Programs

Specifications

**Regular Expression
Connected iteration**

**MSO
Mazurkiewicz Traces**

Distributed Systems

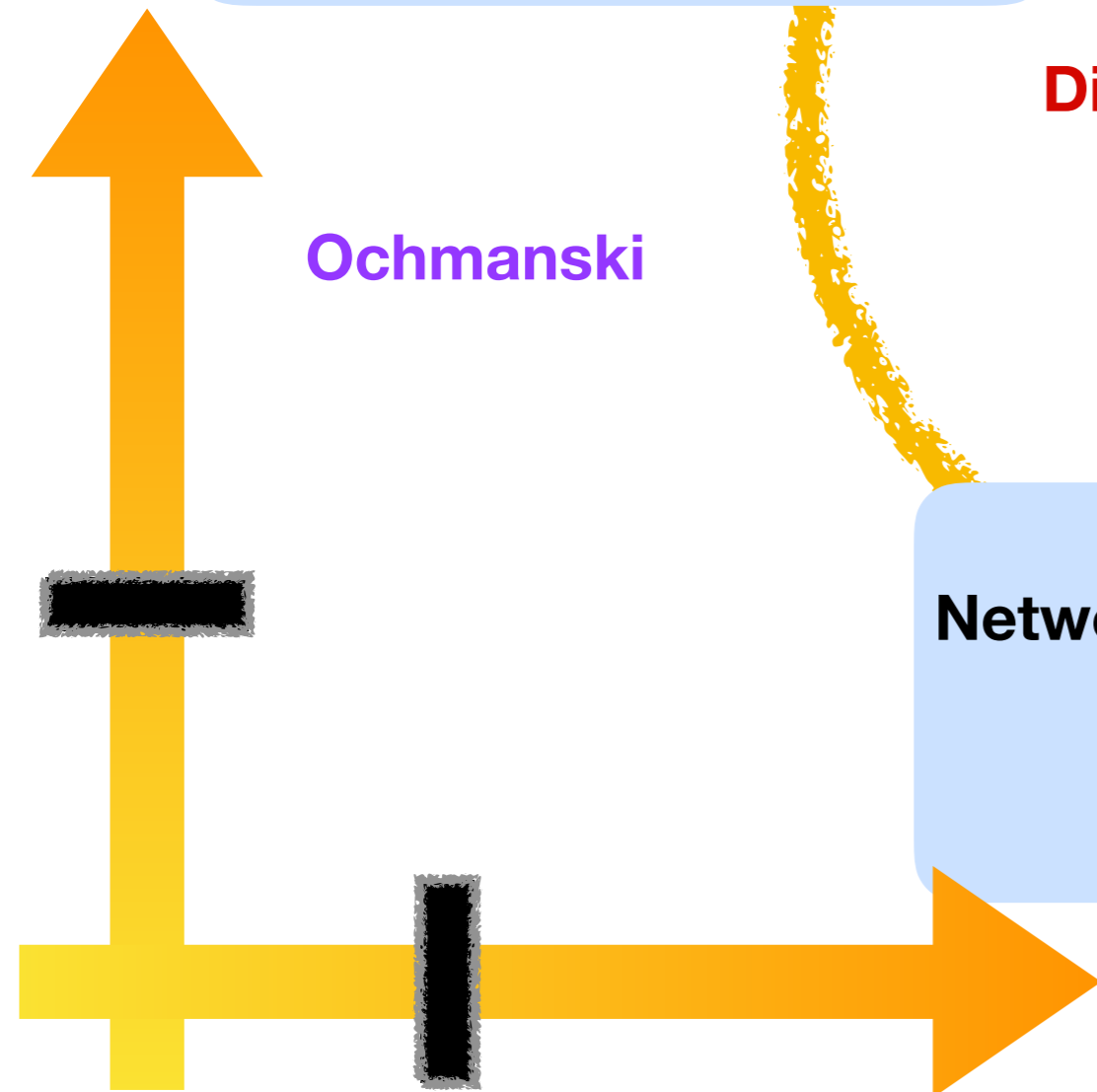
Ochmanski

Thomas

Machines

**Network of Finite Automata
deterministic
asynchronous**

Zielonka



The fundamental Triptych

Programs

Specifications

**Regular Expression
Weighted**

**MSO logic
Weighted**

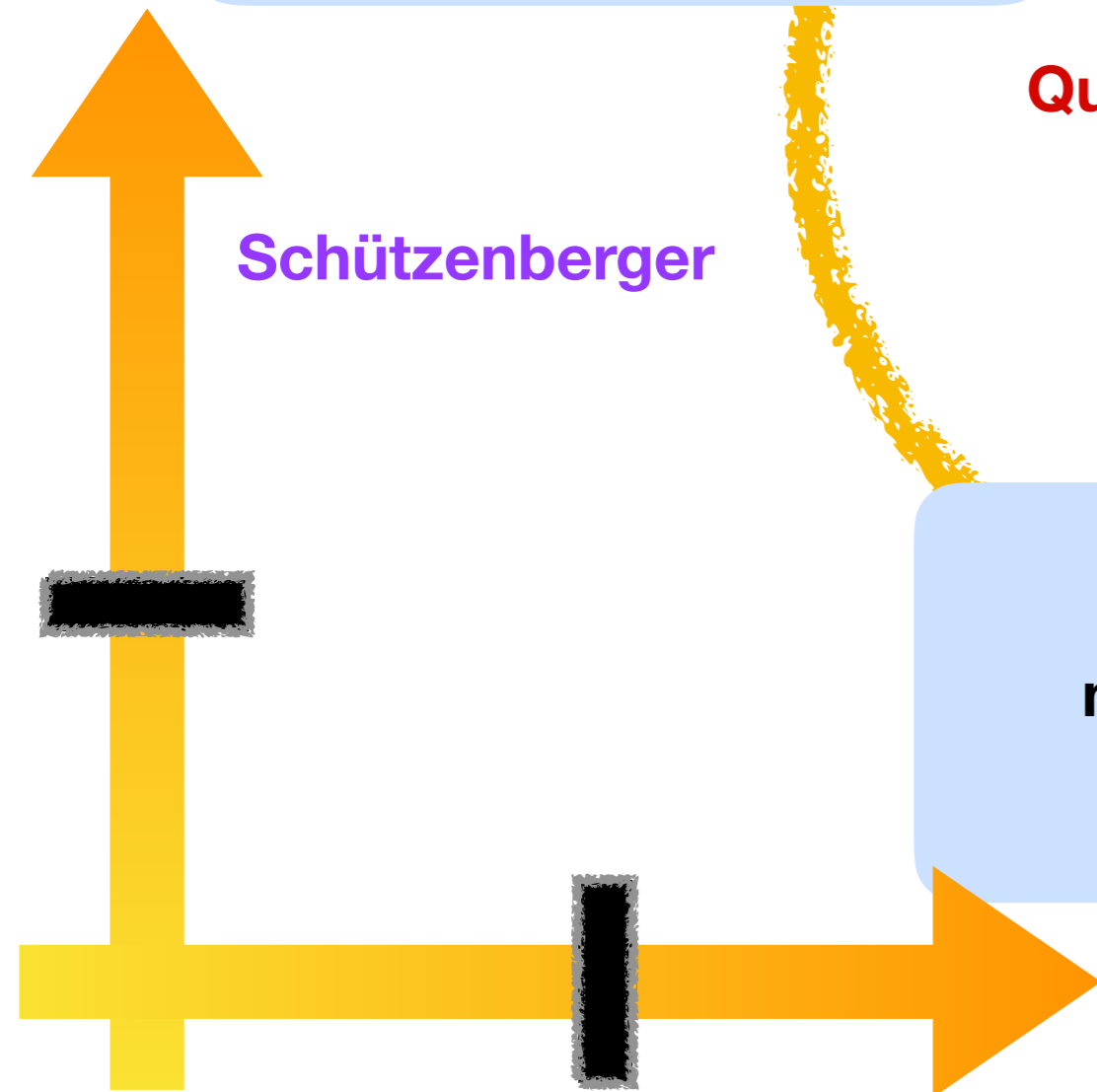
Quantitative Systems

Schützenberger

Droste & Gastin

Machines

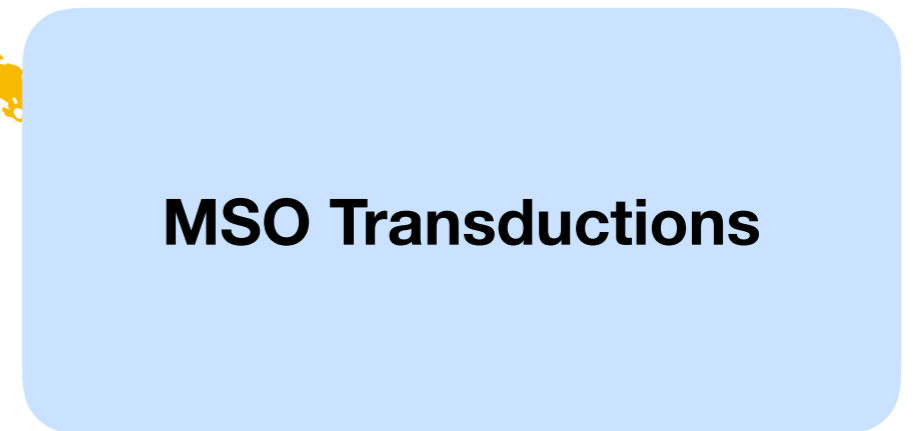
**Finite Automata
non-deterministic
weighted**



The fundamental Triptych

Programs

Specifications

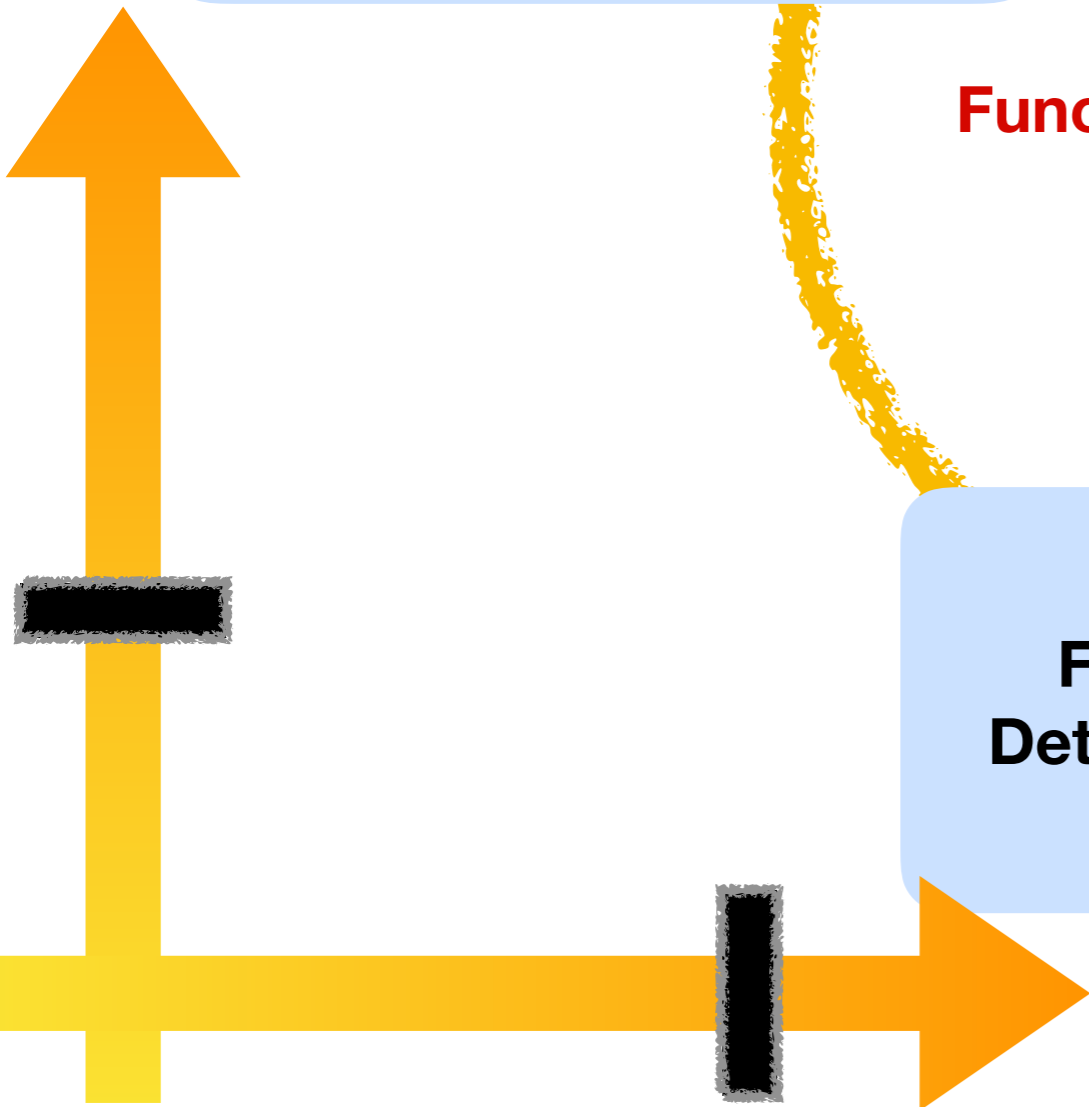


Functional Transductions

Engelfriet & Hoogeboom

Machines

**Finite Transducers
Deterministic, two-way**



The fundamental Triptych

Programs

Specifications

**Combinator Expressions
DReX**

MSO Transductions

Functional Transductions

**Alur &
Freilich &
Raghothaman**

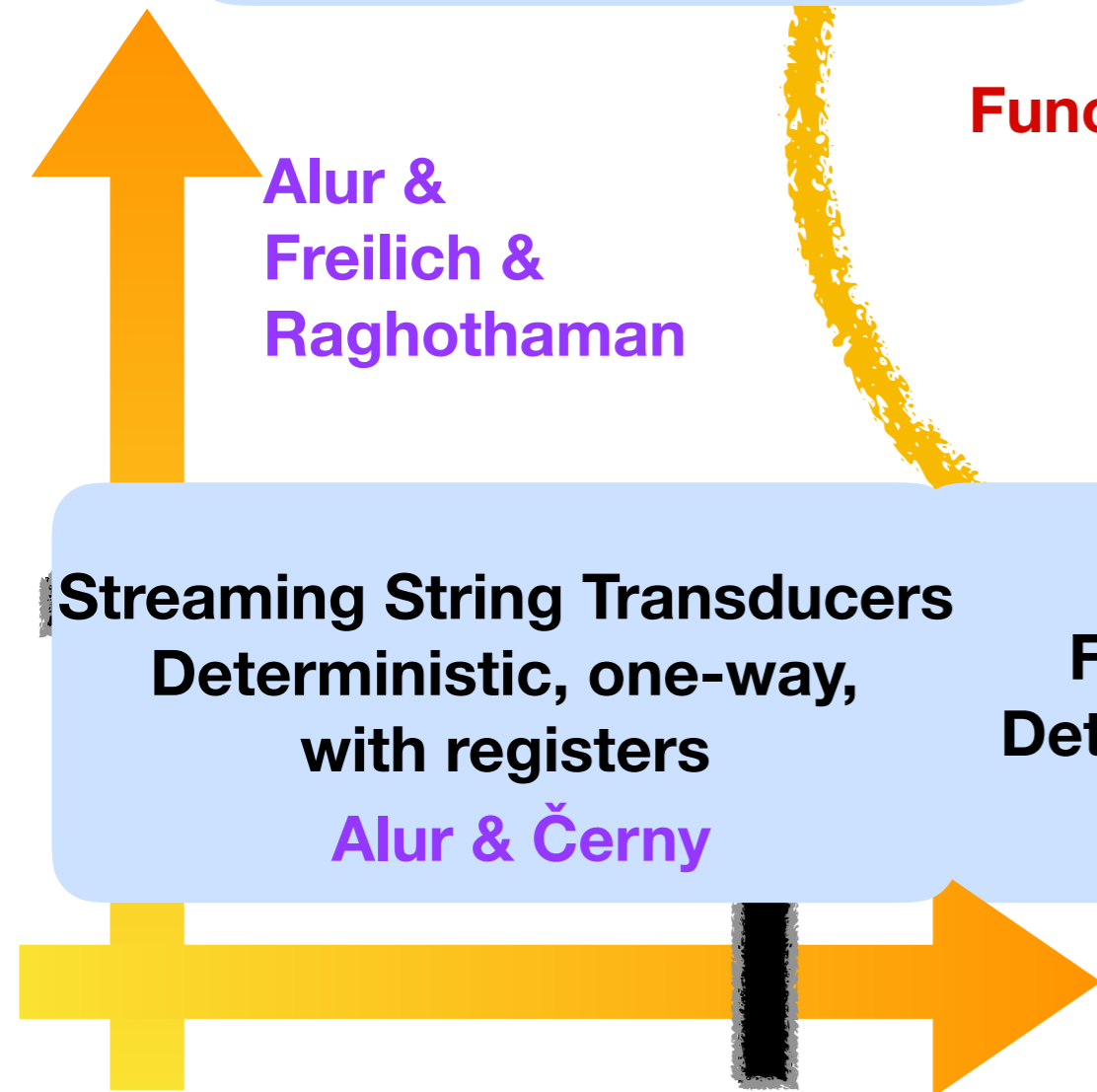
Engelfriet & Hoogeboom

Machines

**Streaming String Transducers
Deterministic, one-way,
with registers**

**Finite Transducers
Deterministic, two-way**

Alur & Černy



The fundamental Triptych

Programs

Specifications

Contribution
This contribution:
Regular Transducer Expressions (RTE)
New proof technique
Works directly with 2DFT
Based on Unambiguous Forest Factorizations
Extension to infinite words

MSO Transductions

Functional Transductions

**Freilich &
Raghothaman**

Engelfriet & Hoogeboom

Machines

Streaming String Transducers
Deterministic, one-way,
with registers

Alur & Černý

Finite Transducers
Deterministic, two-way



Summary

- MSO Transductions
- 2-way Deterministic Transducers (2DFT)
- Regular Transducer Expressions (RTE)
- From RTE to 2DFT
- Transition Monoid
- Unambiguous Forest Factorization
- From 2DFT to RTE
- Conclusion

Functional Transductions

$$f: \Sigma^* \rightarrow \Gamma^*$$

Running example: $\Sigma = \{\#, a\}$, $\Gamma = \{b, c\}$ and $\text{dom}(f) = (\#a^+)^+ \#$

$$f(\#a^{m_1} \#a^{m_2} \dots \#a^{m_{k-1}} \#a^{m_k} \#) = c^{m_2} b^{m_1} c^{m_3} b^{m_2} \dots c^{m_k} b^{m_{k-1}}$$

$$f(\#a^5 \#a^2 \#) = c^2 b^5$$

$$f(\#a \#a^2 \#a^3 \#a^4 \#) = c^2 b^1 c^3 b^2 c^4 b^3$$

$$f(\#a\#) = \varepsilon$$

MSO Transductions (Courcelle)

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$



MSO Transductions (Courcelle)

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$



1



2



MSO Transductions (Courcelle)

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$



1



2



$$\varphi_1(x) = P_a(x) \wedge \exists z_1 < z_2 < x : P_{\#}(z_1) \wedge P_{\#}(z_2)$$

MSO Transductions (Courcelle)

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$



1



2



$$\varphi_1(x) = P_a(x) \wedge \exists z_1 < z_2 < x : P_{\#}(z_1) \wedge P_{\#}(z_2)$$

$$\varphi_{c,1}(x) = \varphi_1(x) \text{ and } \varphi_{c,2}(x) = \perp$$

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1



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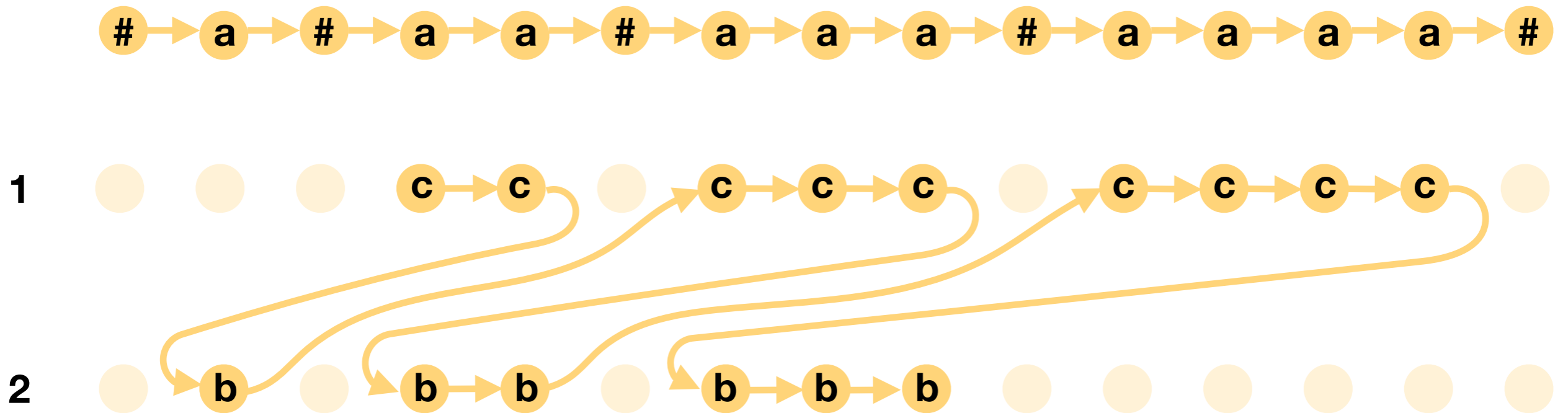
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$$\text{succ}_{1,1}(x, y) = \varphi_1(x) \wedge \varphi_1(y) \wedge x < y$$

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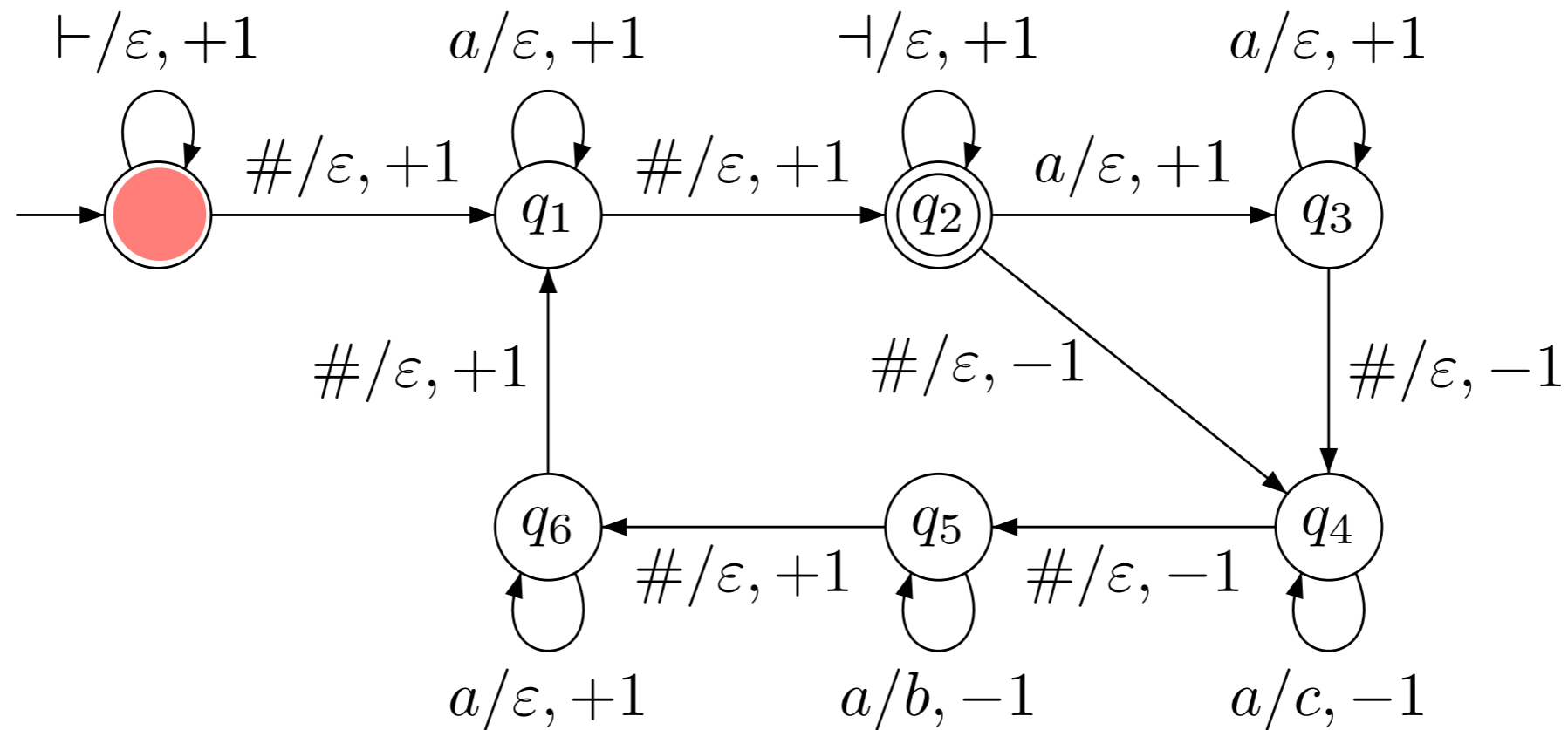
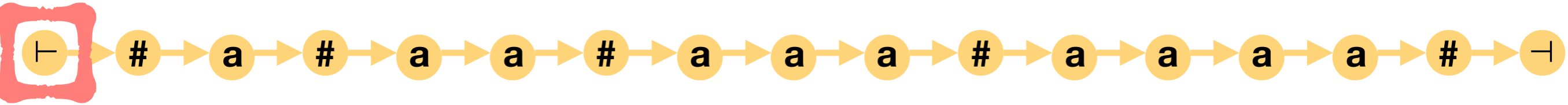
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$$\text{succ}_{1,2}(x, y) = \varphi_1(x) \wedge \varphi_2(y) \wedge \exists z_1 < z_2 < z_3, P_{\#}(z_1) \wedge P_{\#}(z_2) \wedge P_{\#}(z_3) \\ \wedge x < z_3 \wedge z_1 < y \wedge (\forall z, z_1 < z < z_3 \implies (z = z_2 \vee \neg P_{\#}(z)))$$

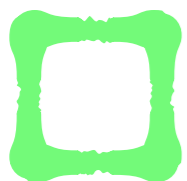
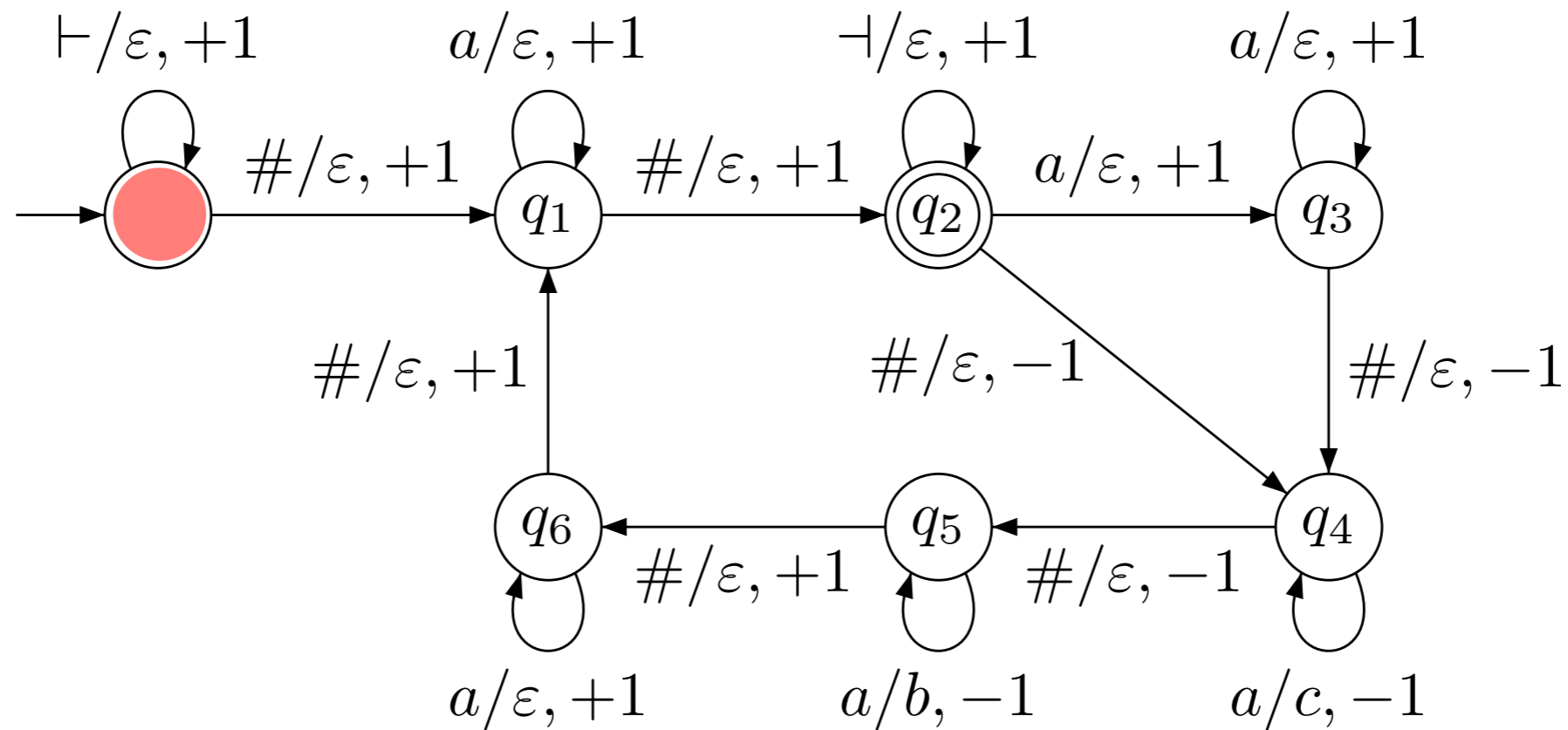
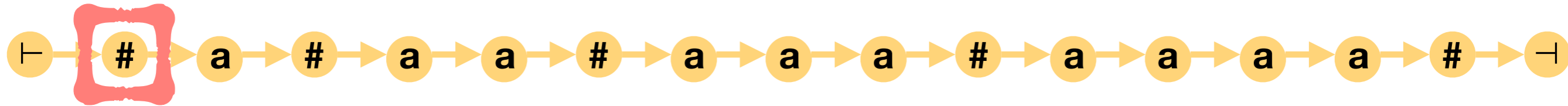
2-way Deterministic Transducers

$$f(\#a\#a^2\#a^3\#a^4\#) = c^2b^1c^3b^2c^4b^3$$



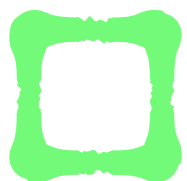
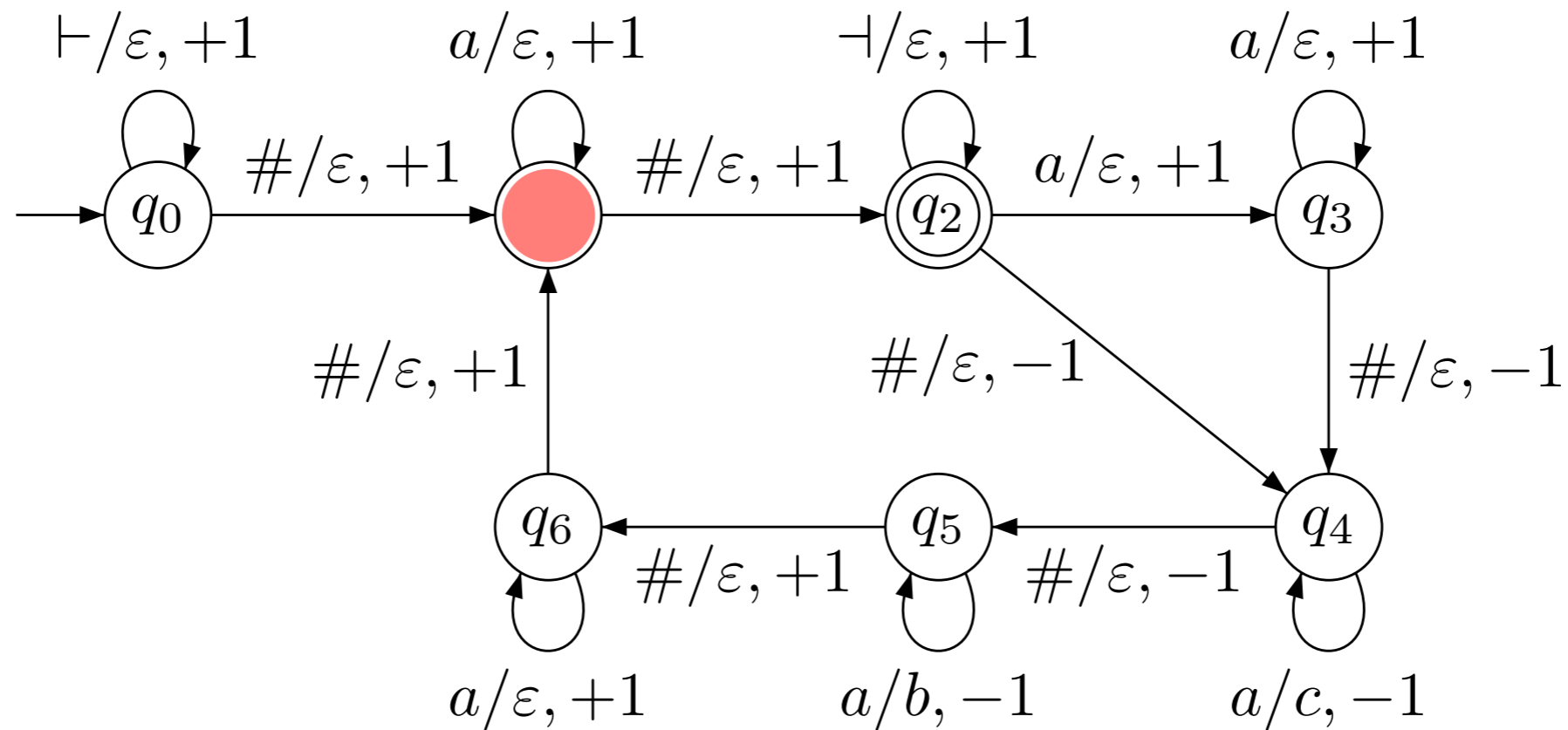
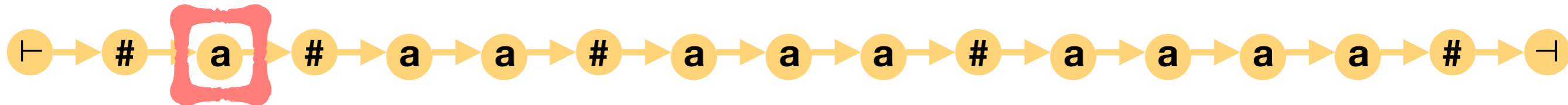
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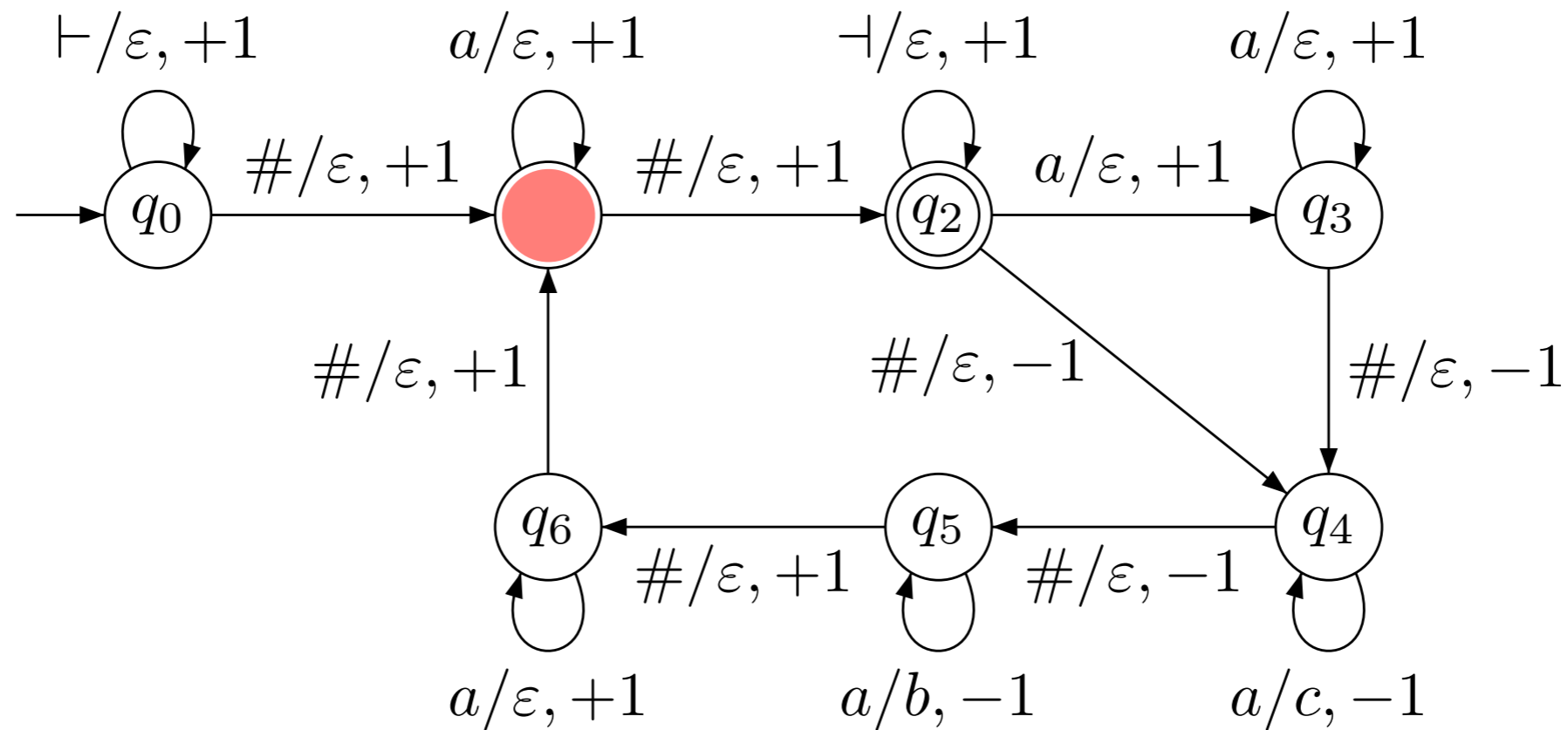
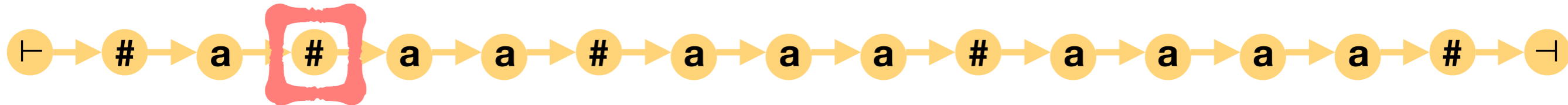
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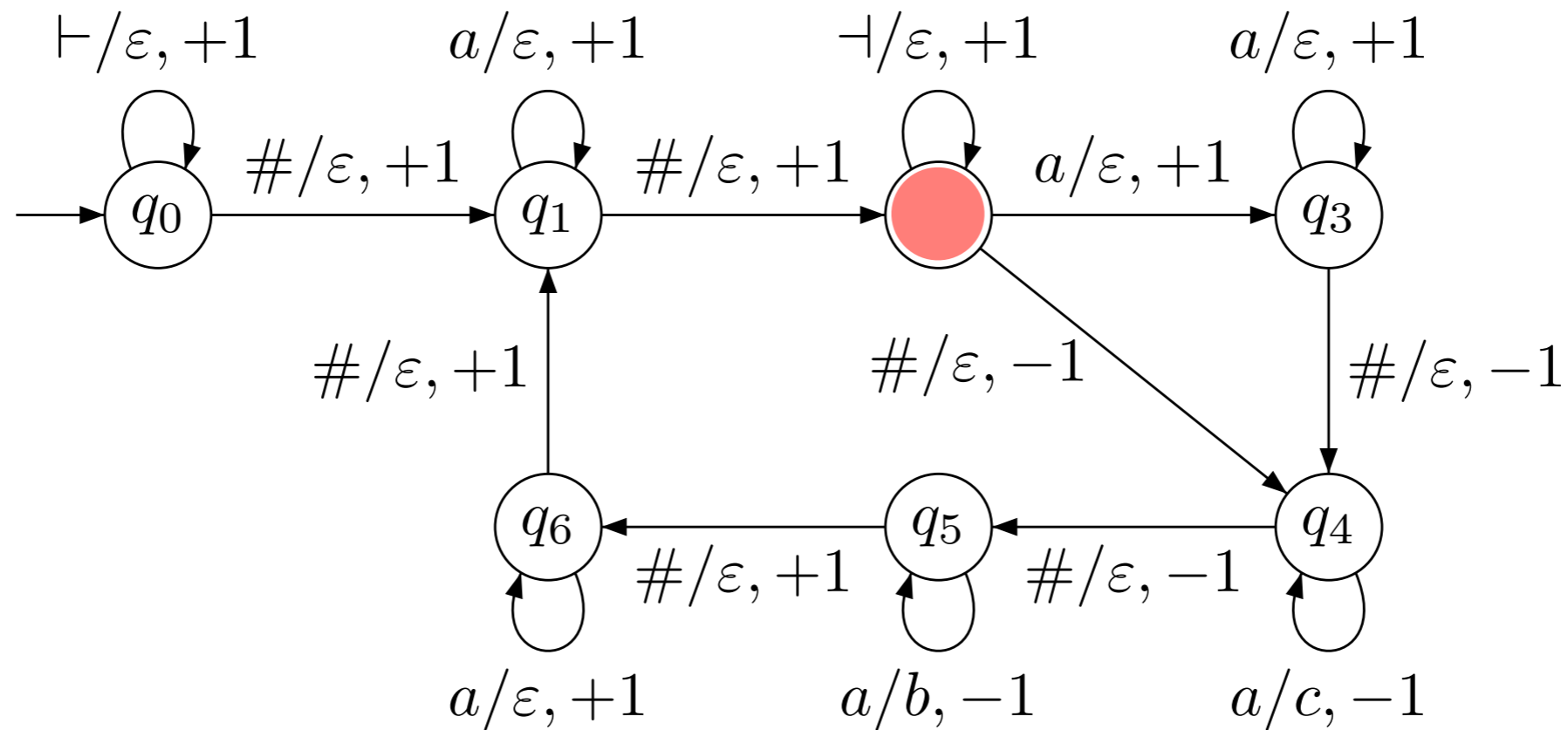
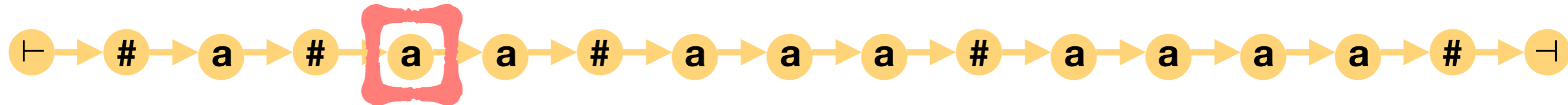
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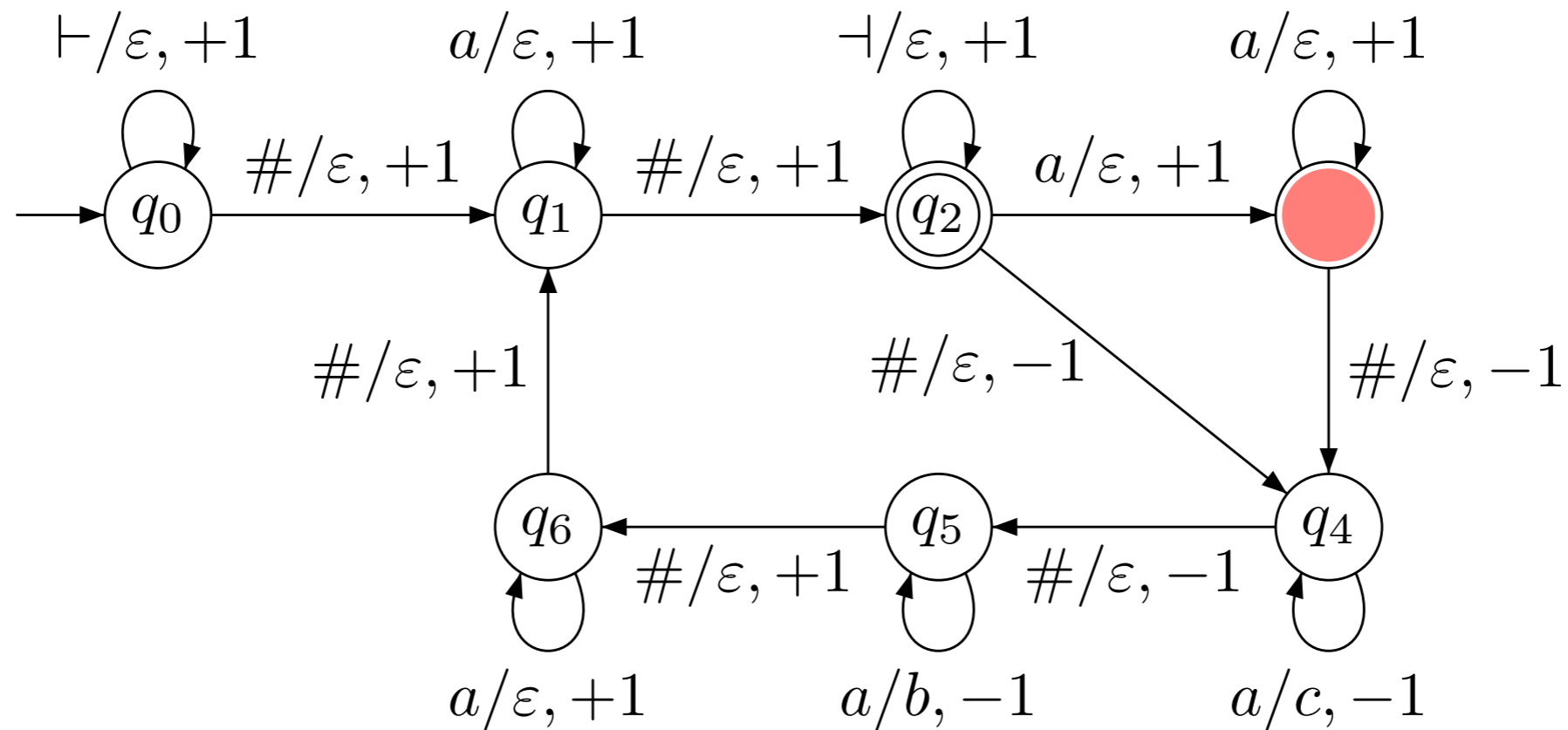
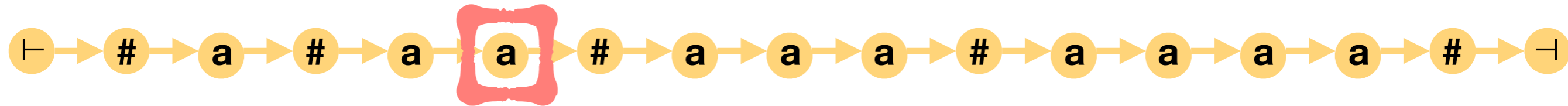
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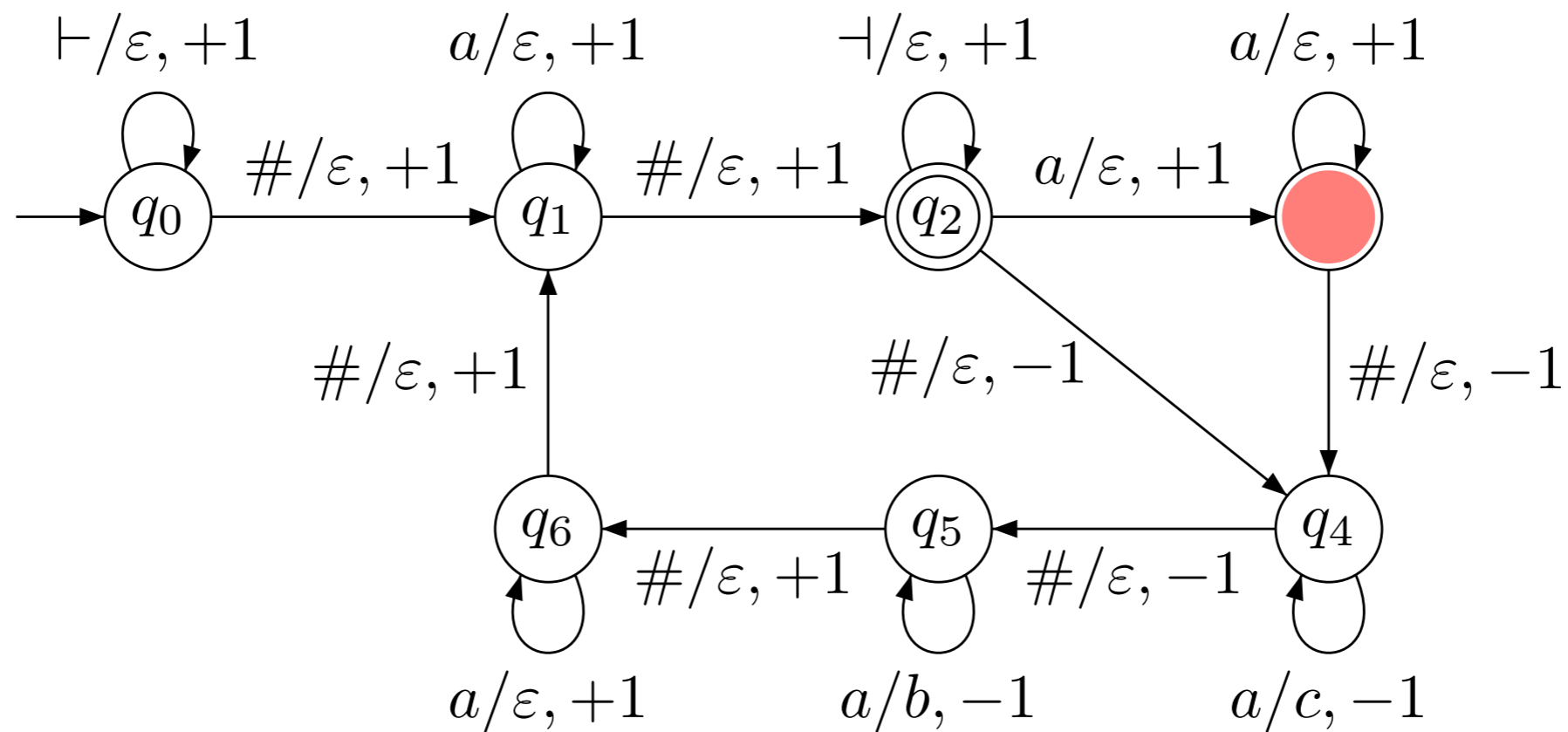
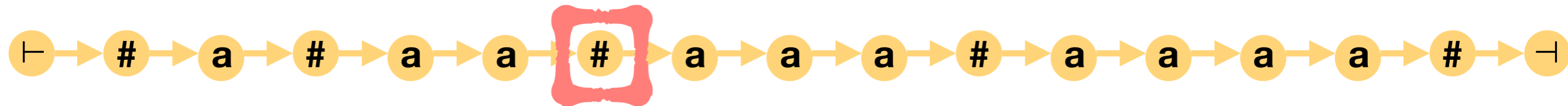
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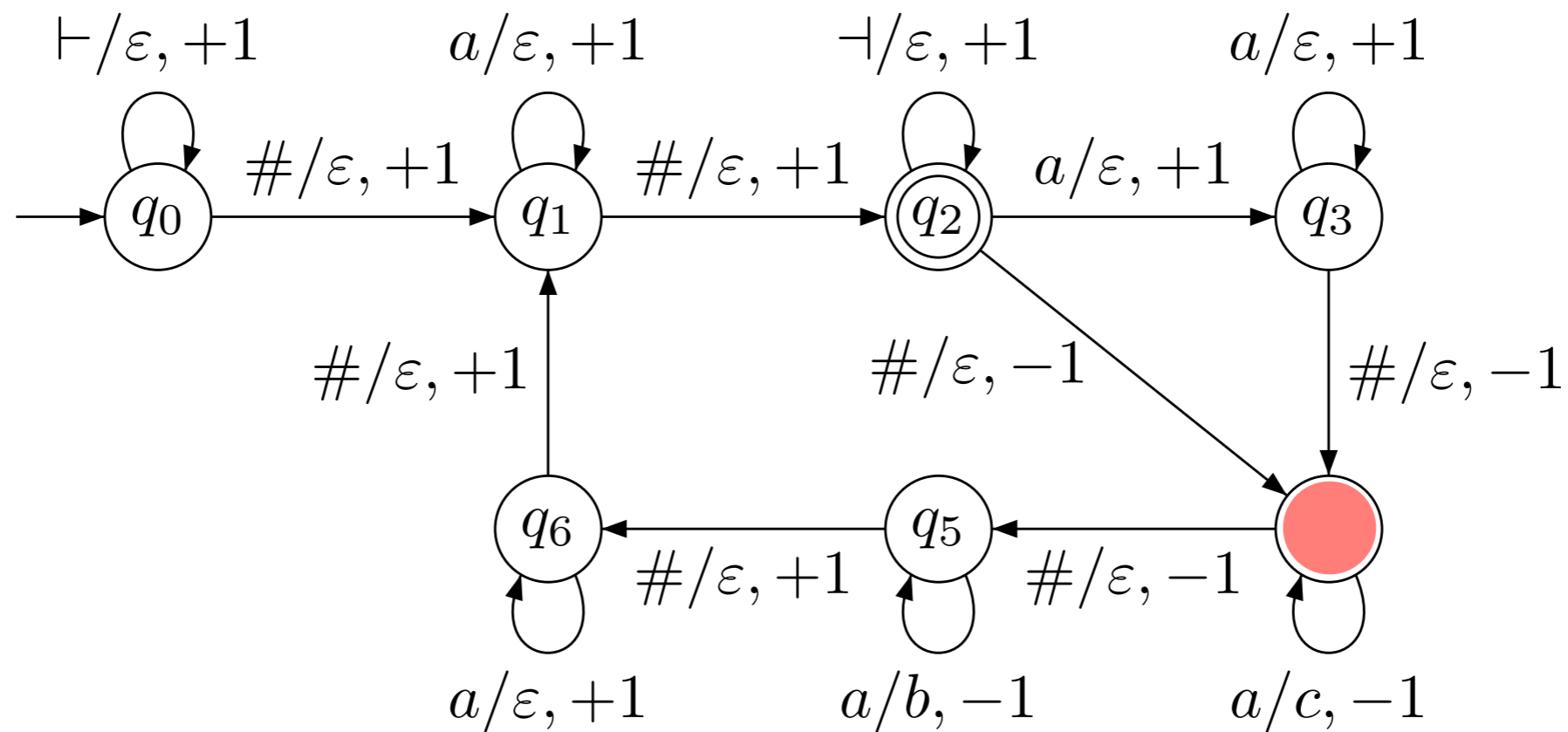
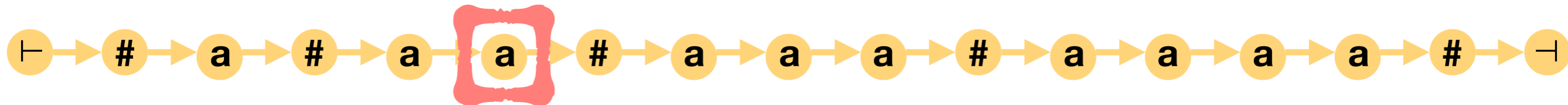
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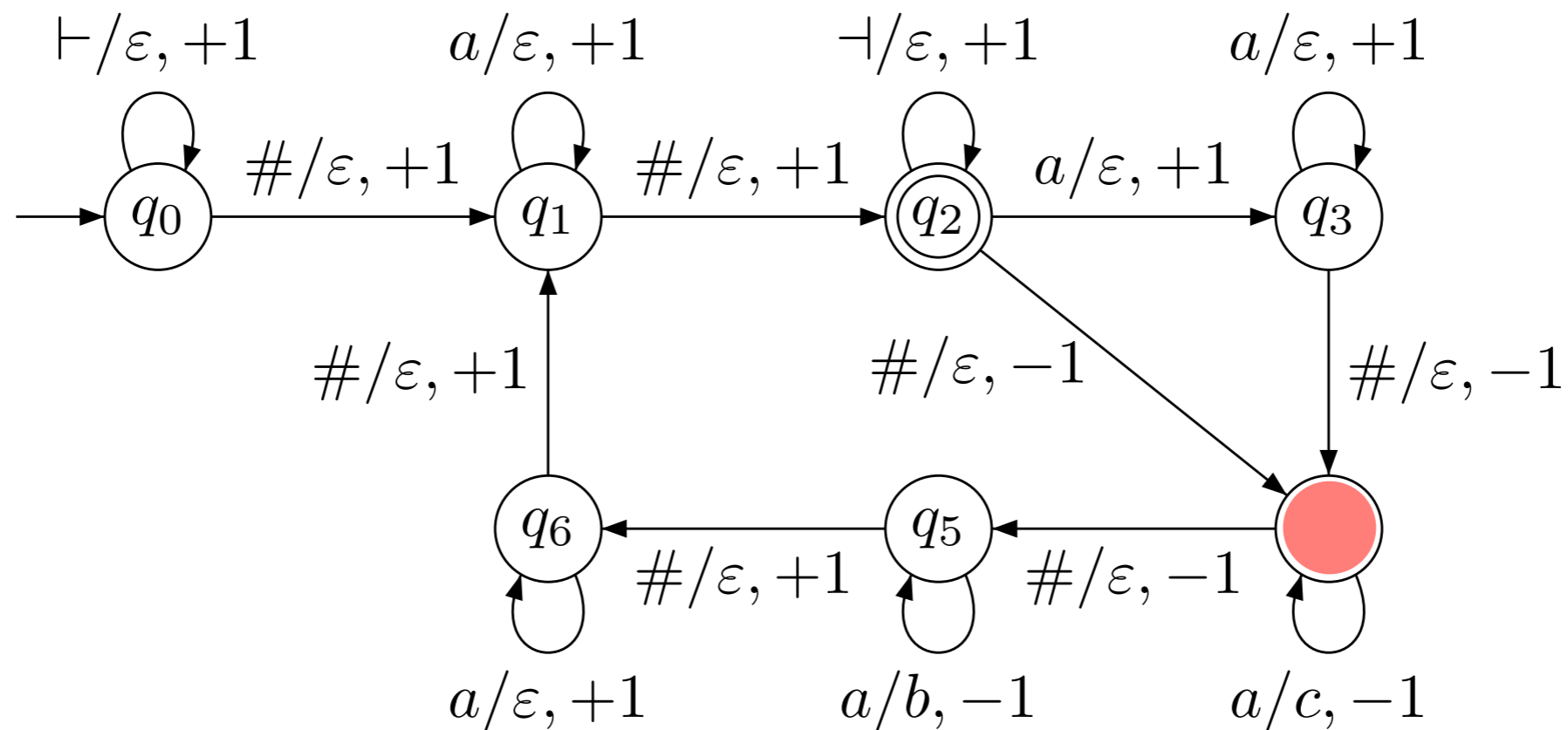
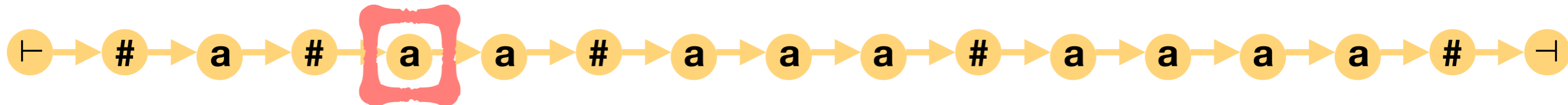
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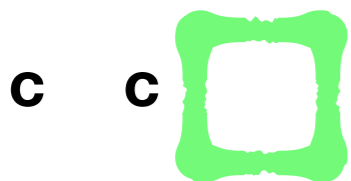
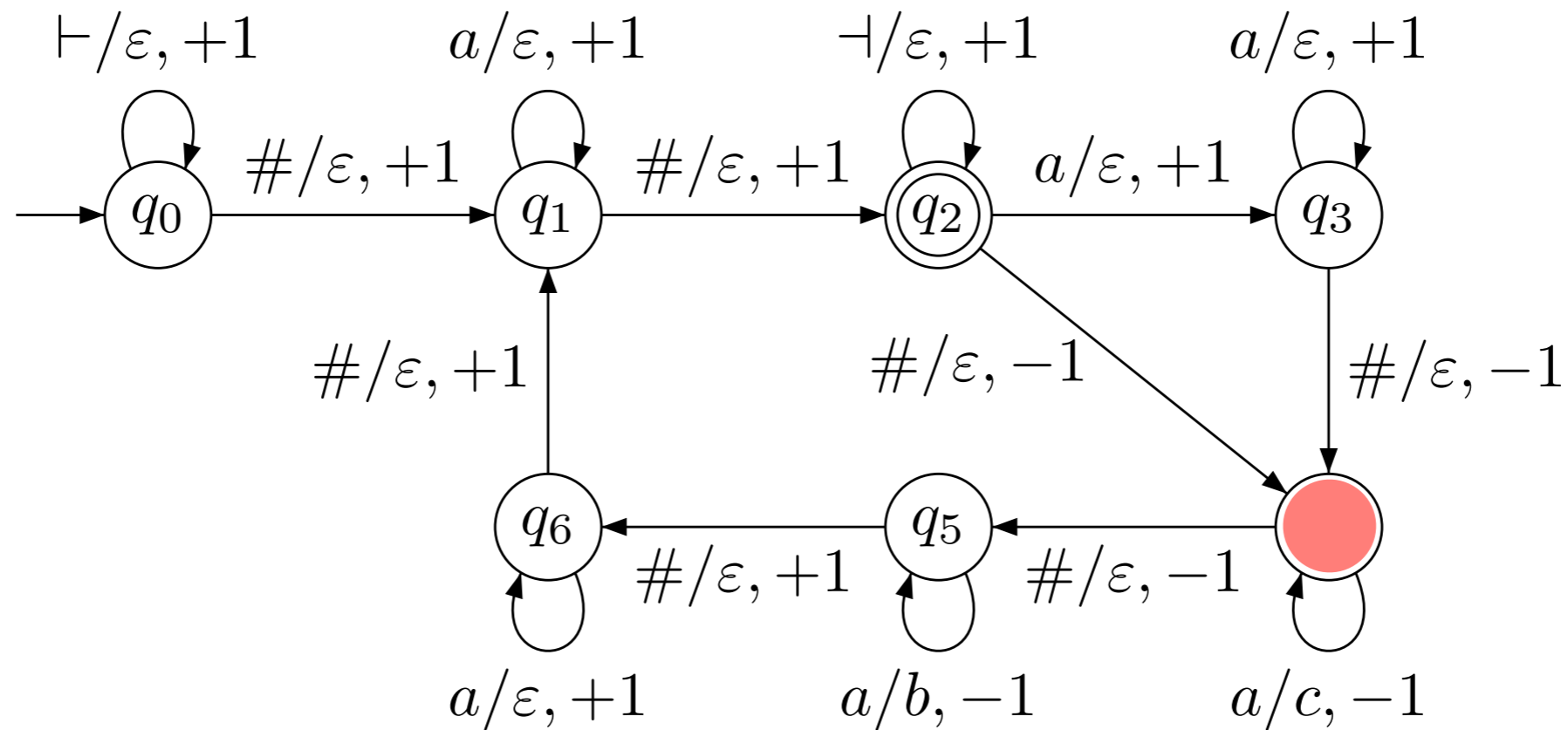
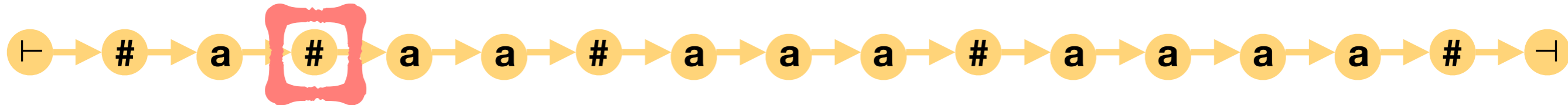
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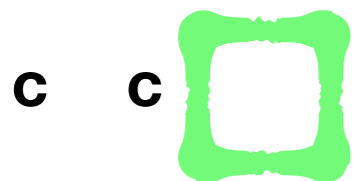
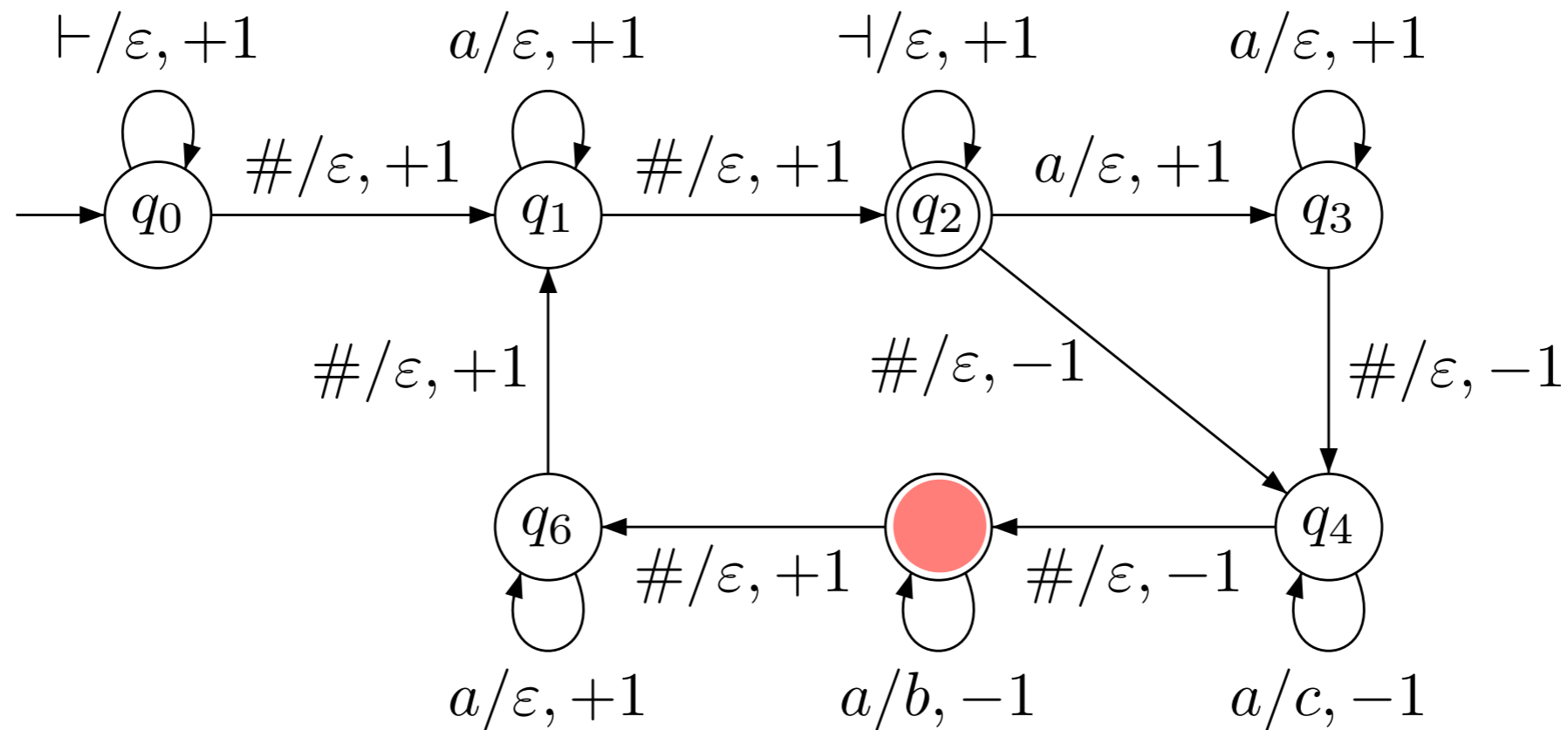
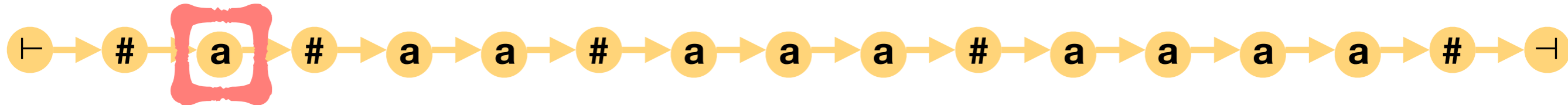
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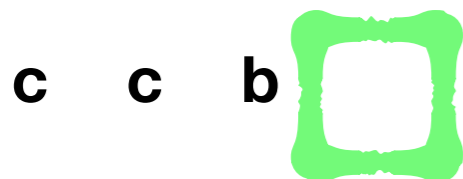
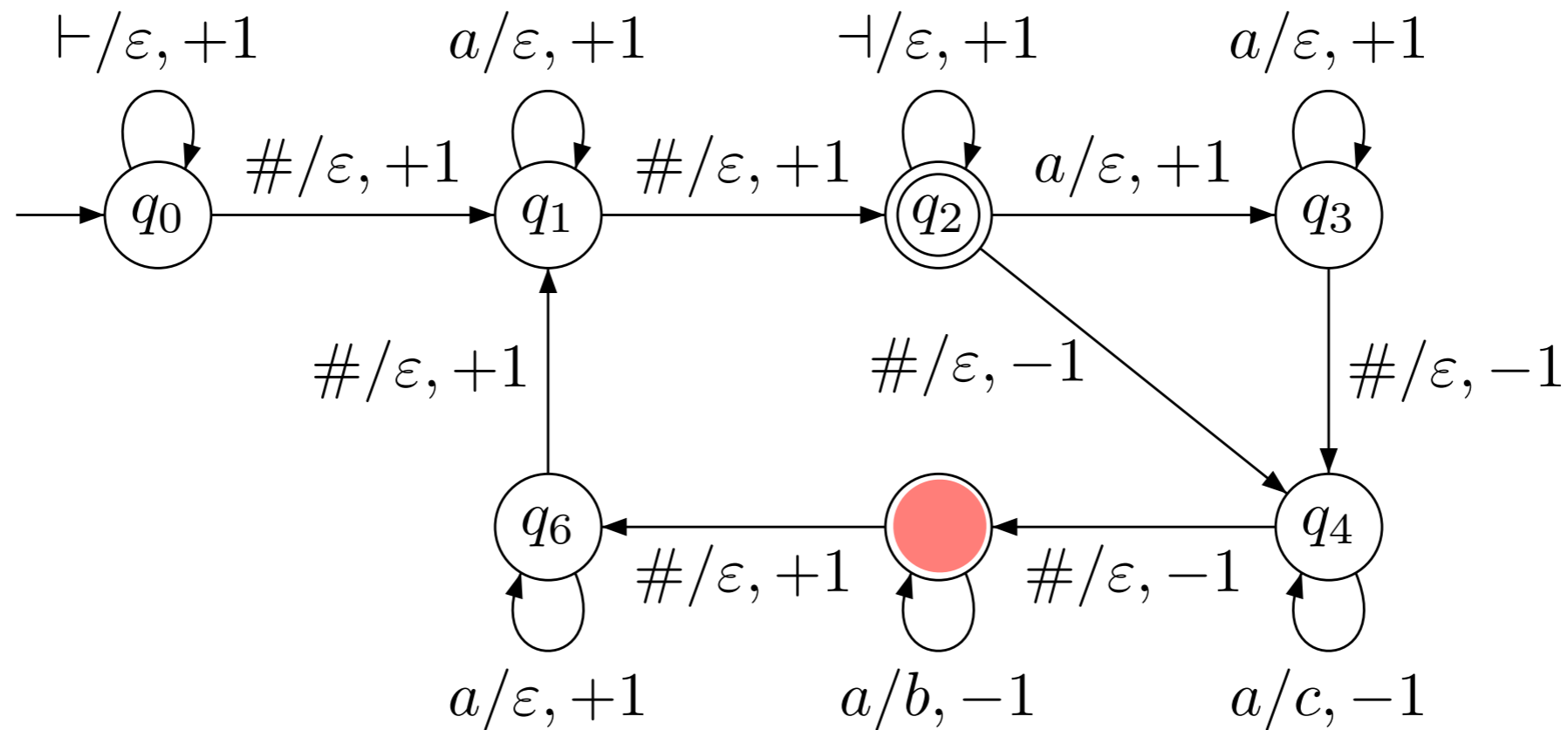
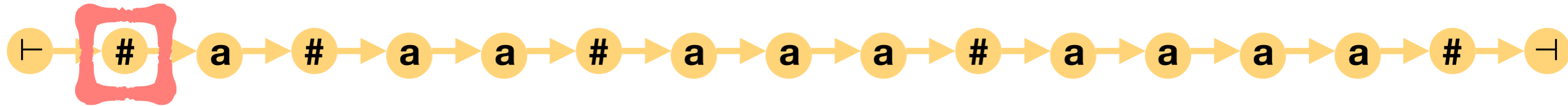
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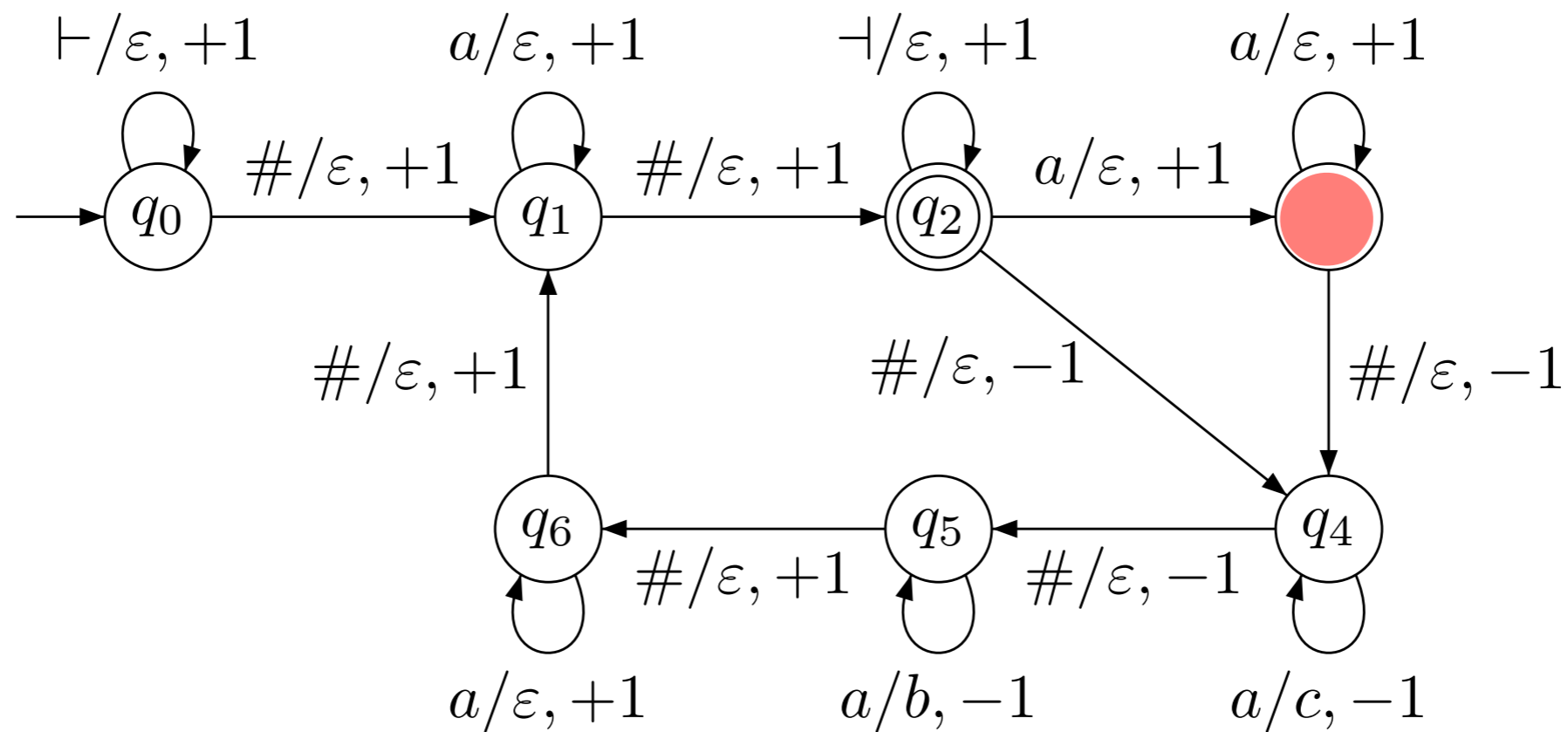
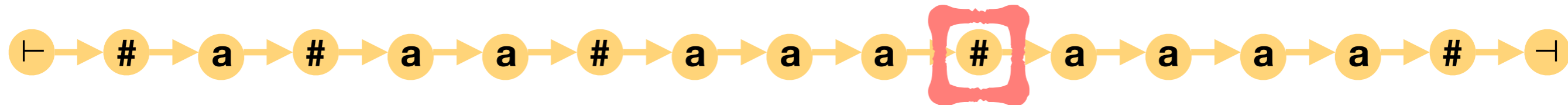
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2-way Deterministic Transducers

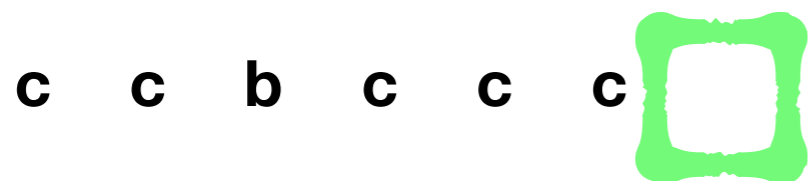
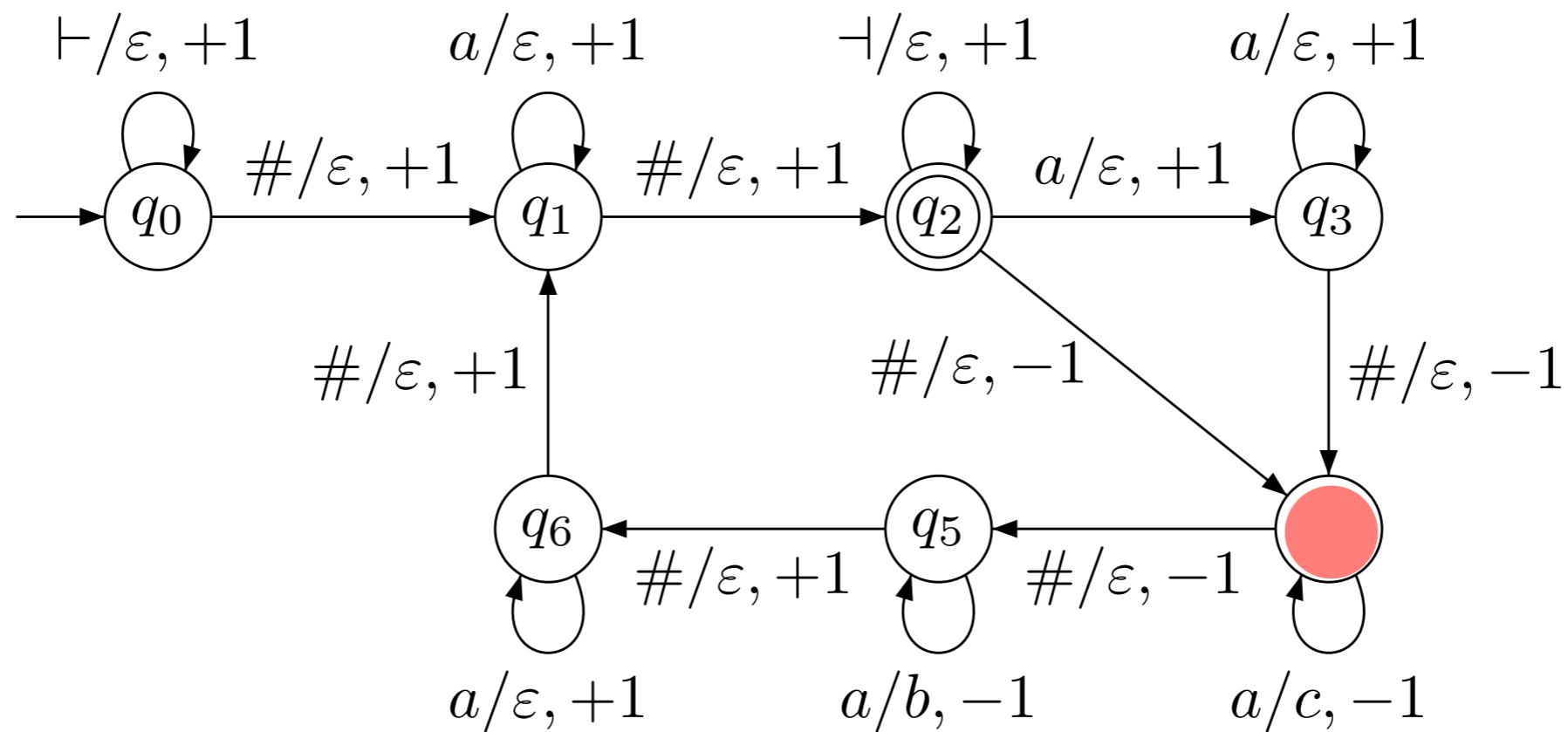
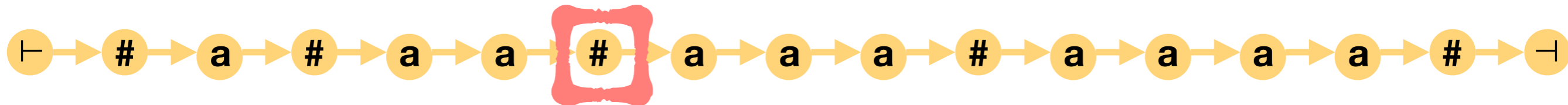
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c c b 

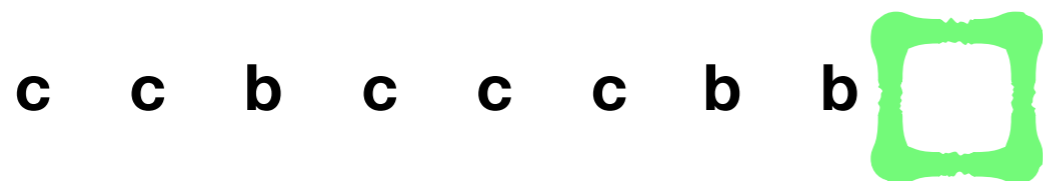
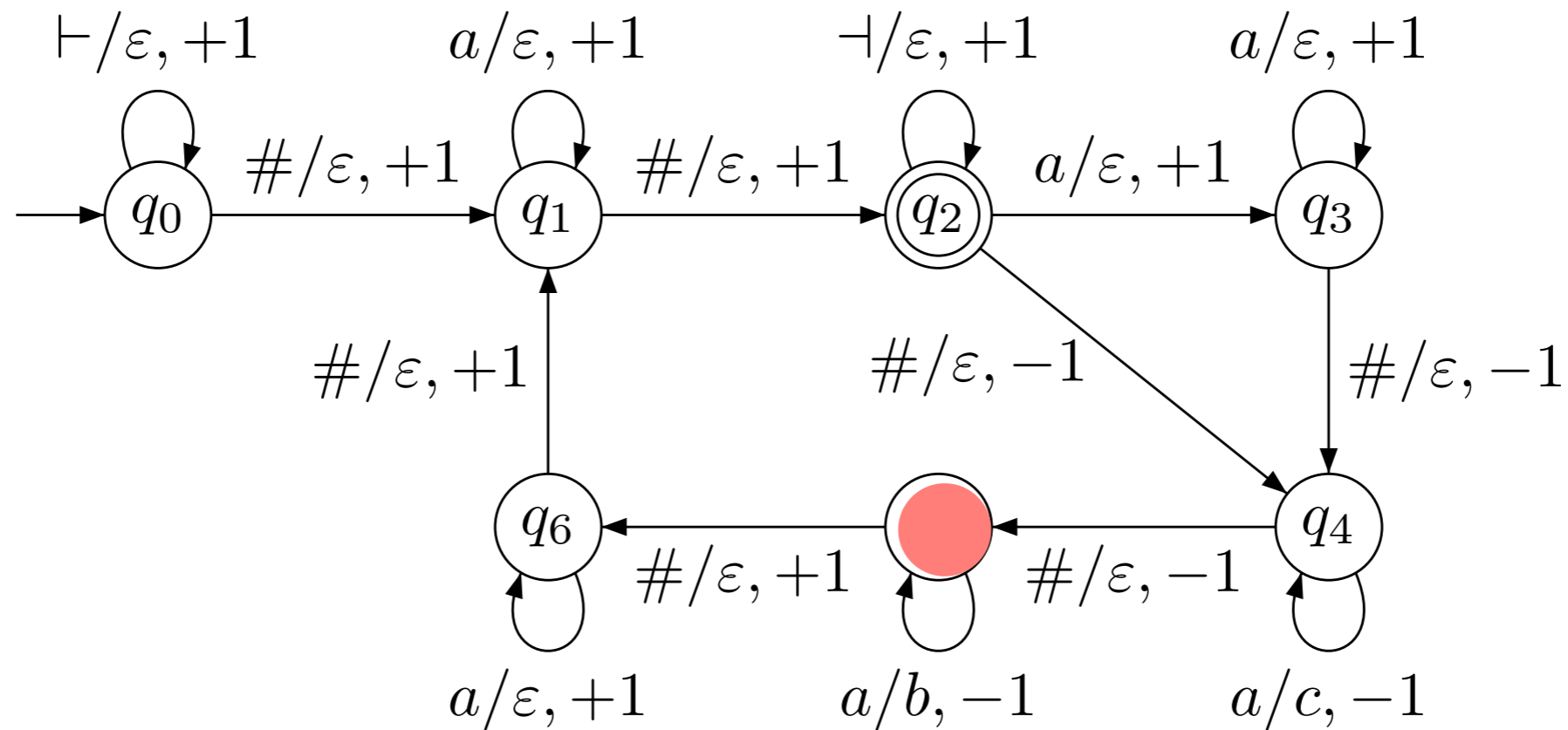
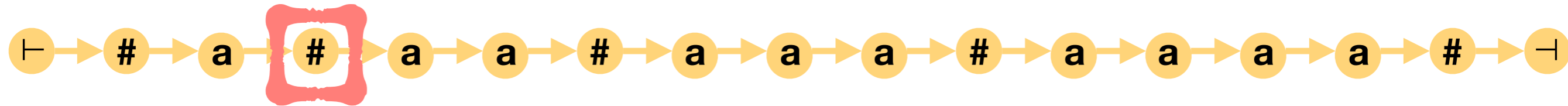
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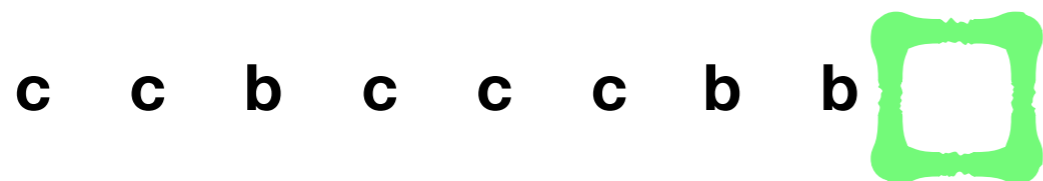
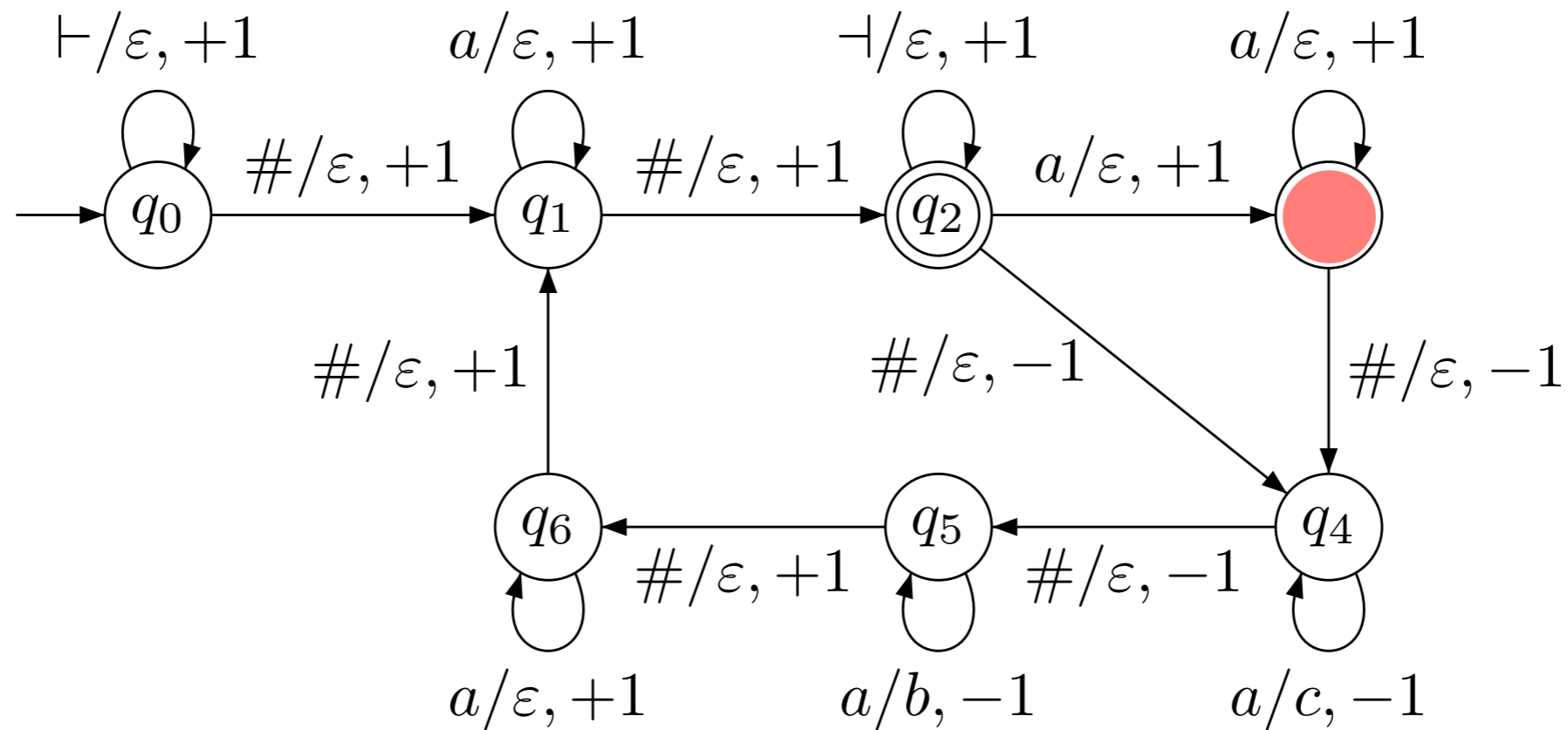
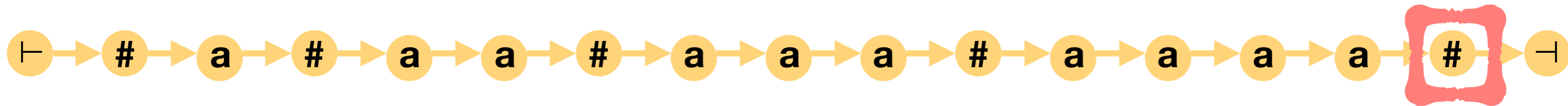
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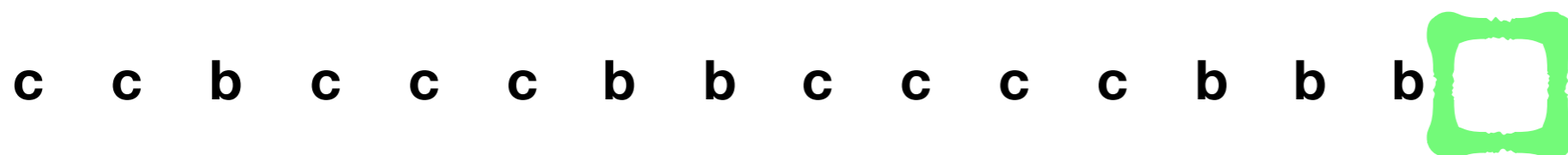
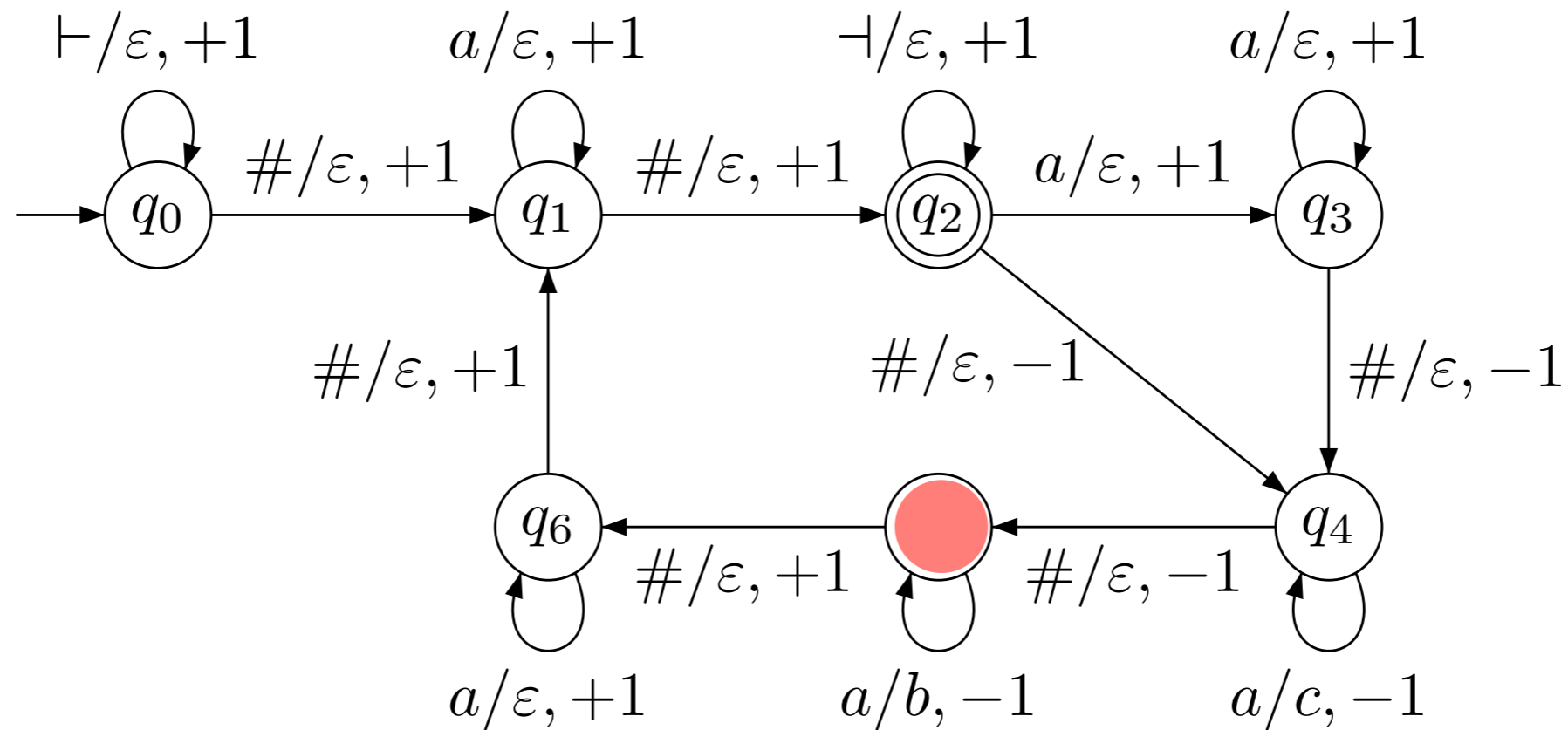
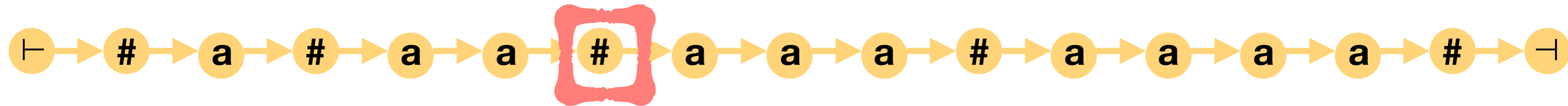
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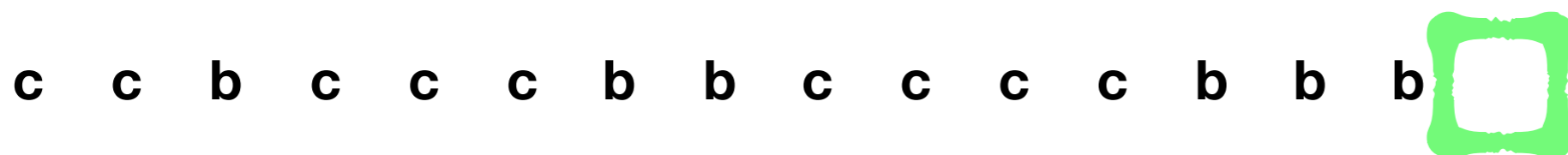
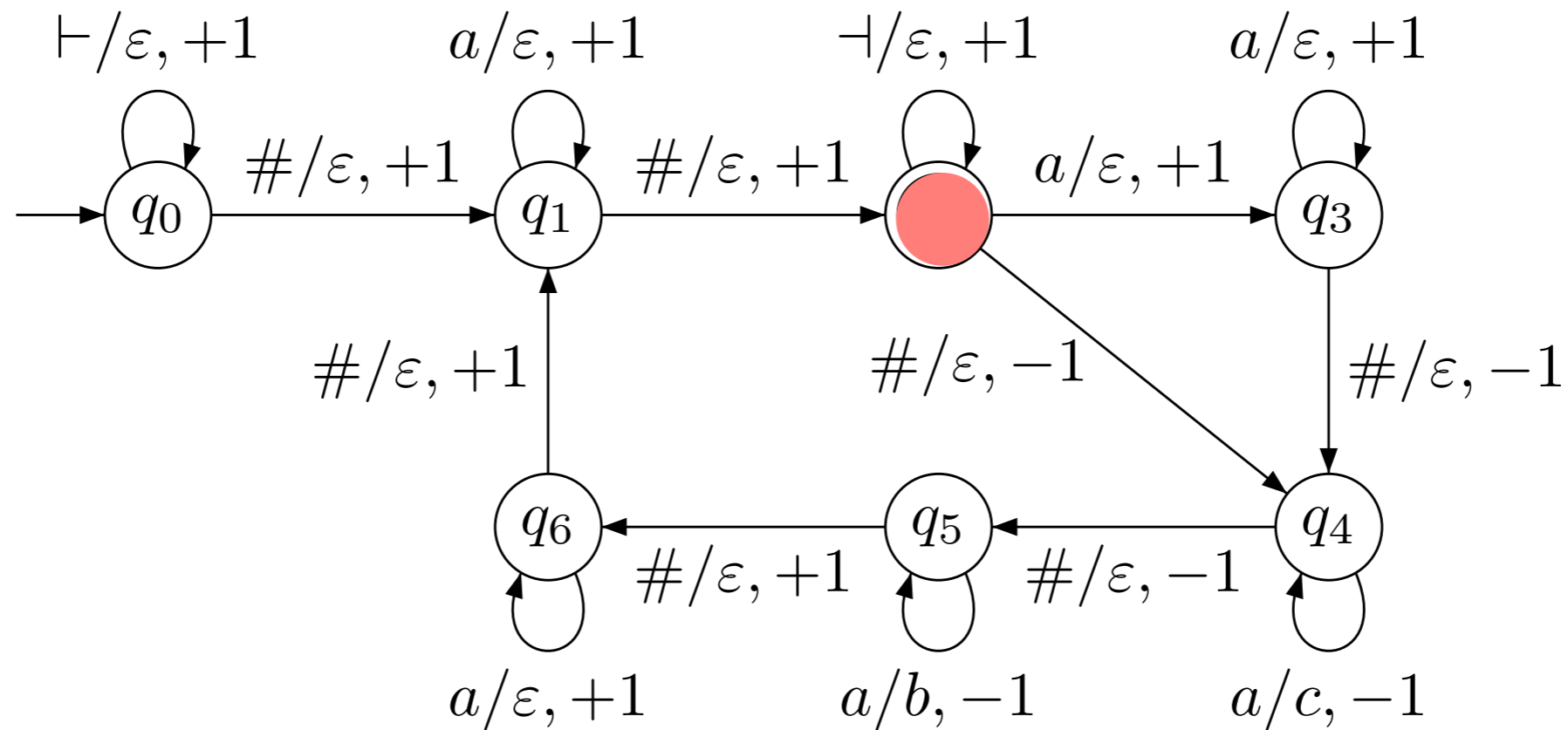
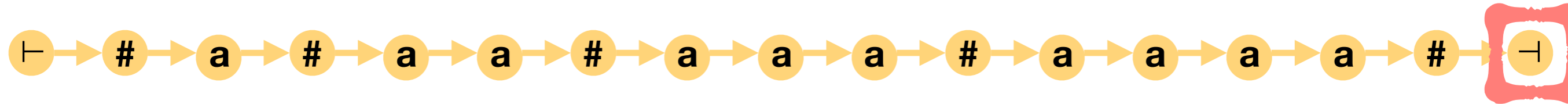
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2-way Deterministic Transducers

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Summary

- MSO Transductions
- 2-way Deterministic Transducers (2DFT)
- Regular Transducer Expressions (RTE)
- From RTE to 2DFT
- Transition Monoid
- Unambiguous Forest Factorization
- From 2DFT to RTE
- Conclusion

Regular Transducer Expressions

$C ::= d \mid K ? C : C \mid C \odot C \mid C \boxtimes C \mid C \overleftarrow{\boxtimes} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

$\llbracket C \rrbracket : \Sigma^* \rightarrow \Gamma^*$ partial function

Regular Transducer Expressions

$$d \in \Gamma^* \uplus \{\perp\}$$

$$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

$$[[C]]: \Sigma^* \rightarrow \Gamma^* \text{ partial function}$$

Regular Transducer Expressions

$d \in \Gamma^* \uplus \{\perp\}$

$K \subseteq \Sigma^*$ regular

$C ::= d \mid K ? C : C \mid C \odot C \mid C \boxplus C \mid C \overset{\leftarrow}{\boxplus} C \mid C^{\boxplus} \mid C^{\overset{\leftarrow}{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overset{\leftarrow}{2\boxplus}}$

$\llbracket C \rrbracket : \Sigma^* \rightarrow \Gamma^*$ partial function

$C_a = a ? a : (b ? \varepsilon : \perp)$

$C_b = b ? b : (a ? \varepsilon : \perp)$

$\text{dom}(C_a) = \text{dom}(C_b) = \{a, b\}$

If then else

$$(K ? f : g)(u) = \begin{cases} f(u) & \text{if } u \in K \\ g(u) & \text{otherwise} \end{cases}$$

Regular Transducer Expressions

$$d \in \Gamma^* \uplus \{\perp\}$$

$$K \subseteq \Sigma^* \text{ regular}$$

$$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overset{\leftarrow}{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

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$$\text{dom}(\llbracket C_a^{\boxplus} \rrbracket) = \{a, b\}^+$$

$$\llbracket C_a^{\boxplus} \rrbracket(baabbab) = a^3$$

If then else

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Unambiguous Kleene-plus

$$f^{\boxplus}(w) = f(u_1)f(u_2) \cdots f(u_n)$$

If $w = u_1u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \dots, u_n \in \text{dom}(f)$

Regular Transducer Expressions

$$d \in \Gamma^* \uplus \{\perp\}$$

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$$\llbracket C_a^{\boxplus} \odot C_b^{\boxplus} \rrbracket(baabbbab) = a^3 b^5$$

If then else

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Unambiguous Kleene-plus

$$f^{\boxplus}(w) = f(u_1) f(u_2) \cdots f(u_n)$$

Hadamard product

$$f \odot g(u) = f(u) g(u)$$

Regular Transducer Expressions

$$d \in \Gamma^* \uplus \{\perp\}$$

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$$\llbracket C \rrbracket (baabbbaab\#abab) = b^5 \# a^2$$

b a a b b b a b # a b a b

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b a a b b b a b # **a b a b**

b b b b b

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b a a b b b a b # a b a b

b b b b b #

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b a a b b b a b # a b a b

b b b b b # a a

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$$C = (\text{copy} \overleftarrow{\boxtimes} (\# ? \# : \perp)) \overleftarrow{\boxtimes} \text{copy}$$

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b a a b b b a b # a b a b

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b a a b b b a b # a b a b

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b a a b b b a b # **a b a b**

a b a b # b a a b b b a b

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$$f(\#a^{m_1} \#a^{m_2} \dots \#a^{m_k-1} \#a^{m_k}) = c^{m_2} b^{m_1} c^{m_3} b^{m_2} \dots c^{m_k} b^{m_k-1}$$

$$C_b = (a ? b : \perp)^{\boxplus} \quad C_c = (a ? c : \perp)^{\boxplus}$$

$$C = [\#a^+, ((\# ? \varepsilon : \perp) \square C_b) \overset{\leftarrow}{\square} ((\# ? \varepsilon : \perp) \square C_c)]^{2\boxplus}$$

Unambiguous 2-chained Kleene-plus

$$[K, f]^{2\boxplus}(w) = f(u_1 u_2) f(u_2 u_3) \cdots f(u_{n-1} u_n)$$

If $w = u_1 u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \dots, u_n \in K$

Regular Transducer Expressions

$$d \in \Gamma^* \uplus \{\perp\}$$

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$$\llbracket C \rrbracket (\#a\#a^2\#a^3\#a^4) = c^2b^1c^3b^2c^4b^3$$

a # a a # a a a # a a a a

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a # a a # a a a # a a a a

c c b

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a # a a # a a a # a a a a

c c b c c c b b

$$C_b = (a ? b : \perp)^{\boxplus} \quad C_c = (a ? c : \perp)^{\boxplus}$$

$$C = [\#a^+ \cdot ((\# ? \varepsilon : \perp) \square C_b) \overset{\leftarrow}{\square} ((\# ? \varepsilon : \perp) \square C_c)]^{2\boxplus}$$

Unambiguous 2-chained Kleene-plus

$$[K, f]^{2\boxplus}(w) = f(u_1u_2)f(u_2u_3) \cdots f(u_{n-1}u_n)$$

If $w = u_1u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \dots, u_n \in K$

Regular Transducer Expressions

$$d \in \Gamma^* \uplus \{\perp\}$$

$$K \subseteq \Sigma^* \text{ regular}$$

$$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overset{\leftarrow}{\square} C \mid C^{\boxplus} \mid C^{\boxplus\boxplus} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overset{\leftarrow}{2\boxplus}}$$

$$\llbracket C \rrbracket (\#a\#a^2\#a^3\#a^4) = c^2b^1c^3b^2c^4b^3$$

a # a a # a a a # a a a a

c c b c c c b b c c c c b b b

$$C_b = (a ? b : \perp)^{\boxplus} \quad C_c = (a ? c : \perp)^{\boxplus}$$

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If $w = u_1u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \dots, u_n \in K$

RTE and 2DFT

Main Theorem:

2DFTs and RTEs define the same class of functions. More precisely,

1. given an RTE C , we can construct a 2DFT \mathcal{A} such that $[[\mathcal{A}]] = [[C]]$,
2. given a 2DFT \mathcal{A} , we can construct an RTE C such that $[[\mathcal{A}]] = [[C]]$.

Summary

- MSO Transductions
- 2-way Deterministic Transducers (2DFT)
- Regular Transducer Expressions (RTE)
- From RTE to 2DFT
- Transition Monoid
- Unambiguous Forest Factorization
- From 2DFT to RTE
- Conclusion

RTE to 2DFT

$$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

Main Lemma:

Let $K \subseteq \Sigma^*$ be regular, and let f and g be RTEs with $\llbracket f \rrbracket = \llbracket M_f \rrbracket$ and $\llbracket g \rrbracket = \llbracket M_g \rrbracket$ for 2DFTs M_f and M_g respectively. Then, one can construct

1. a 2DFT \mathcal{A} such that $\llbracket K ? f : g \rrbracket = \llbracket \mathcal{A} \rrbracket$.
2. a 2DFT \mathcal{A} such that $\llbracket \mathcal{A} \rrbracket = \llbracket f \odot g \rrbracket$.
3. 2DFTs \mathcal{A}, \mathcal{B} such that $\llbracket \mathcal{A} \rrbracket = \llbracket f \square g \rrbracket$ and $\llbracket \mathcal{B} \rrbracket = \llbracket f \overleftarrow{\square} g \rrbracket$.
4. 2DFTs \mathcal{A}, \mathcal{B} such that $\llbracket \mathcal{A} \rrbracket = \llbracket f^{\boxplus} \rrbracket$ and $\llbracket \mathcal{B} \rrbracket = \llbracket f^{\overleftarrow{\boxplus}} \rrbracket$.
5. 2DFTs \mathcal{A}, \mathcal{B} such that $\llbracket \mathcal{A} \rrbracket = \llbracket [K, f]^{2\boxplus} \rrbracket$ and $\llbracket \mathcal{B} \rrbracket = \llbracket [K, f]^{\overleftarrow{2\boxplus}} \rrbracket$.

RTE to 2DFT: If then else

$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b a a b b b b a b a a a b b b b ⊣

Main Lemma:

Let $K \subseteq \Sigma^*$ be regular, and let f and g be RTEs with $\llbracket f \rrbracket = \llbracket M_f \rrbracket$ and $\llbracket g \rrbracket = \llbracket M_g \rrbracket$ for 2DFTs M_f and M_g respectively. Then, one can construct

1. a 2DFT \mathcal{A} such that $\llbracket K ? f : g \rrbracket = \llbracket \mathcal{A} \rrbracket$.

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$$C ::= d \mid K?C:C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

⊢ a a b a b b a a a b a a b b b b a b a a a b b b b ⊢

DFA
K?

Main Lemma:

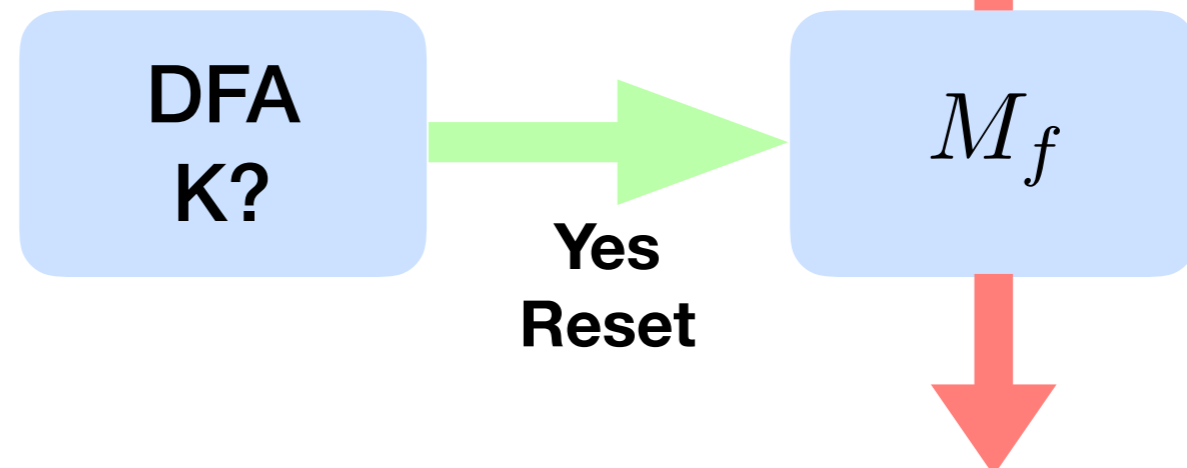
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RTE to 2DFT: If then else

$$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

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⊢

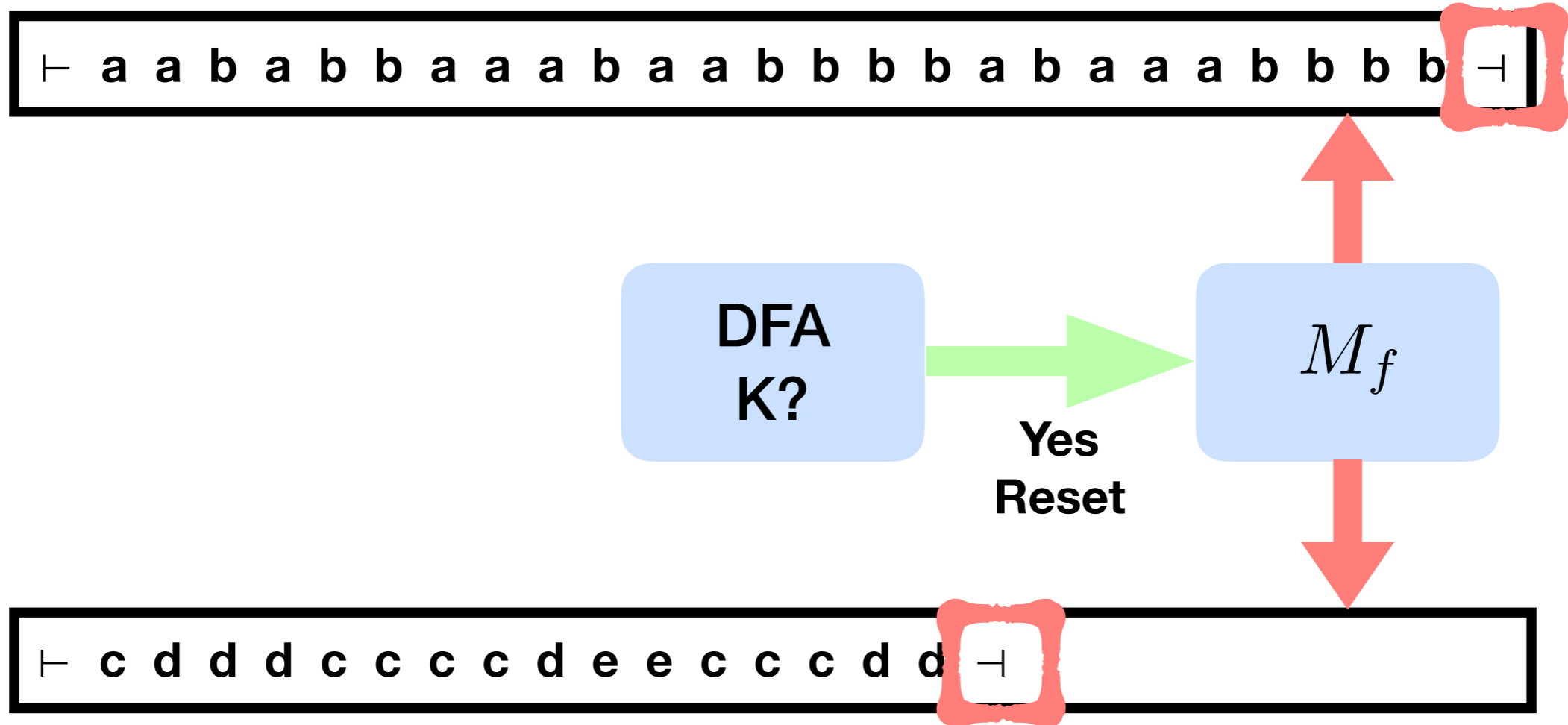
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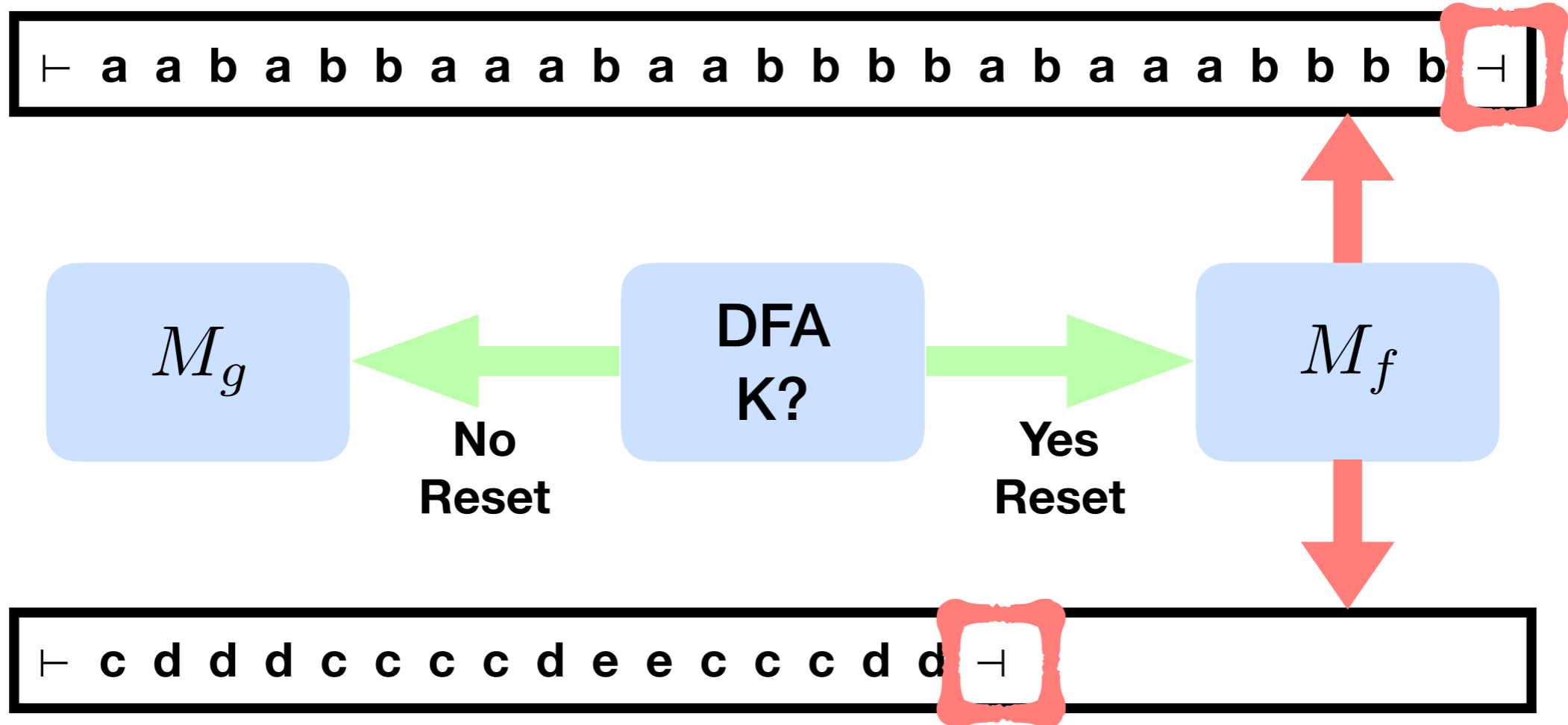
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Main Lemma:

Let $K \subseteq \Sigma^*$ be regular, and let f and g be RTEs with $\llbracket f \rrbracket = \llbracket M_f \rrbracket$ and $\llbracket g \rrbracket = \llbracket M_g \rrbracket$ for 2DFTs M_f and M_g respectively. Then, one can construct

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RTE to 2DFT: Hadamard product

$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b a a b b b b a b a a a b b b b ⊣

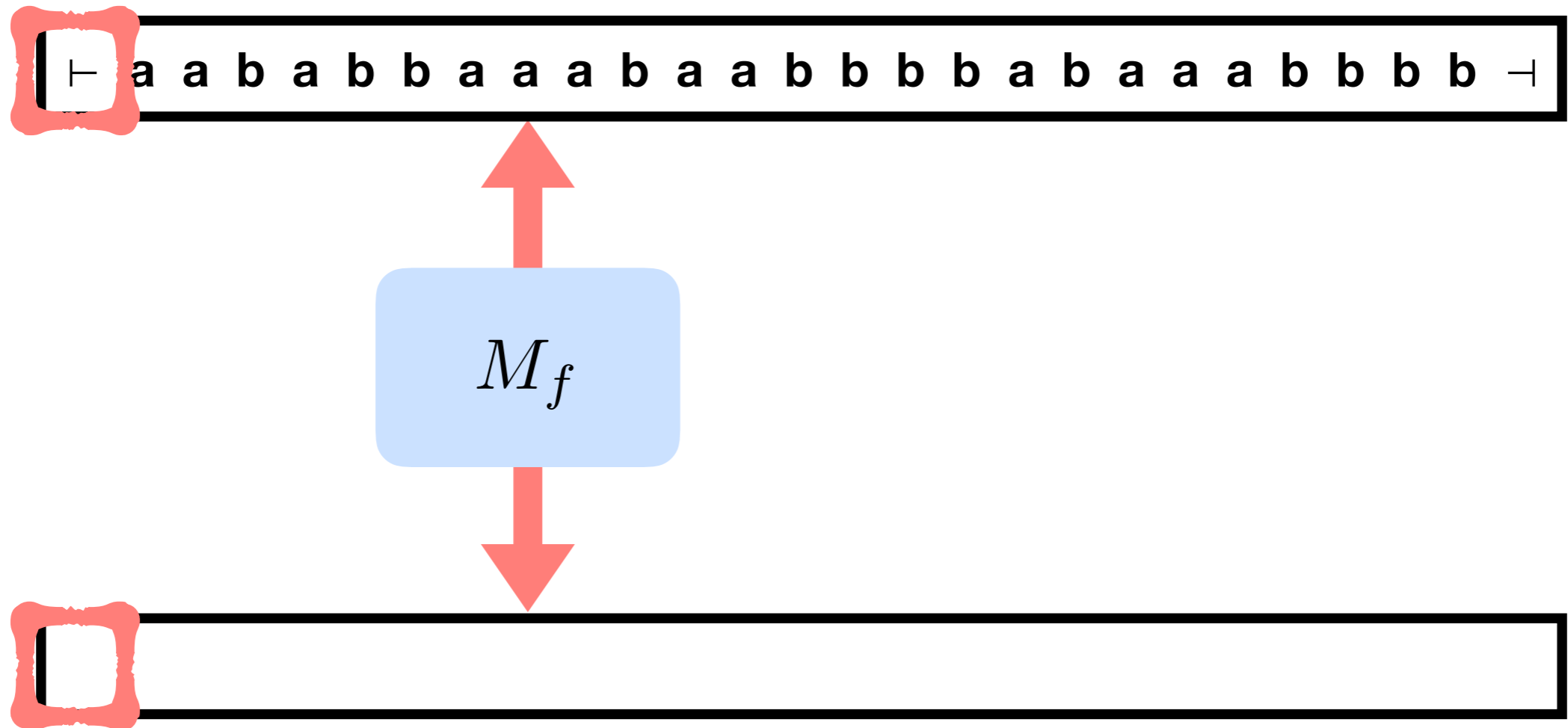
Main Lemma:

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2. a 2DFT \mathcal{A} such that $\llbracket \mathcal{A} \rrbracket = \llbracket f \odot g \rrbracket$.

RTE to 2DFT: Hadamard product

$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$



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⊢ a a b a b b a a a b a a b b b b a b a a a b b b b ⊢

M_f

⊢ c d d d c c c c d e e c c c d c ⊢

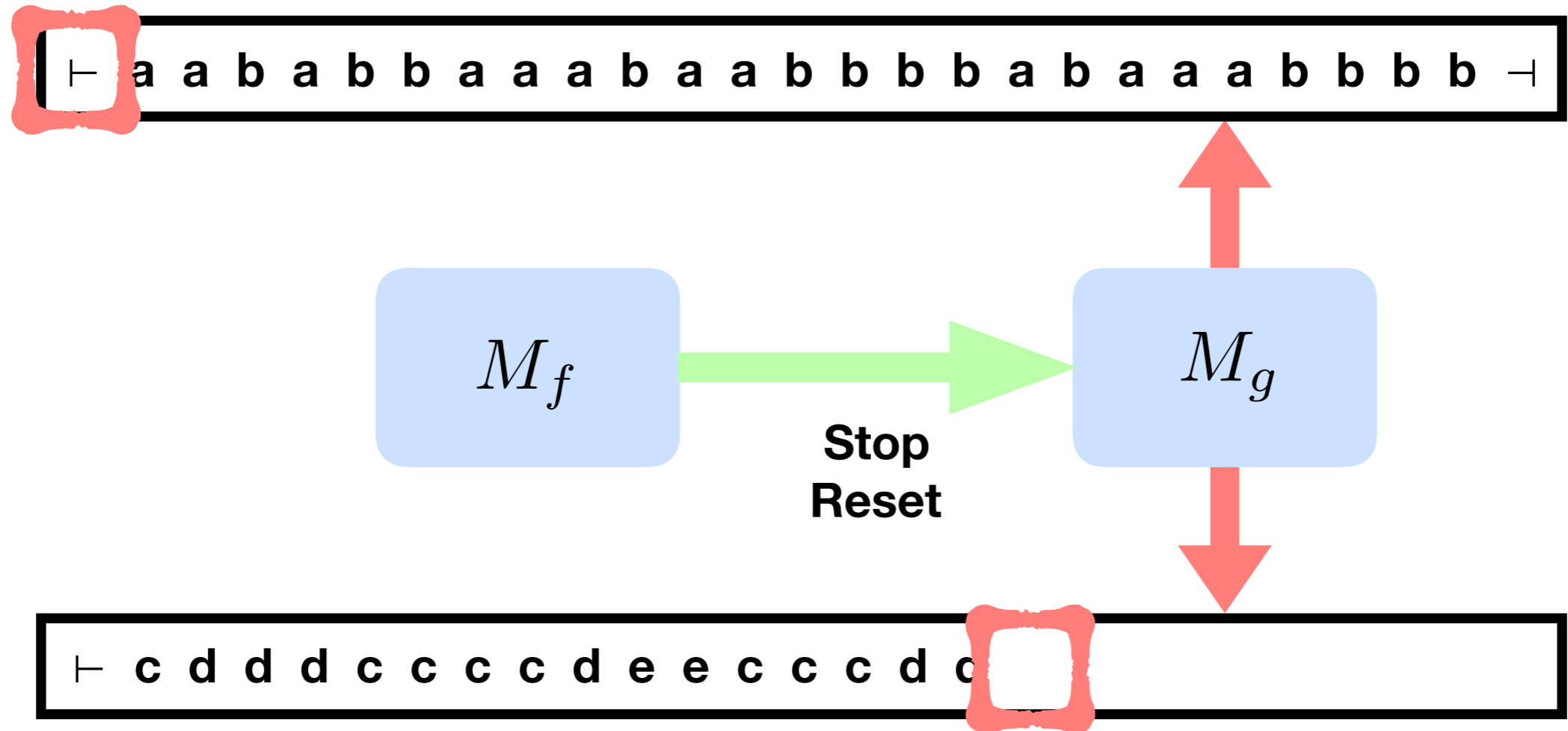
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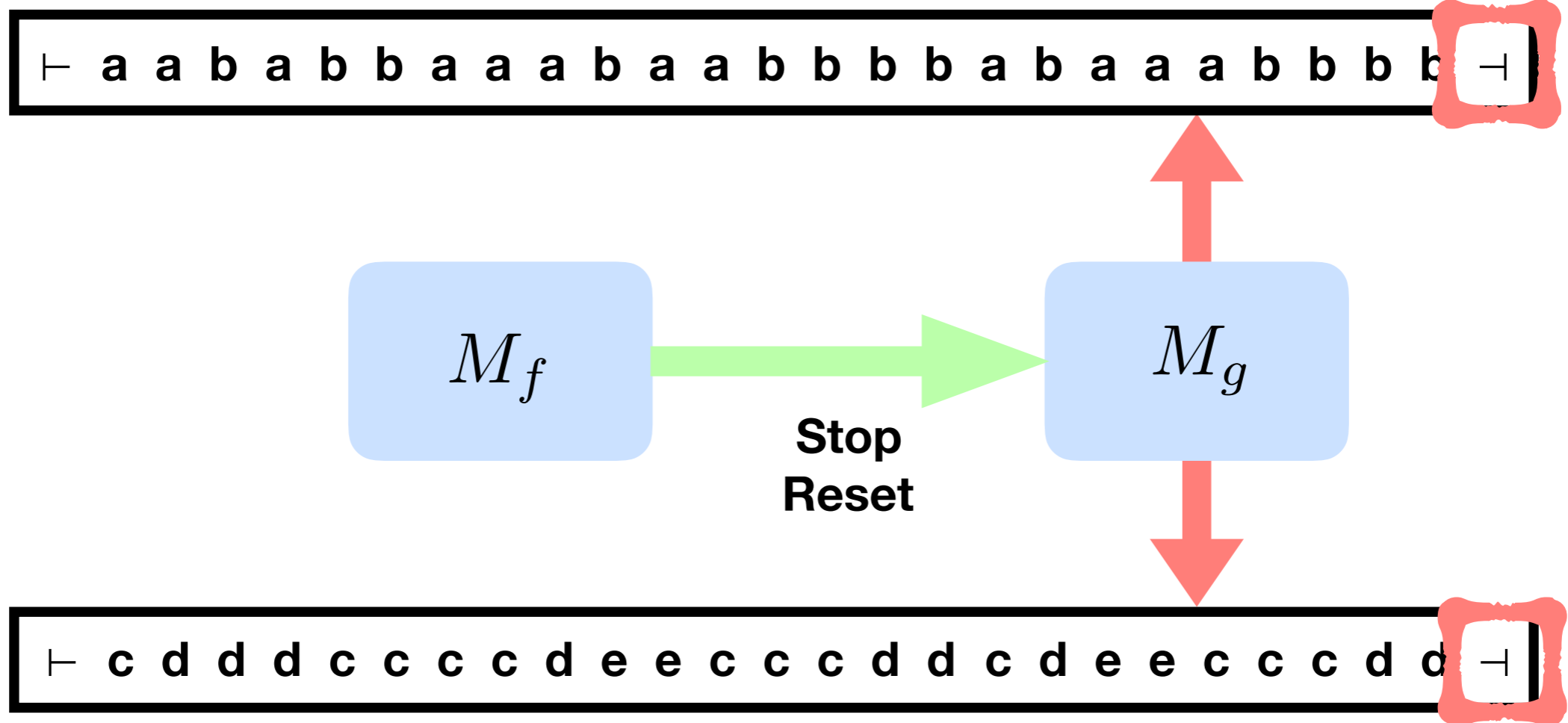
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RTE to 2DFT: Cauchy product

$C ::= d \mid K?C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b a a b b b a b a a a b b b b ⊣

Main Lemma:

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RTE to 2DFT: Cauchy product

$C ::= d \mid K?C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b a a b b b a b a a a b b b b ⊣

DFA
 $\text{dom}(f) \cdot \text{dom}(g)$

Unambiguous

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⊢ a a b a b b a a a b a a b b b a b a a a b b b b ⊢

DFA
 $\text{dom}(f) \cdot \text{dom}(g)$

Unambiguous

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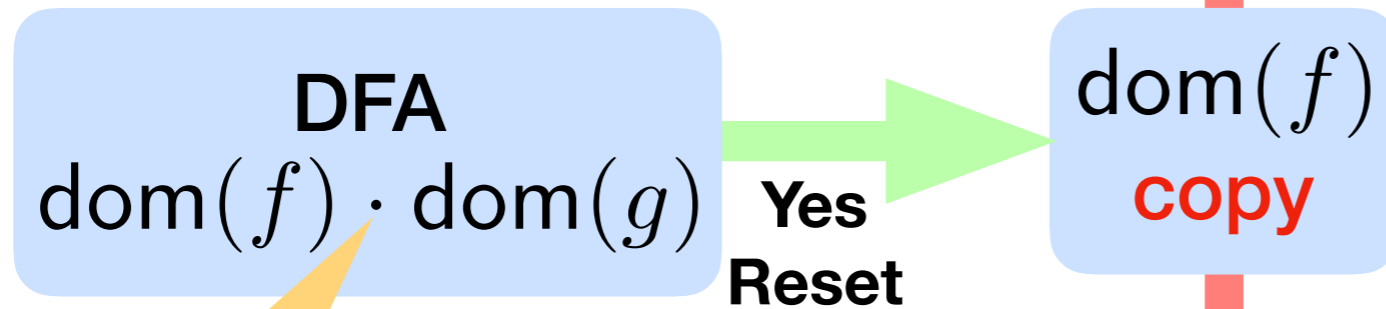
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RTE to 2DFT: Cauchy product

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⊢ a a b a b b a a a b a a b b b a b a a a b b b b ⊣



Unambiguous

⊢

Main Lemma:

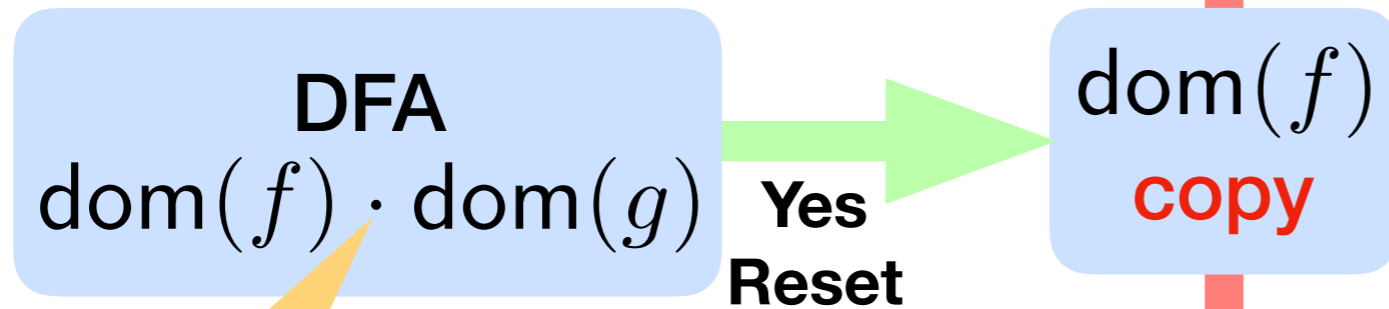
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RTE to 2DFT: Cauchy product

$$C ::= d \mid K?C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

┌ a a b a b b a a a b a a b b b a b a a a b b b b ─┐



Unambiguous

┌ a a b a b b a a a b ─┐

Main Lemma:

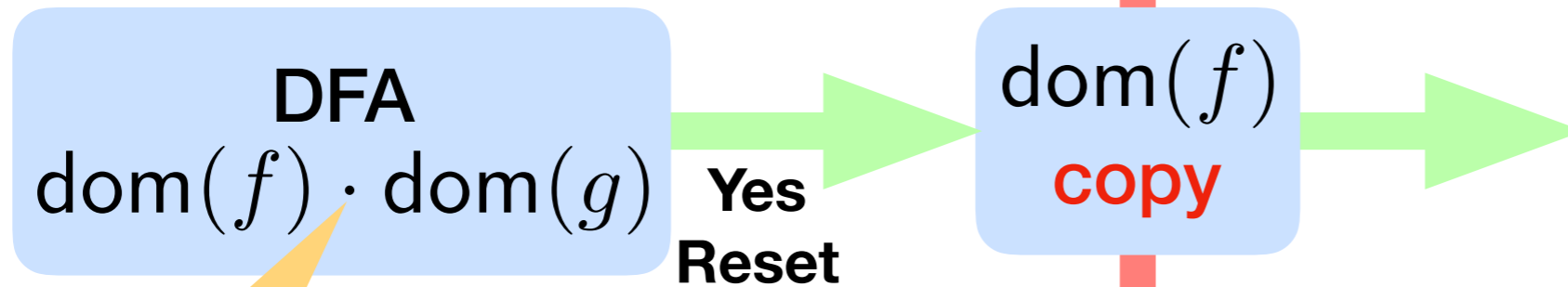
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RTE to 2DFT: Cauchy product

$$C ::= d \mid K?C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

┌ a a b a b b a a a b a a b b b a b a a a b b b b ─┐



Unambiguous

┌ a a b a b b a a a b # ─┐

Main Lemma:

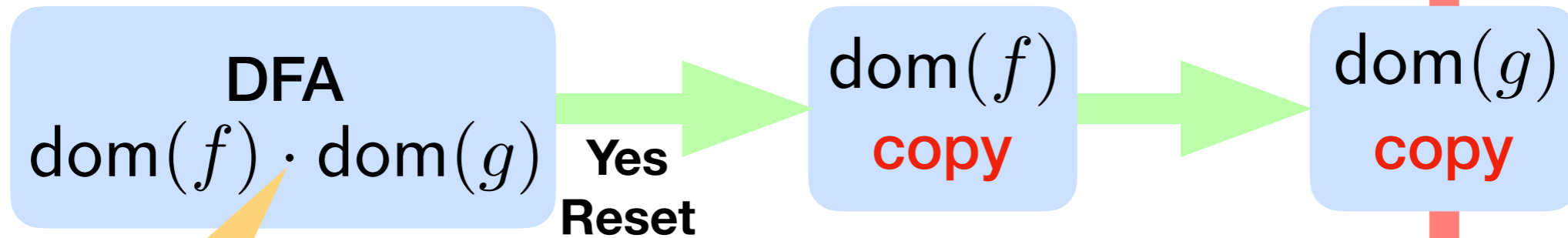
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RTE to 2DFT: Cauchy product

$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b a a b b b a b a a a b b b b ⊣



Unambiguous

⊢ a a b a b b a a a b #

Main Lemma:

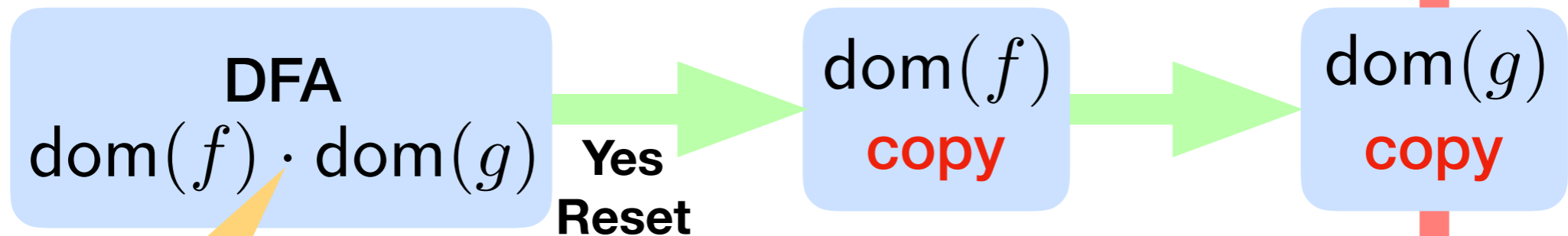
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RTE to 2DFT: Cauchy product

$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b a a b b b a b a a a b b b b ⊢



Unambiguous

⊢ a a b a b b a a a b # a a b b b a b a a a b b b b ⊢

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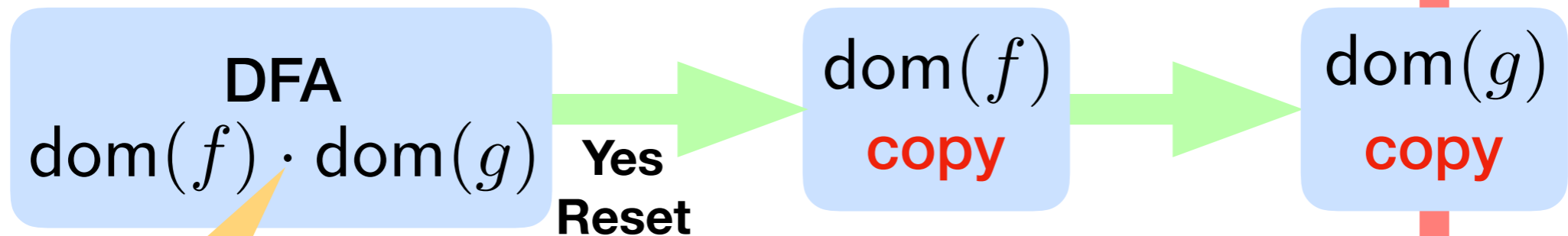
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RTE to 2DFT Cauchy product

2-way Unambiguous Transducer (2UFT)

$$C \mid C \overset{\leftarrow}{\square} C \mid C^{\boxplus} \mid C^{\boxplus} \mid [K, C]^{2\boxplus} \mid [K, C]^{2\boxplus}$$

a a b a b b a a a b a a b b b a b a a a b b b b +



Unambiguous

+ a a b a b b a a a b # a a b b b a b a a a b b b b +

Main Lemma:

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RTE to 2DFT: Cauchy product

$C ::= d \mid K?C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

\vdash a a b a b b a a a b # a a b b b a b a a a b b b b \dashv

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RTE to 2DFT: Cauchy product

$C ::= d \mid K?C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

\vdash a a b a b b a a a b # a a b b b a b a a a b b b b \dashv

M_f

\square

Main Lemma:

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RTE to 2DFT: Cauchy product

$C ::= d \mid K ? C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b # a a b b b a b a a a b b b b ⊣

M_f

⊢ c d d d c c c c d e e c c c d c □ ⊣

Main Lemma:

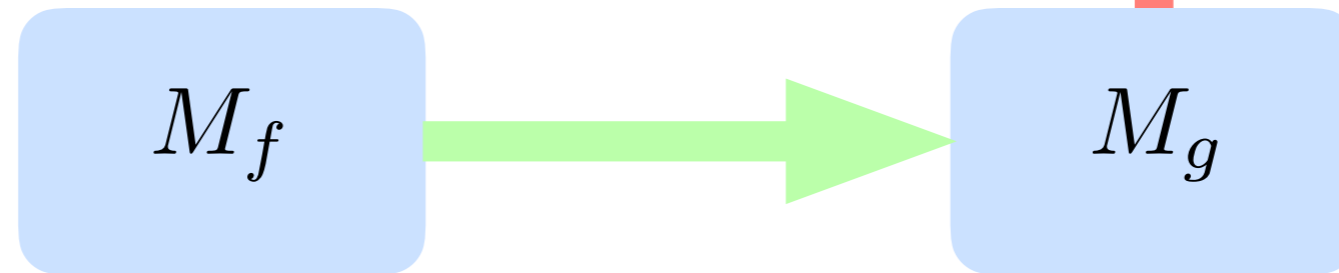
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RTE to 2DFT: Cauchy product

$C ::= d \mid K?C : C \mid C \odot C \mid C \square C \mid C \overleftarrow{\square} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$

⊢ a a b a b b a a a b # a a b b b a b a a a b b b b ⊣



⊢ c d d d c c c c d e e c c c d c □

Main Lemma:

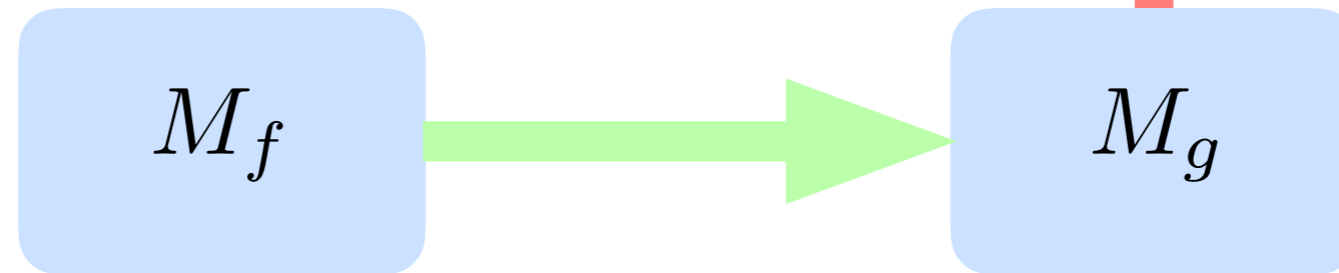
Let f and g be RTEs with $\llbracket f \rrbracket = \llbracket M_f \rrbracket$ and $\llbracket g \rrbracket = \llbracket M_g \rrbracket$ for 2DFTs M_f and M_g . Then, one can construct

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$\vdash a a b a b b a a a b \# a a b b b a b a a a b b b b \vdash$



$\vdash c d d d c c c c d e e c c c d d c d e e c c c d d \vdash$

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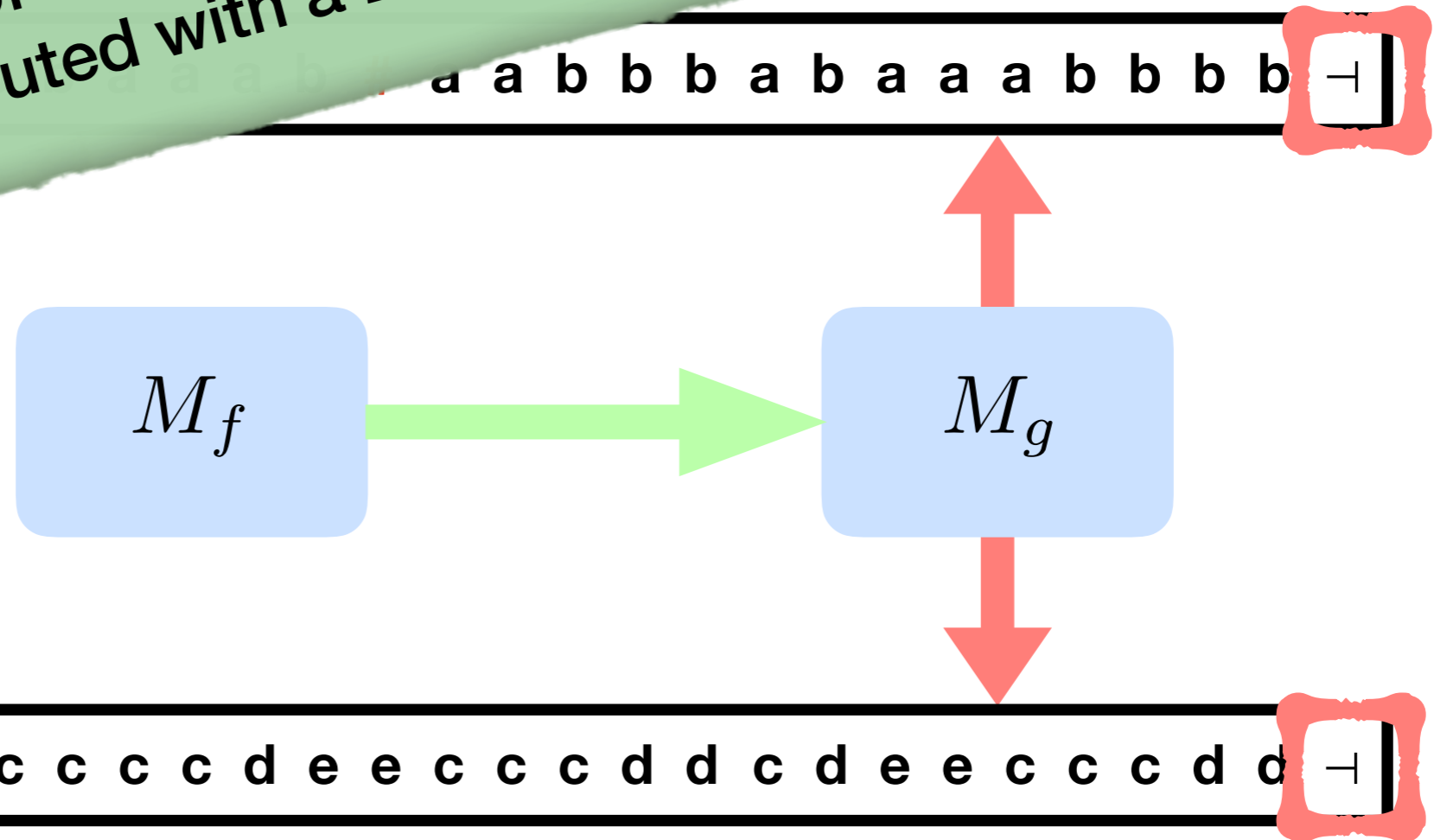
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RTE to 2DFT

Chyhy product

Theorem: (Chytil & Jákl, 1977)
 The composition of a 2UFT with a 2DFT can be computed with a 2DFT

$$C \boxplus | C \overset{\leftarrow}{\boxplus} | [K, C]^{2\boxplus} | [K, C]^{2\overset{\leftarrow}{\boxplus}}$$



Main Lemma:

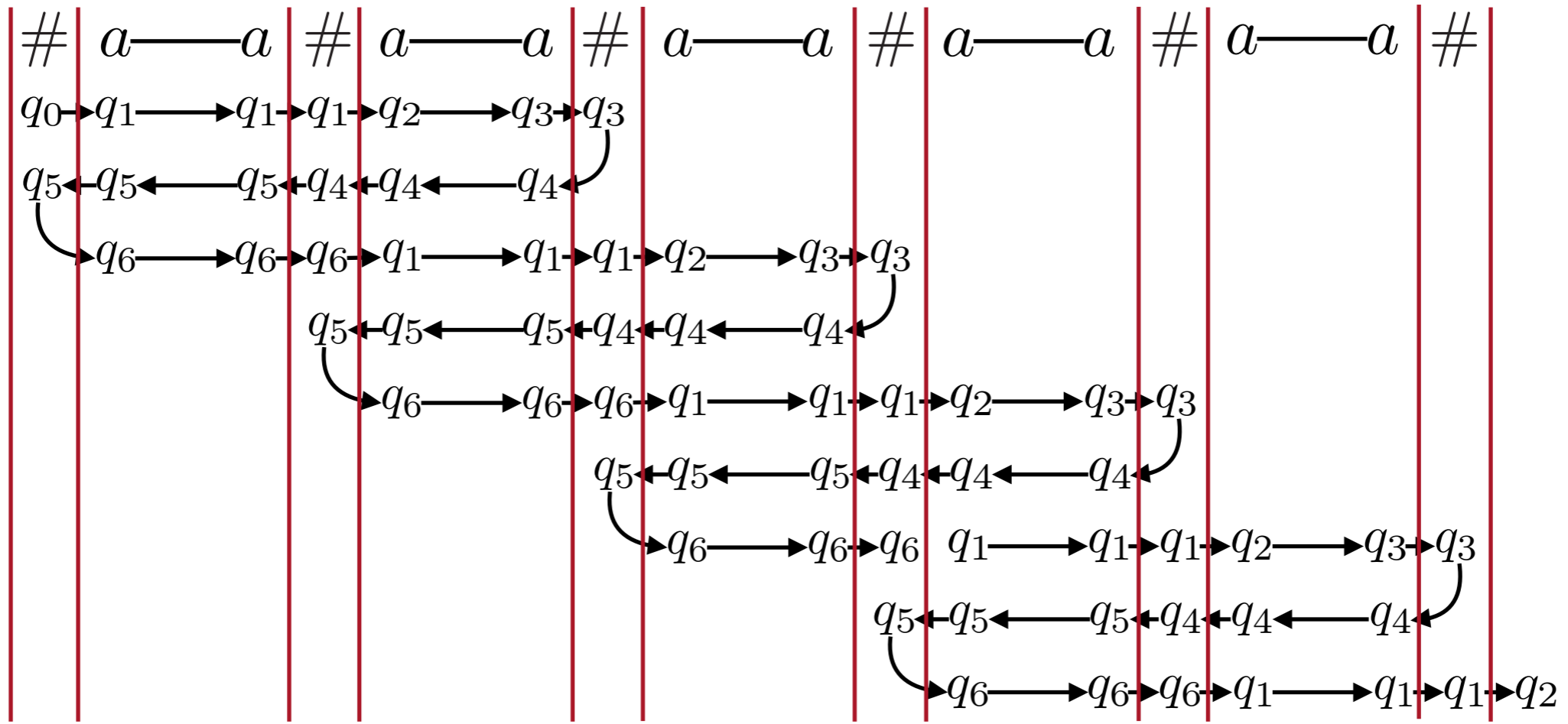
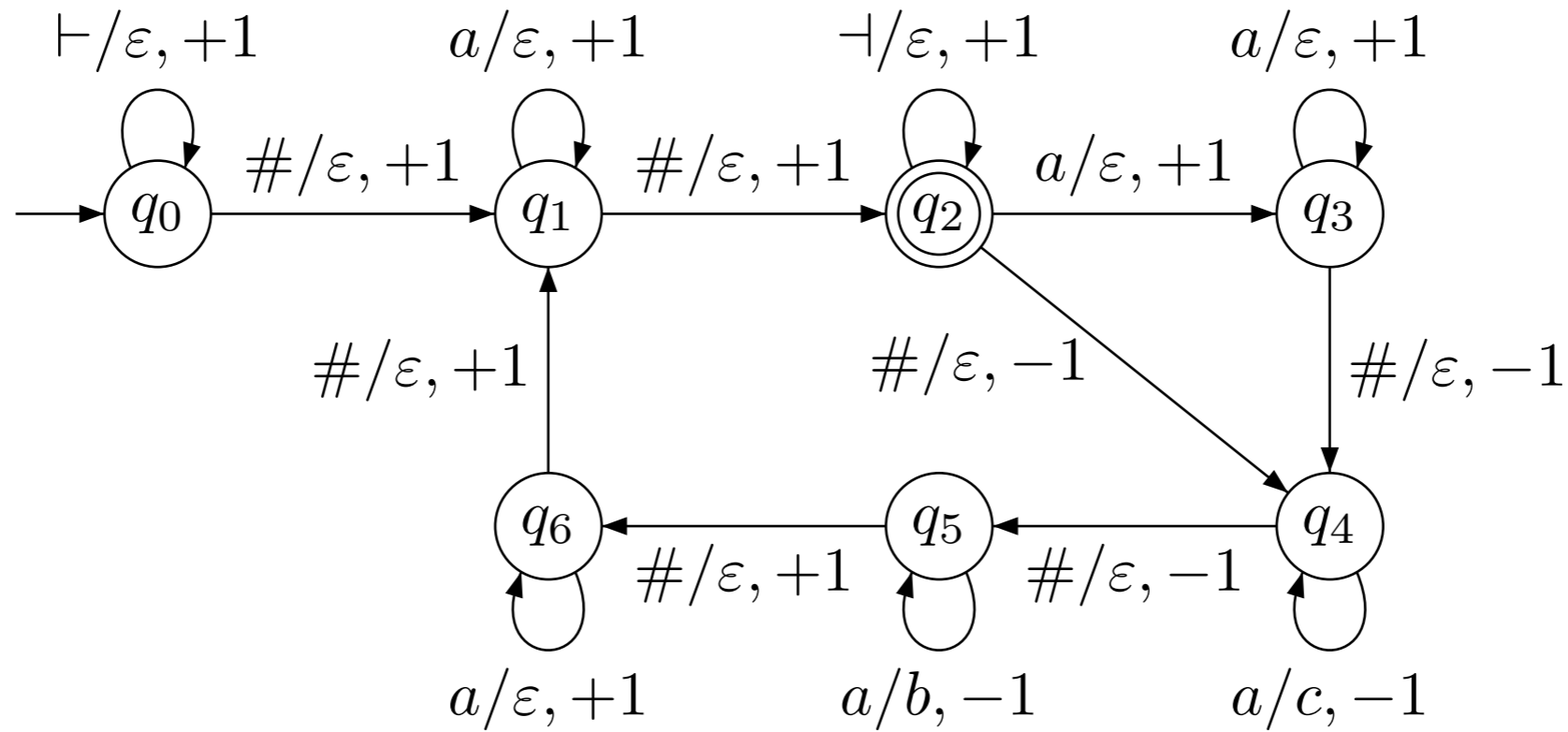
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3. 2DFTs \mathcal{A} , \mathcal{B} such that $\llbracket \mathcal{A} \rrbracket = \llbracket f \boxplus g \rrbracket$ and $\llbracket \mathcal{B} \rrbracket = \llbracket f \overset{\leftarrow}{\boxplus} g \rrbracket$.

Summary

- MSO Transductions
- 2-way Deterministic Transducers (2DFT)
- Regular Transducer Expressions (RTE)
- From RTE to 2DFT
- Transition Monoid
- Unambiguous Forest Factorization
- From 2DFT to RTE
- Conclusion

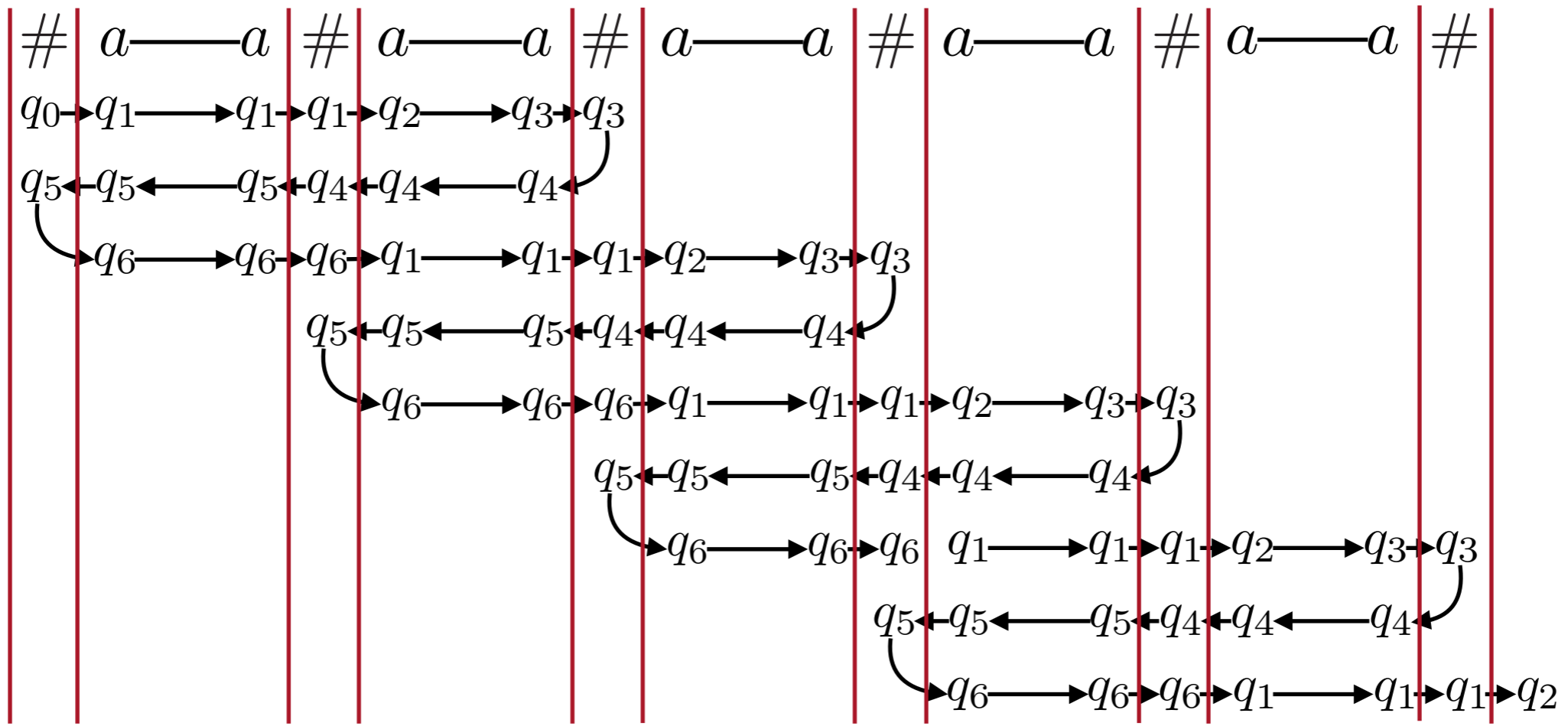
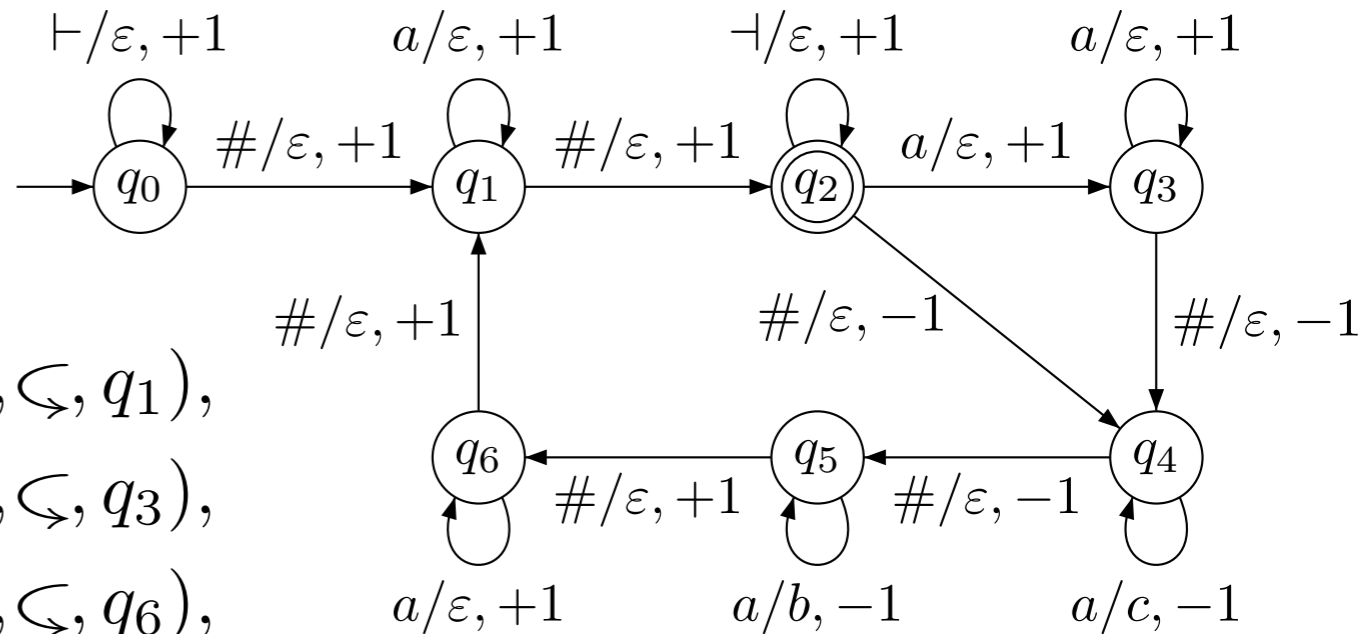
Transition Monoid



Transition Monoid

$$\text{Tr}: \Sigma^* \rightarrow \text{TrM}$$

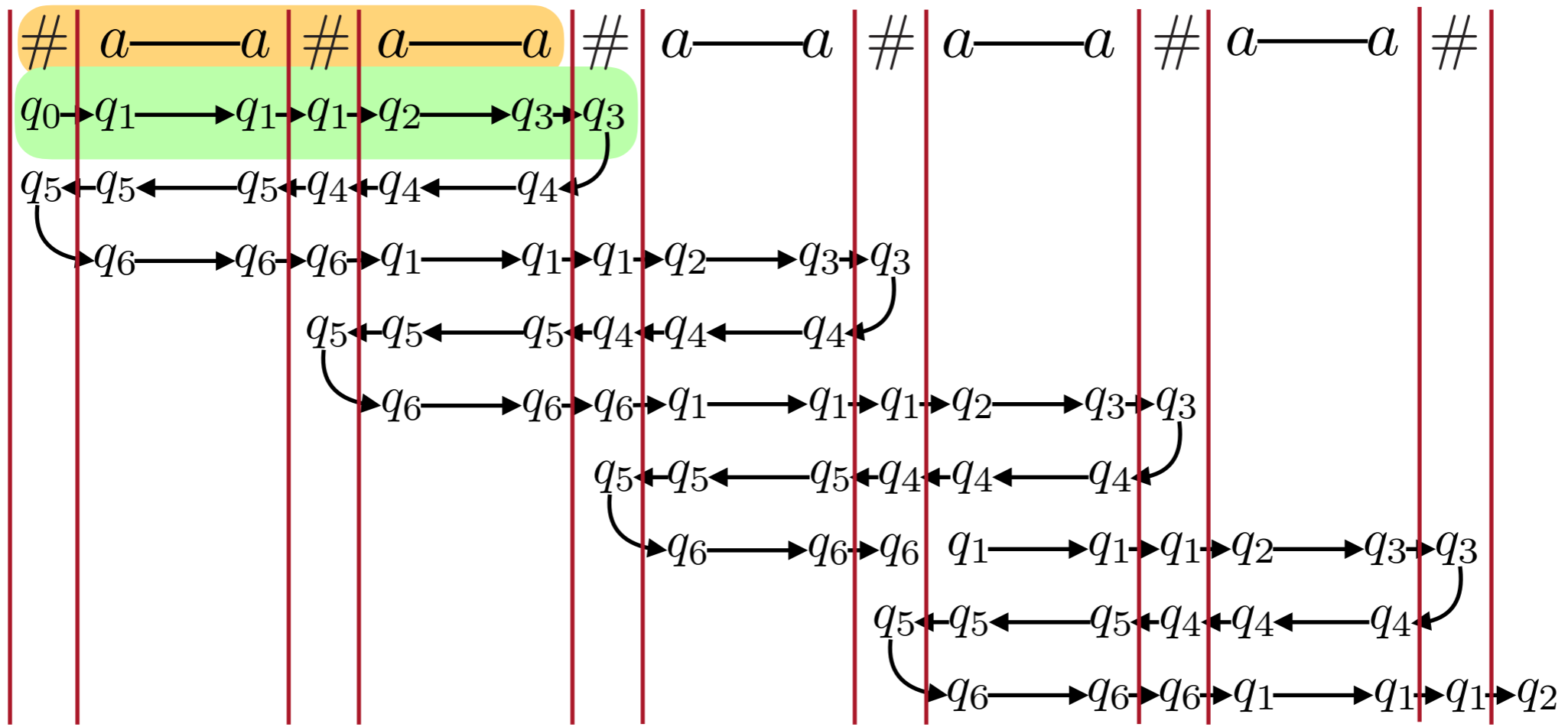
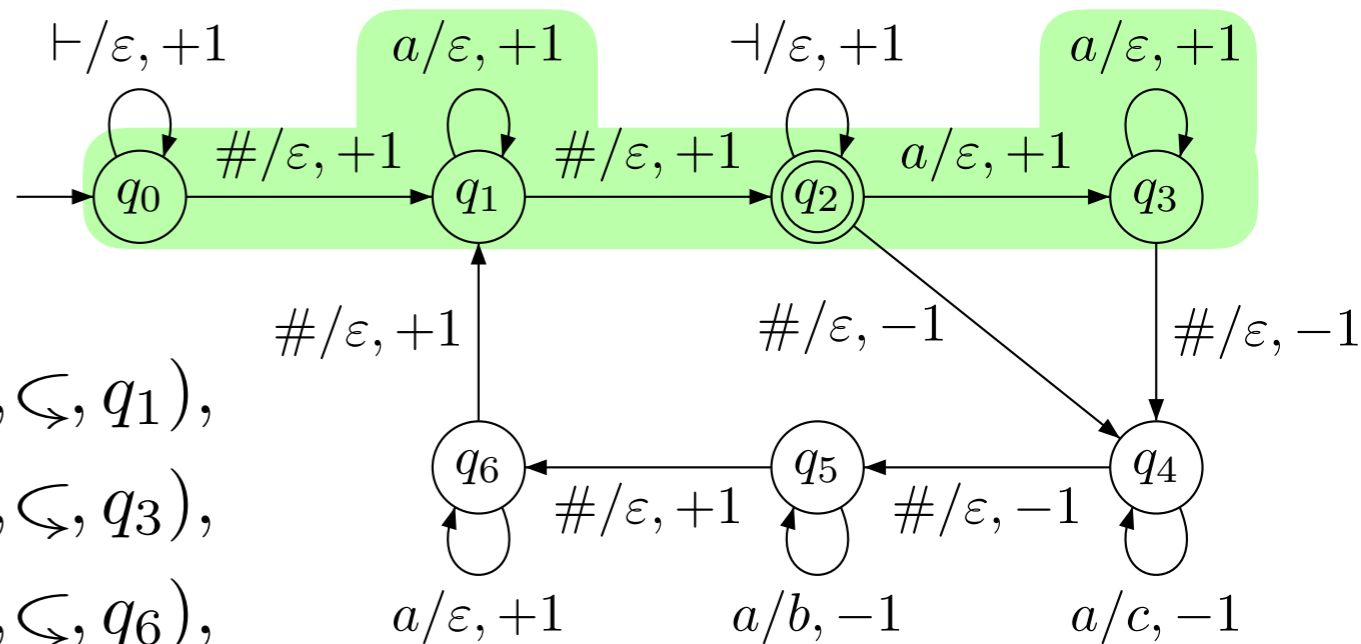
$$\begin{aligned} \text{Tr}(\#a^+ \#a^+) = \{ & (q_0, \rightarrow, q_3), (q_1, \rhd, q_5), (q_1, \hookleftarrow, q_1), \\ & (q_2, \rhd, q_4), (q_2, \hookleftarrow, q_3), (q_3, \rhd, q_4), (q_3, \hookleftarrow, q_3), \\ & (q_4, \rhd, q_5), (q_4, \hookleftarrow, q_1), (q_5, \rightarrow, q_1), (q_5, \hookleftarrow, q_6), \\ & (q_6, \rightarrow, q_3), (q_6, \hookleftarrow, q_6) \} \end{aligned}$$



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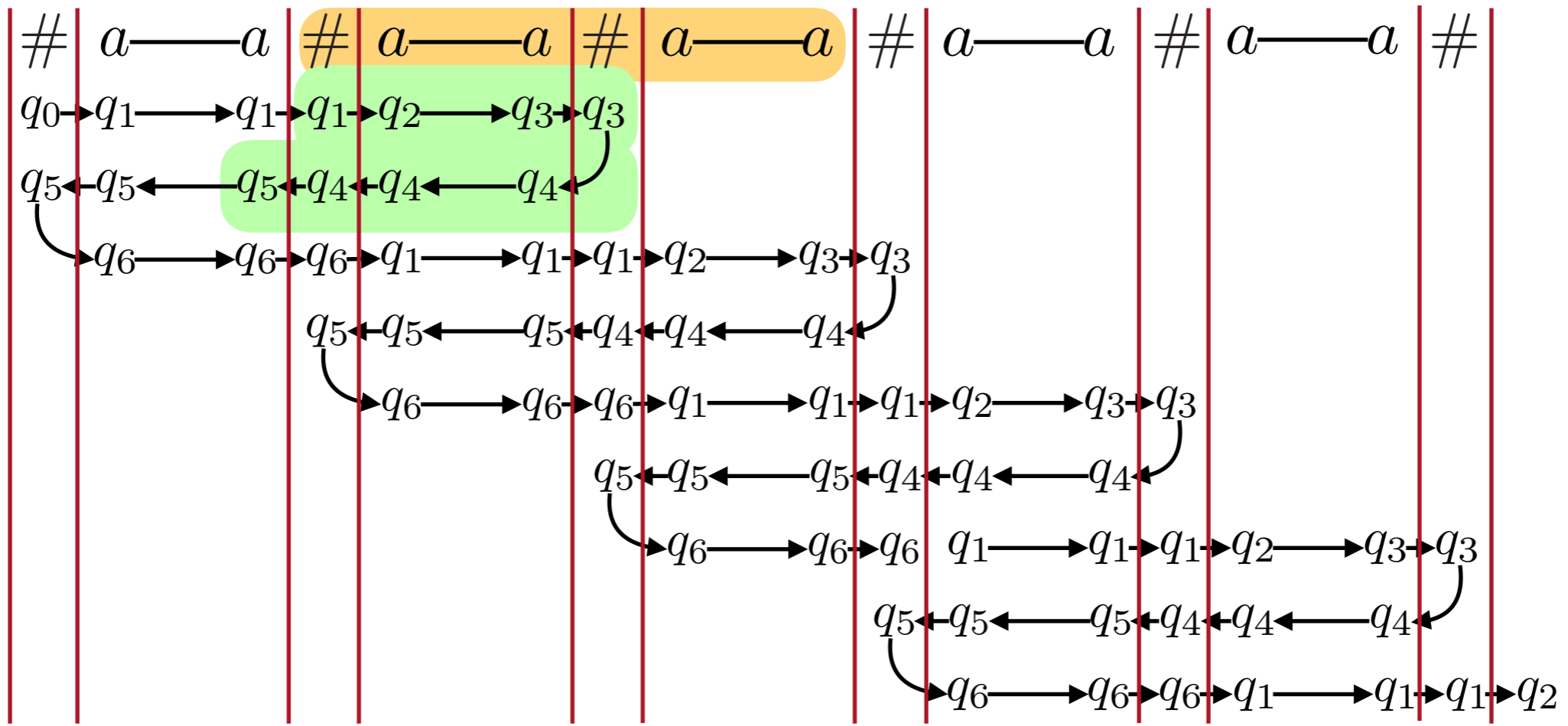
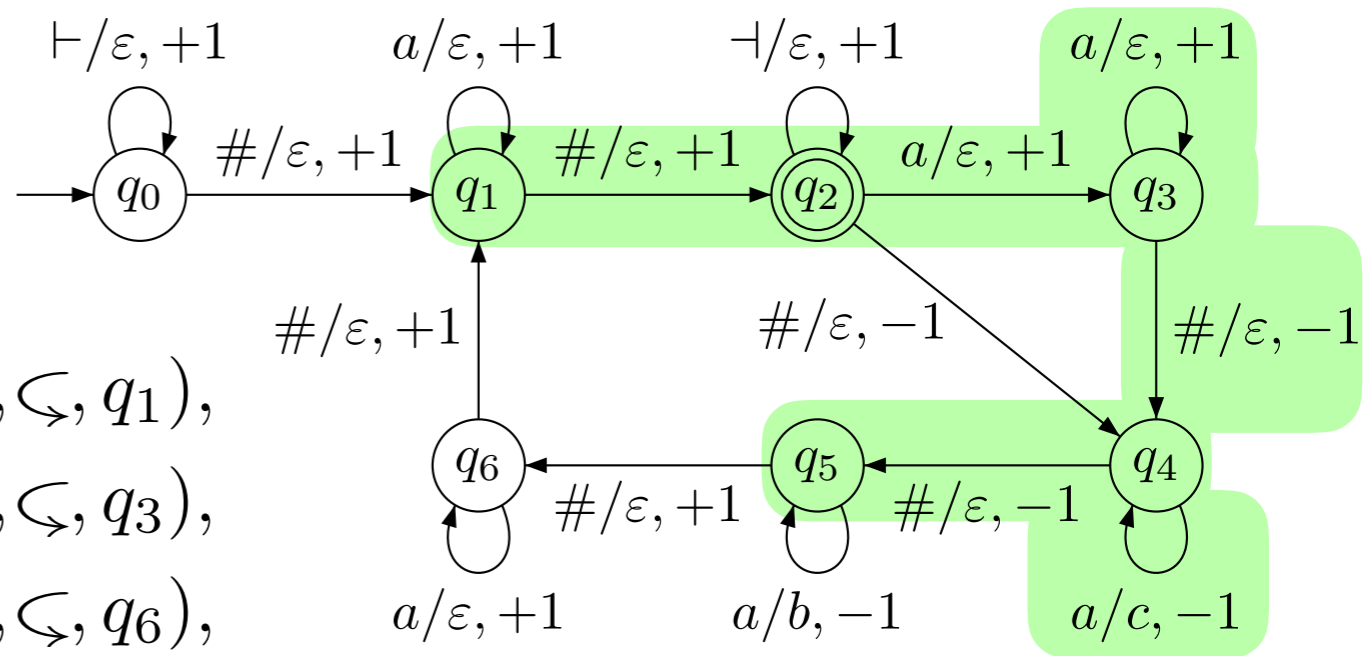
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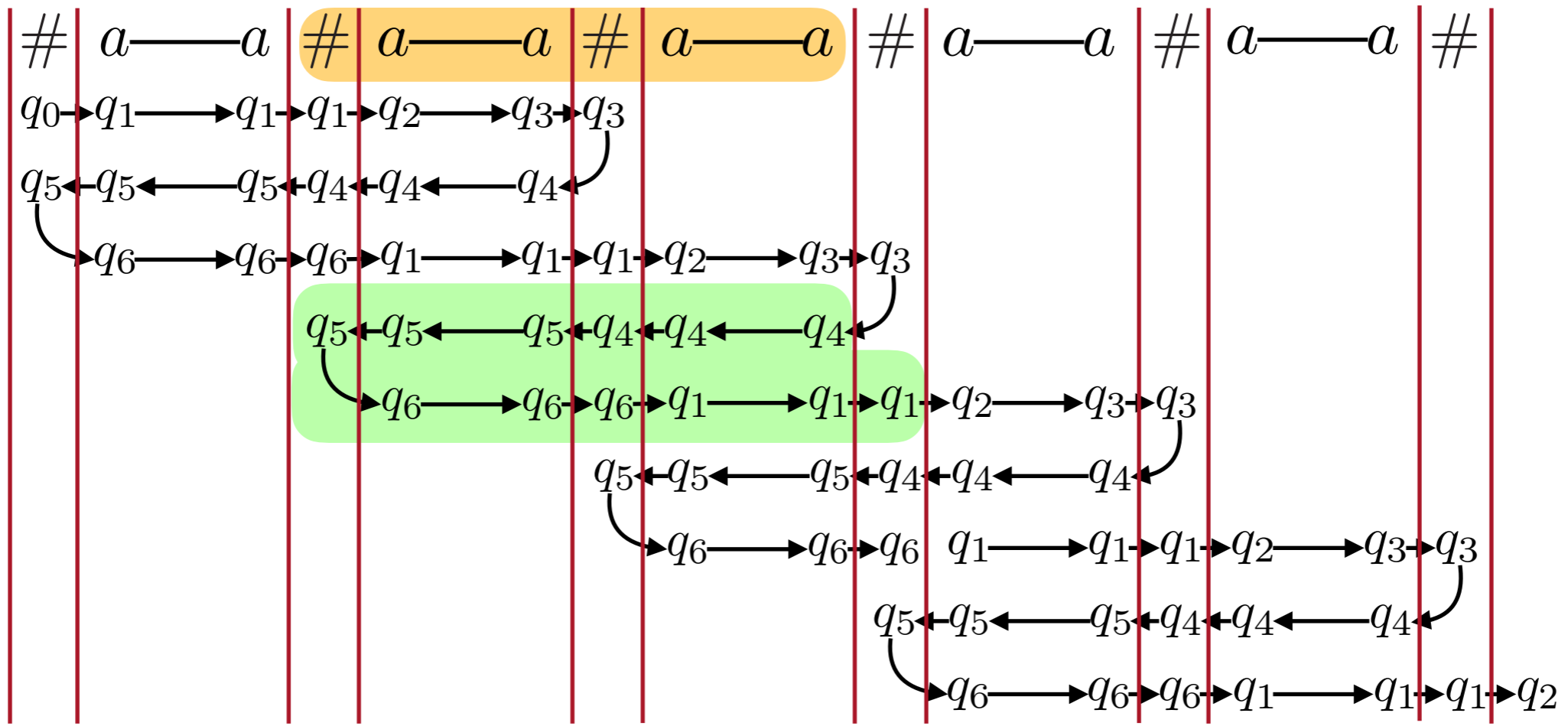
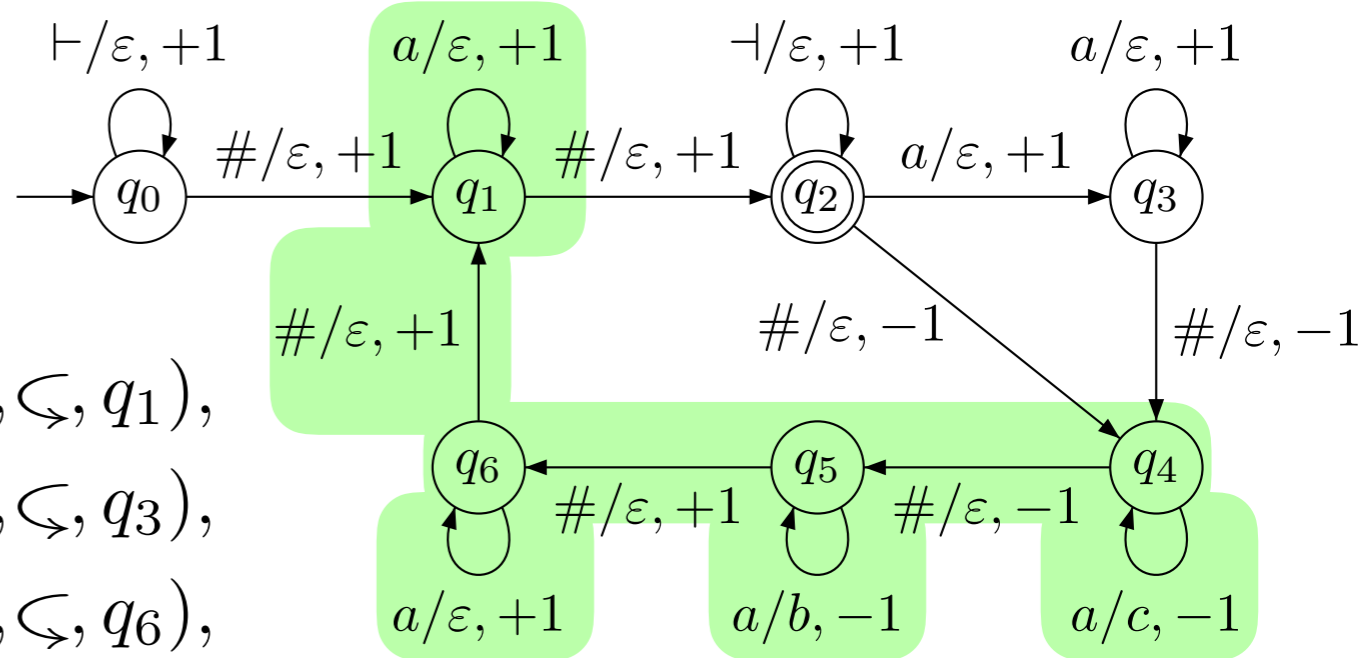
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Summary

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Forest Factorization

$\varphi: \Sigma^* \rightarrow S$ morphism

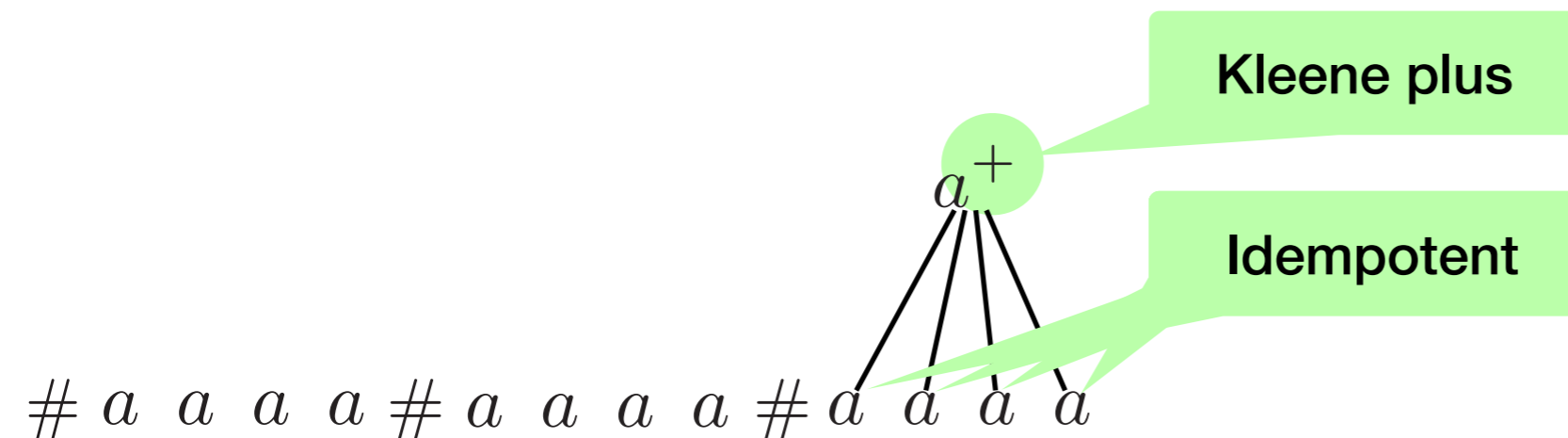
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Concatenation

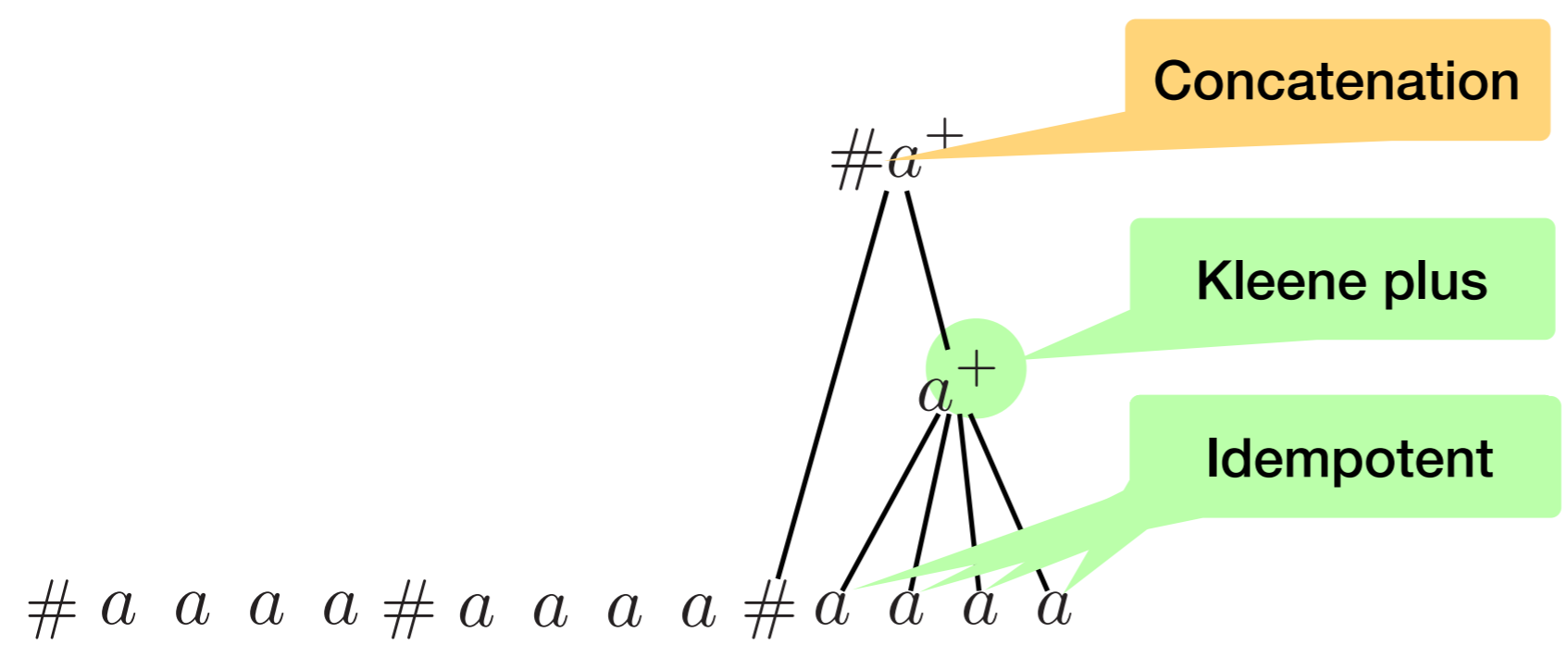
Kleene plus

Idempotent

$\#a^+$

a^+

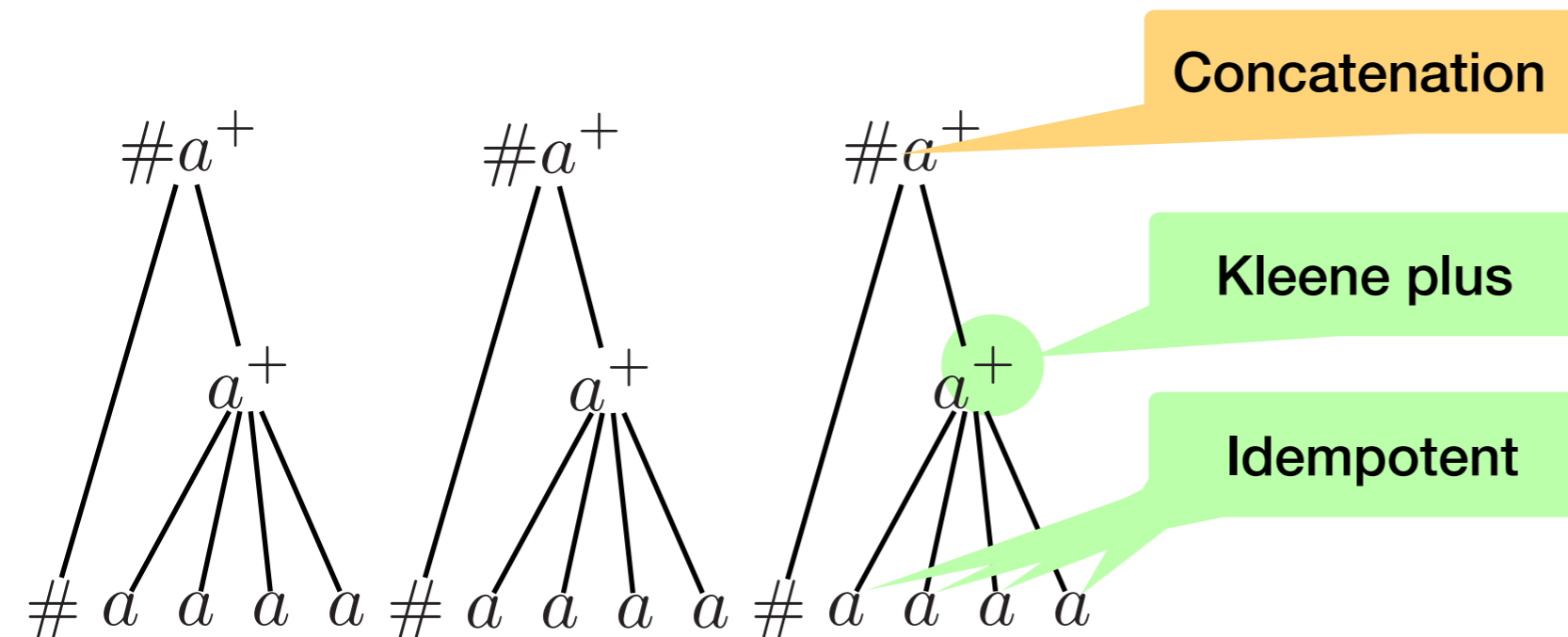
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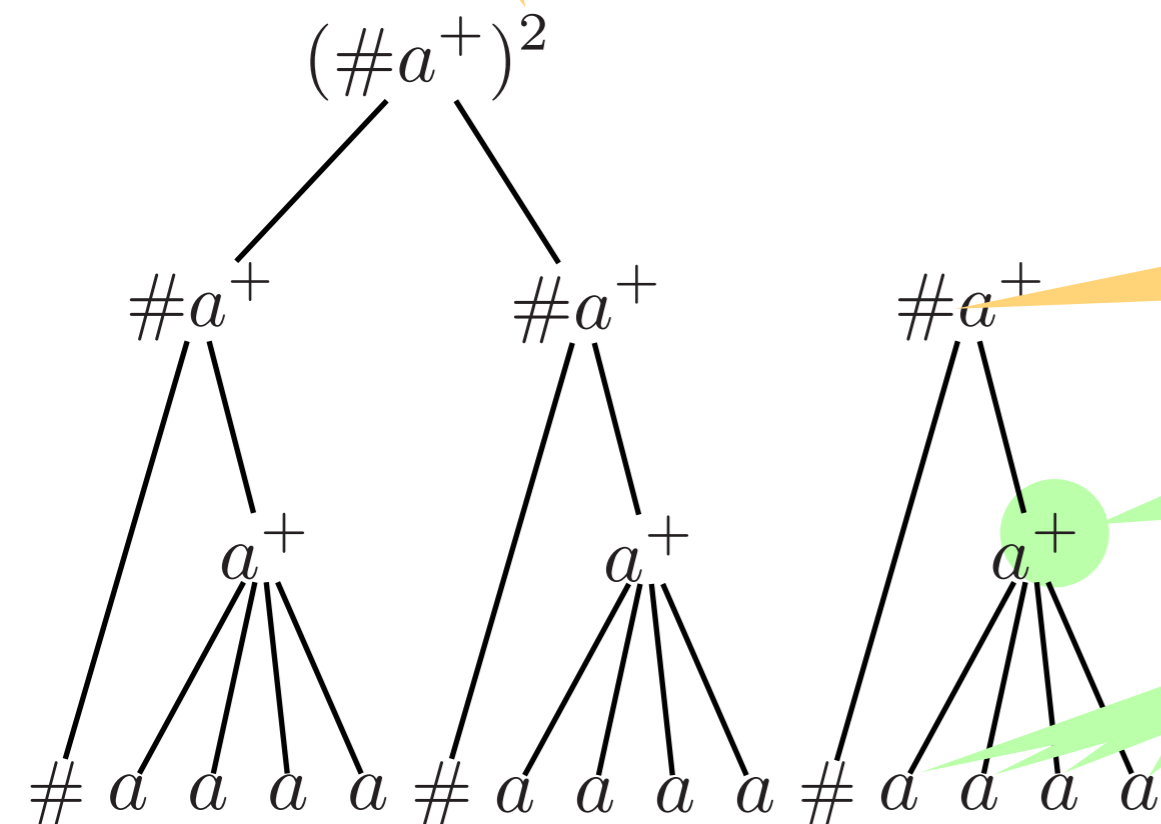
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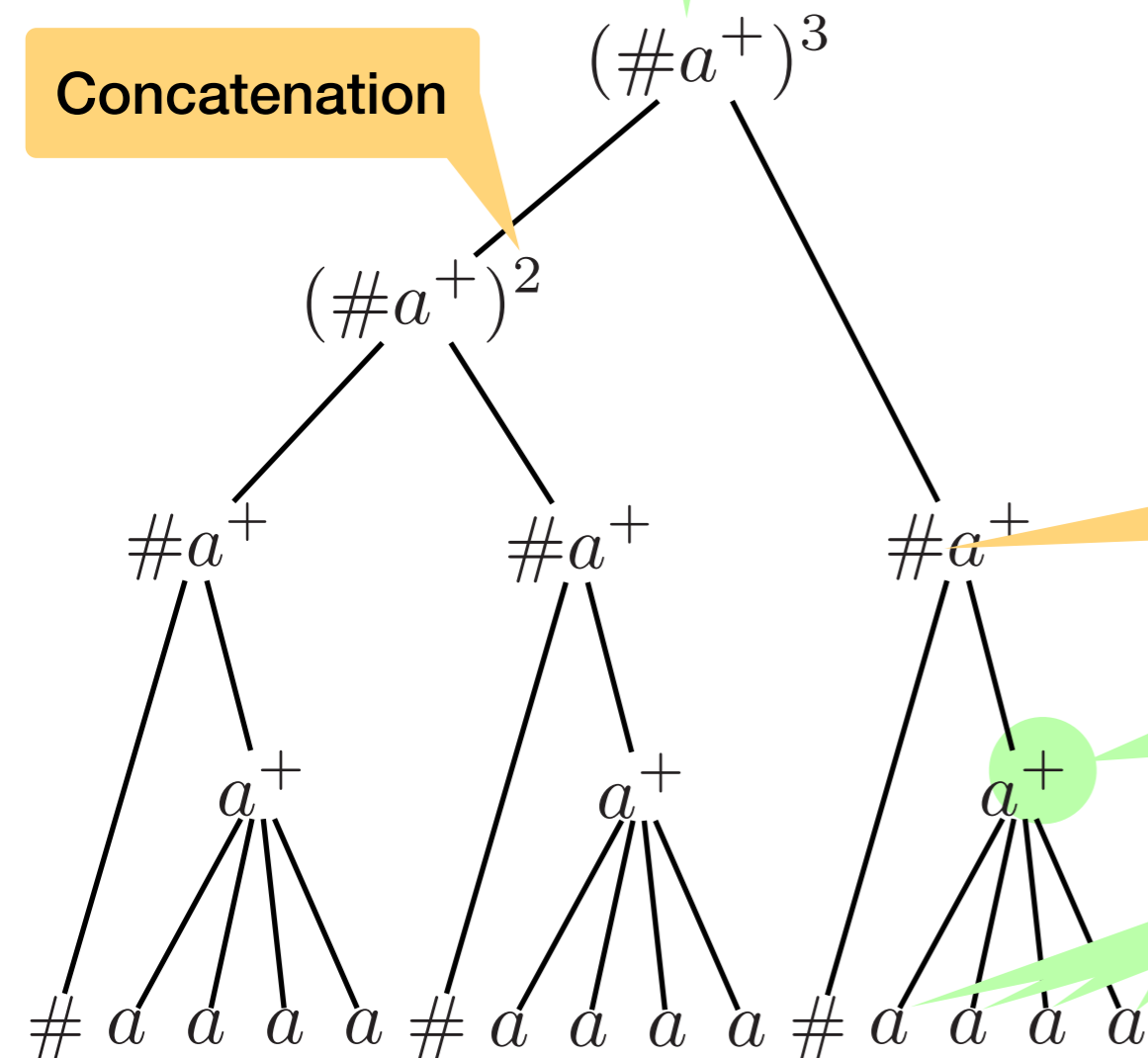
Idempotent

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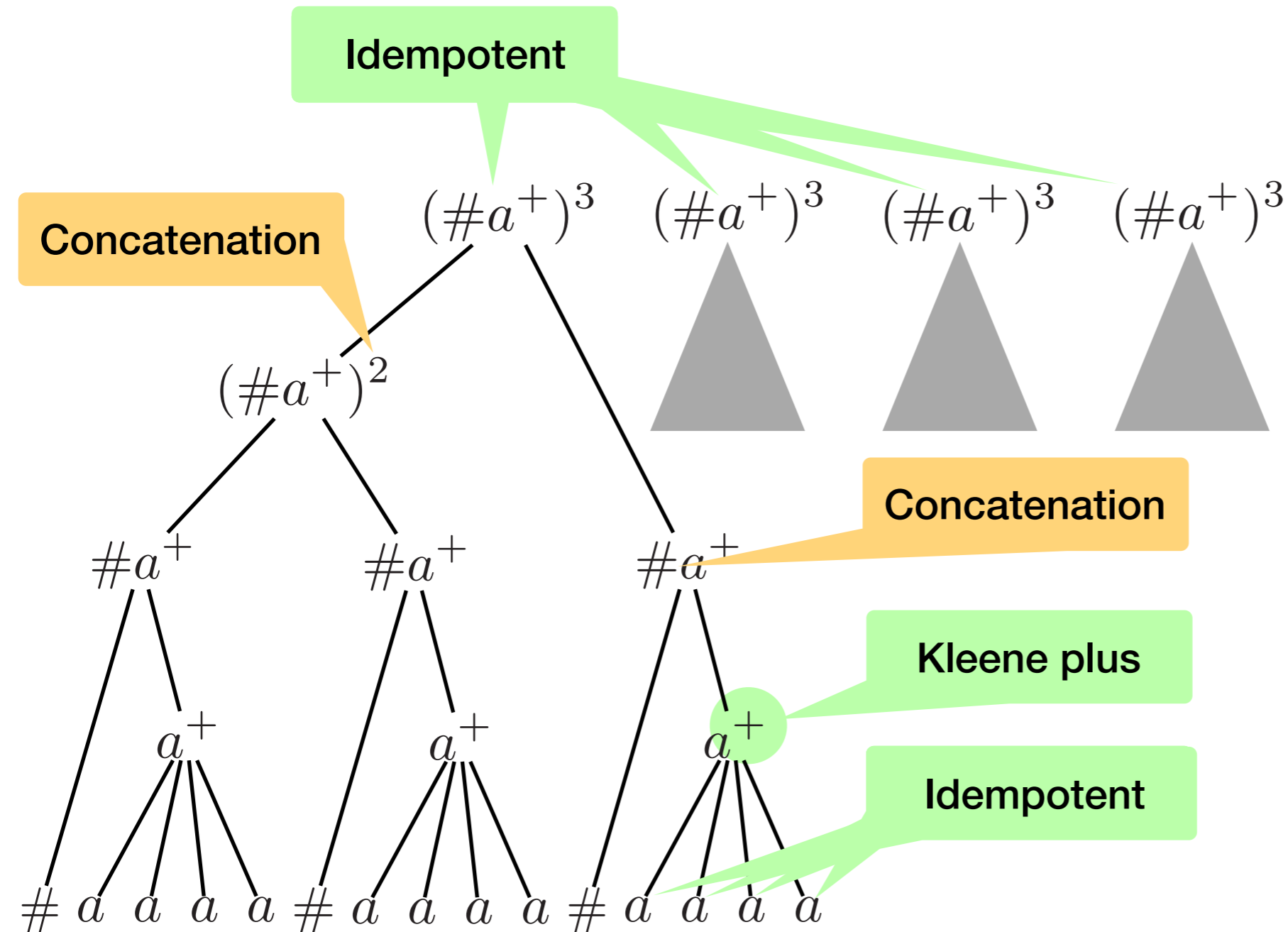
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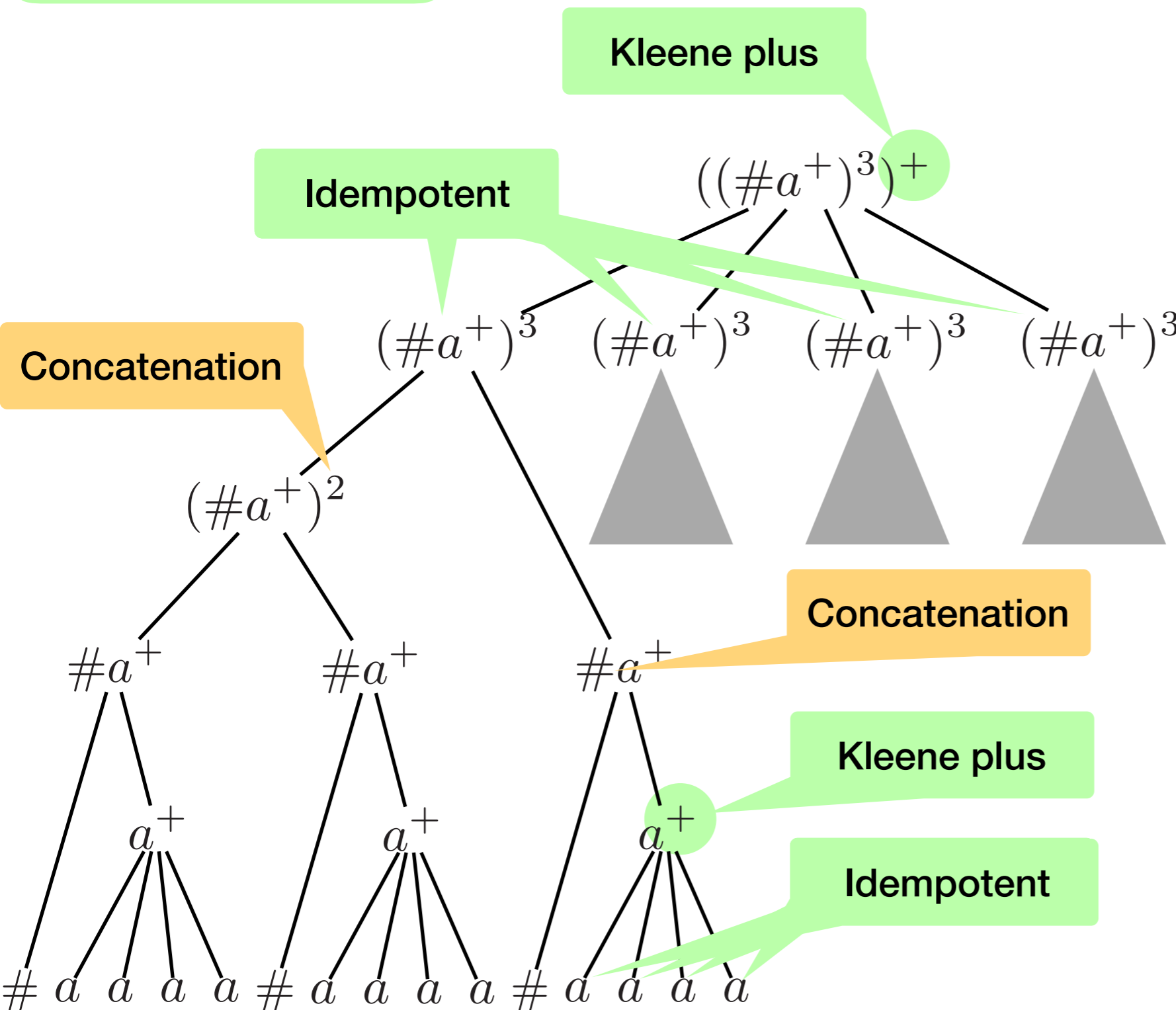
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Concatenation

$((\#a^+)^3)^+ \#a^+ \#$

Kleene plus

$((\#a^+)^3)^+ \#a^+$

Idempotent

$((\#a^+)^3)^+$

$\#a^+$

Concatenation

$(\#a^+)^3$

$(\#a^+)^3$

$(\#a^+)^3$

$(\#a^+)^3$

$(\#a^+)^2$

Concatenation

$\#a^+$

$\#a^+$

$\#a^+$

Kleene plus

Idempotent

a^+

a^+

a^+

$\# a a a a \# \# a a a a \# \# a a a a \#$

Forest Factorization

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Idempotent

$((\#a^+)^3)^+$

Concatenation

$(\#a^+)^3$

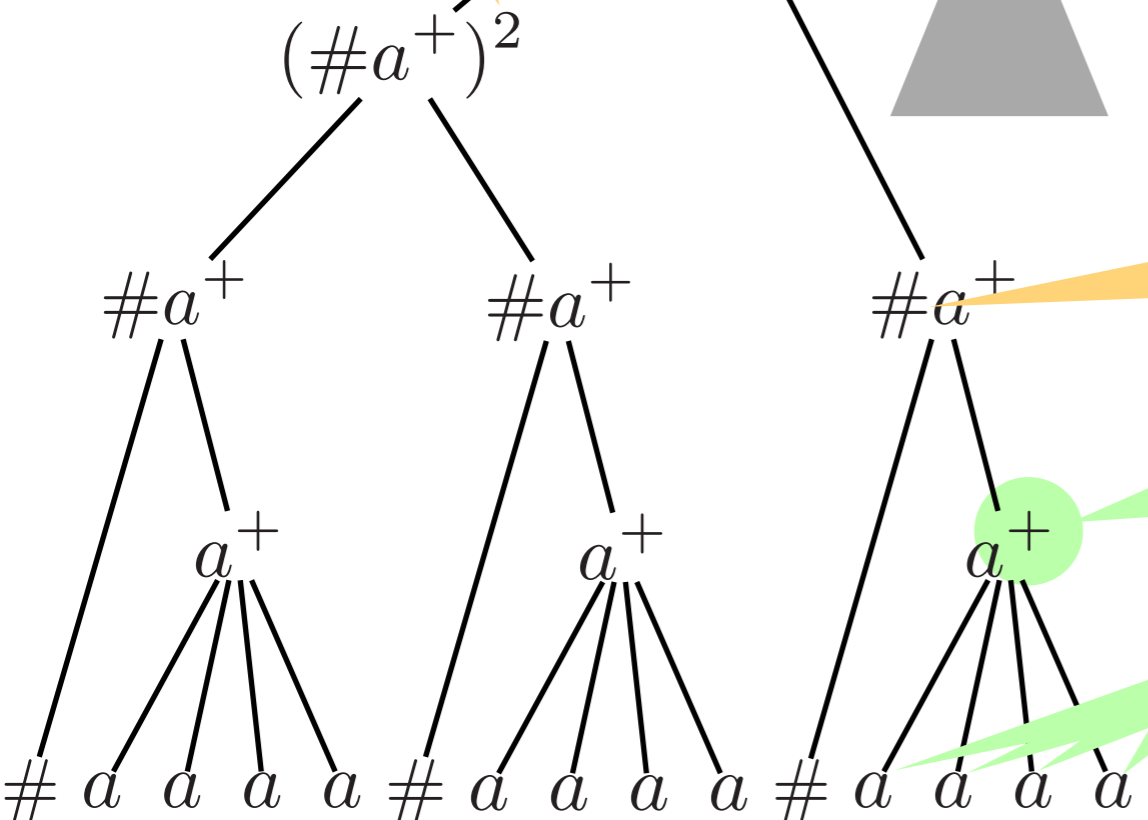
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$(\#a^+)^3$

$(\#a^+)^3$

$\#a^+$

Concatenation



Theorem: (Simon 1990)
 Every word can be factorized (parsed)
 with a tree of height at most $9|S|$

Good Rational Expressions

$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

F is *good* wrt. $\varphi: \Sigma^* \rightarrow S$ morphism to a finite monoid S if

1. F is unambiguous
2. If E is a subexpression of F then $\varphi(\mathcal{L}(E)) = \{s_E\}$ is a singleton
3. If E^+ is a subexpression of F then s_E is an idempotent.

Theorem: (PG, S.Krishna)

For each $s \in S$, there is an ε -free *good* rational expression F_s such that

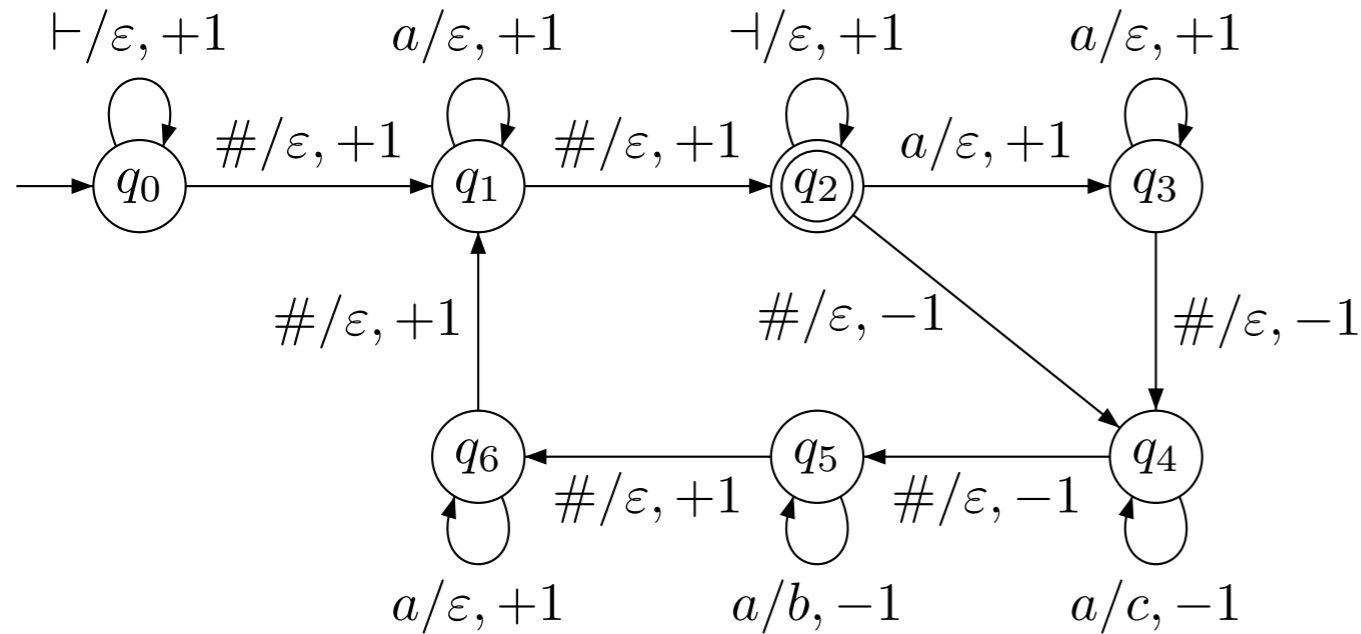
$$\mathcal{L}(F_s) = \varphi^{-1}(s) \setminus \{\varepsilon\} \subseteq \Sigma^+$$

Therefore, $G = \varepsilon \cup \bigcup_{s \in S} F_s$ is an *unambiguous* rational expression over Σ such that $\mathcal{L}(G) = \Sigma^*$.

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$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

$$\text{Tr}: \Sigma^* \rightarrow \text{TrM}$$

$$F_1 = \#a^+\#$$

$$F_2 = \#a^+\#a^+\#$$

$$F_3 = ((\#a^+)^3)^+\#$$

$$F_4 = ((\#a^+)^3)^+\#a^+\#$$

$$F_5 = ((\#a^+)^3)^+\#a^+\#a^+\#$$

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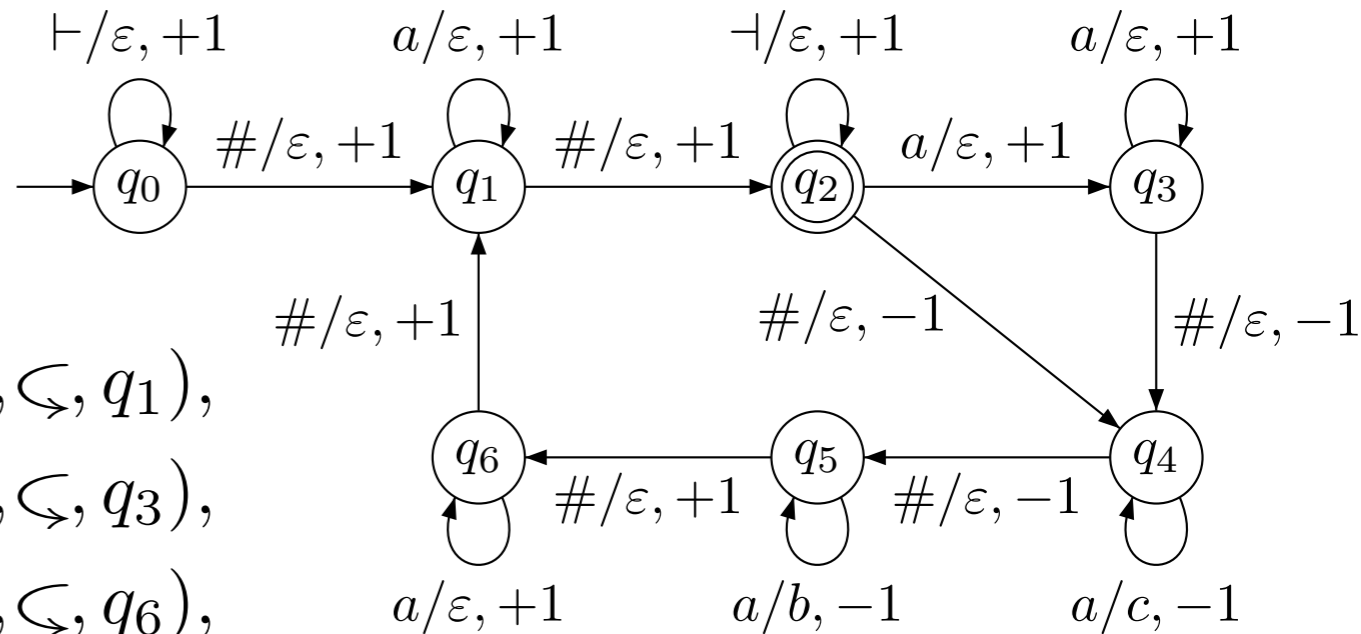
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2DFA to RTE

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Main Lemma:

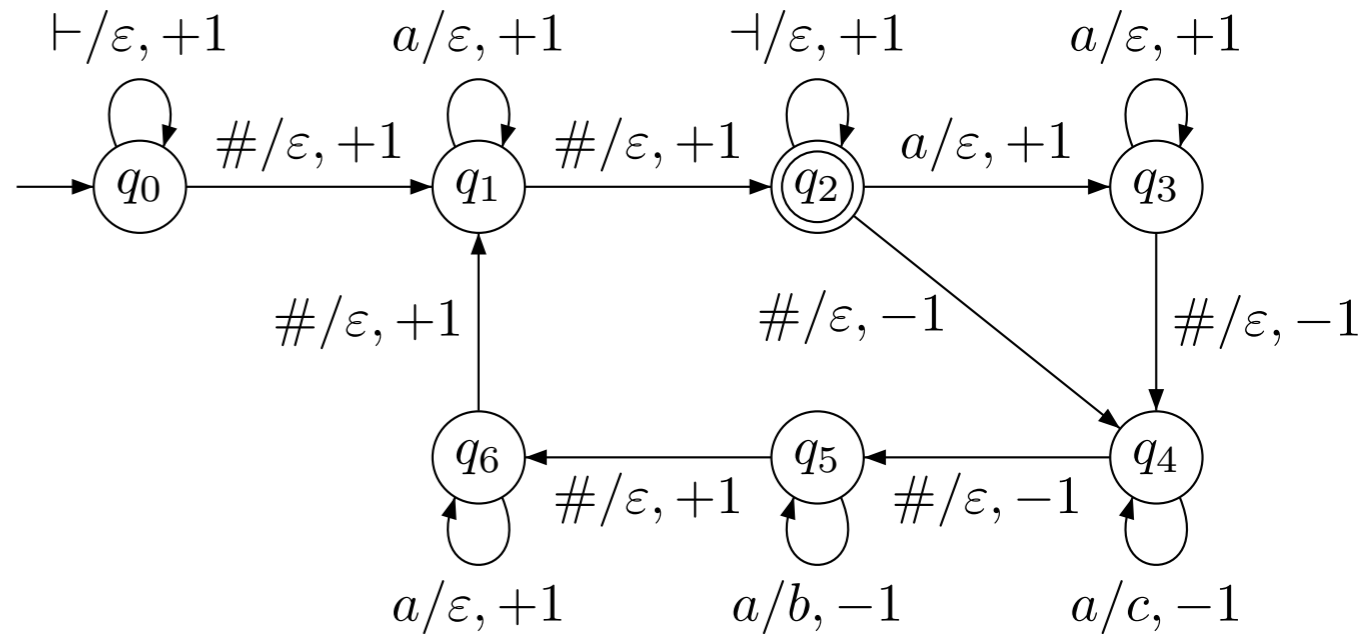
$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

Let F be an ε -free Tr-good rational expression with $\text{Tr}(F) = s_F$. We can construct a map $C_F: s_F \rightarrow \text{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\text{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $\llbracket C_F(x) \rrbracket(u)$ is the output produced by \mathcal{A} when running step x on u (i.e., running \mathcal{A} on u from p to q following direction d).

2DFA to RTE: atomic

$\text{Tr}: \Sigma^* \rightarrow \text{TrM}$



$\text{Tr}(a) = \{(q_1, \rightarrow, q_1), \dots, (q_4, \leftarrow, q_4), \dots, (q_5, \leftarrow, q_5), \dots\}$

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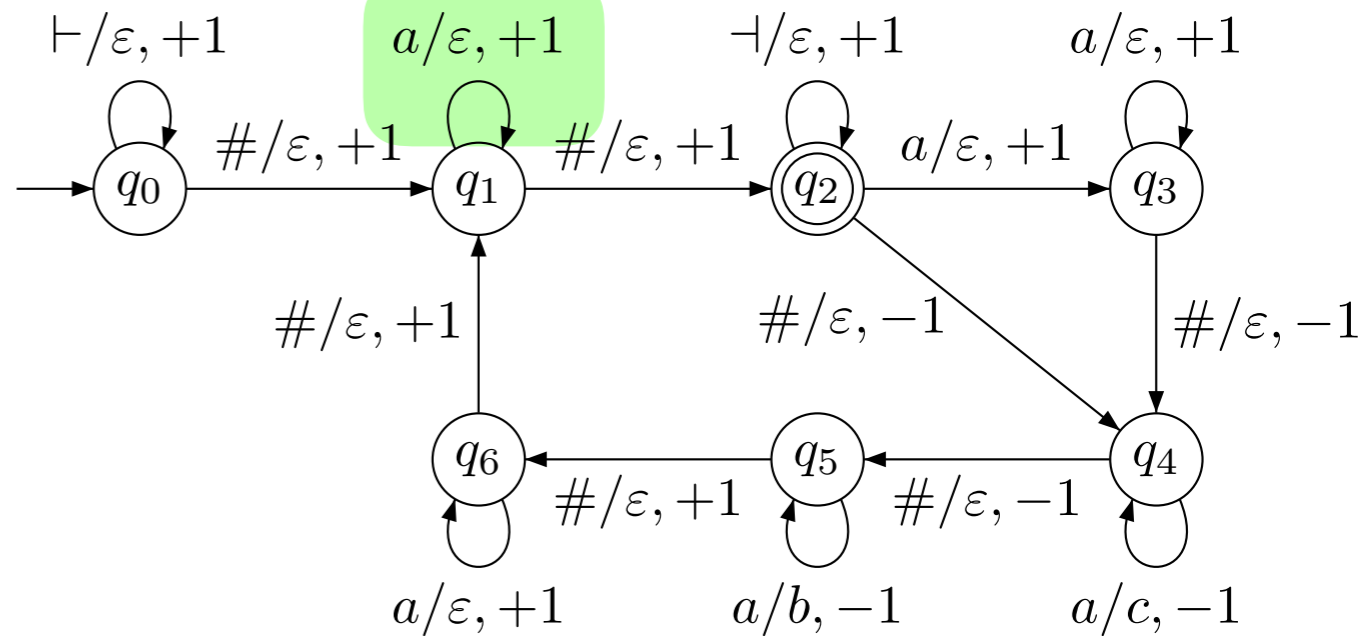
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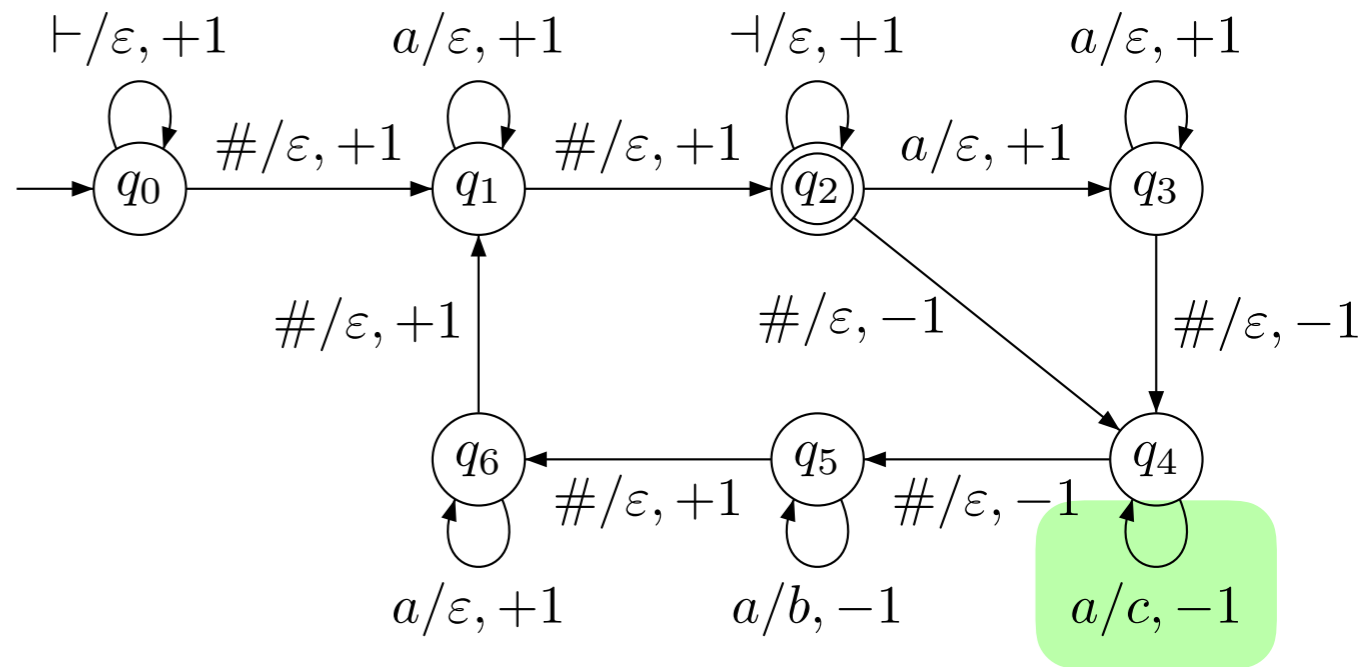
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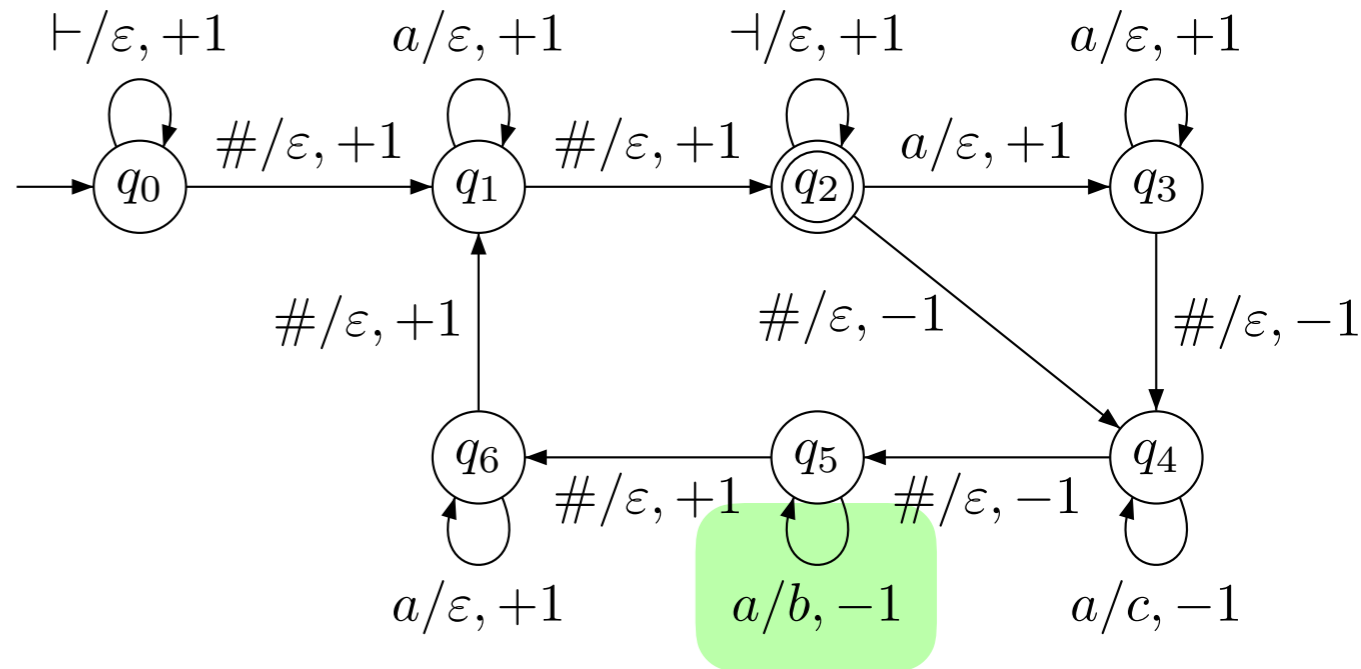
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$$C_a(q_5, \leftarrow, q_5) = (a ? b : \perp)$$



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Main Lemma:

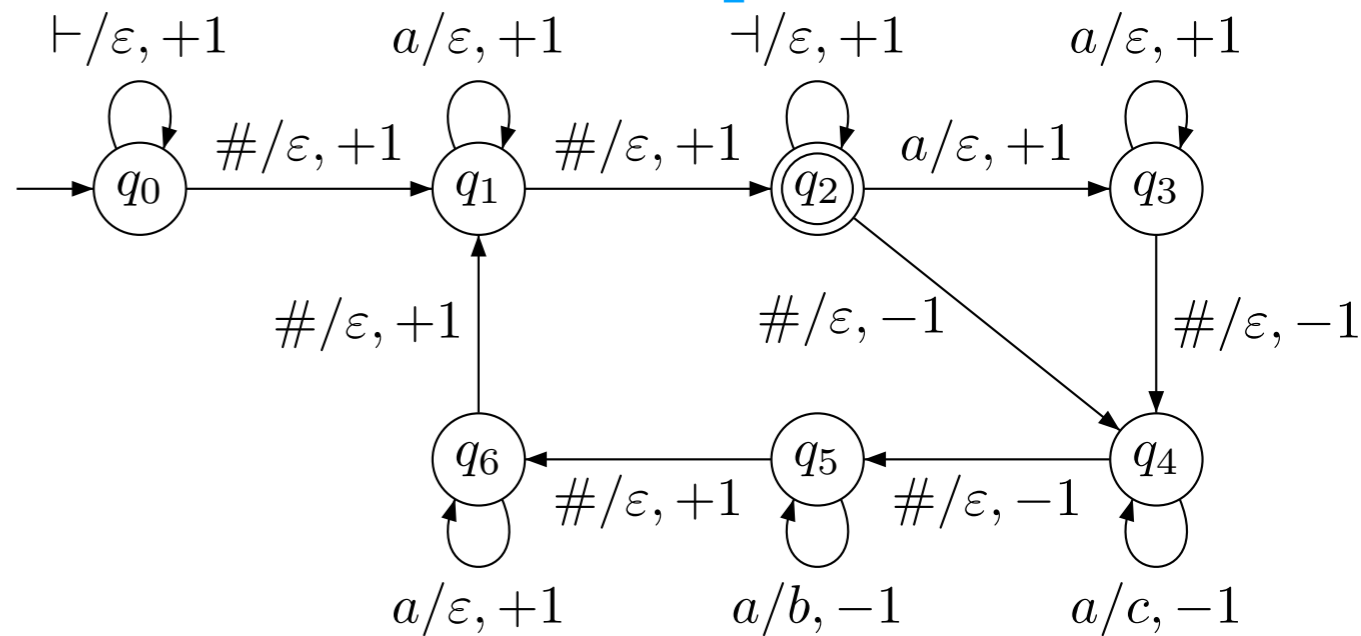
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2DFA to RTE: Kleene-plus

$\text{Tr}: \Sigma^* \rightarrow \text{TrM}$



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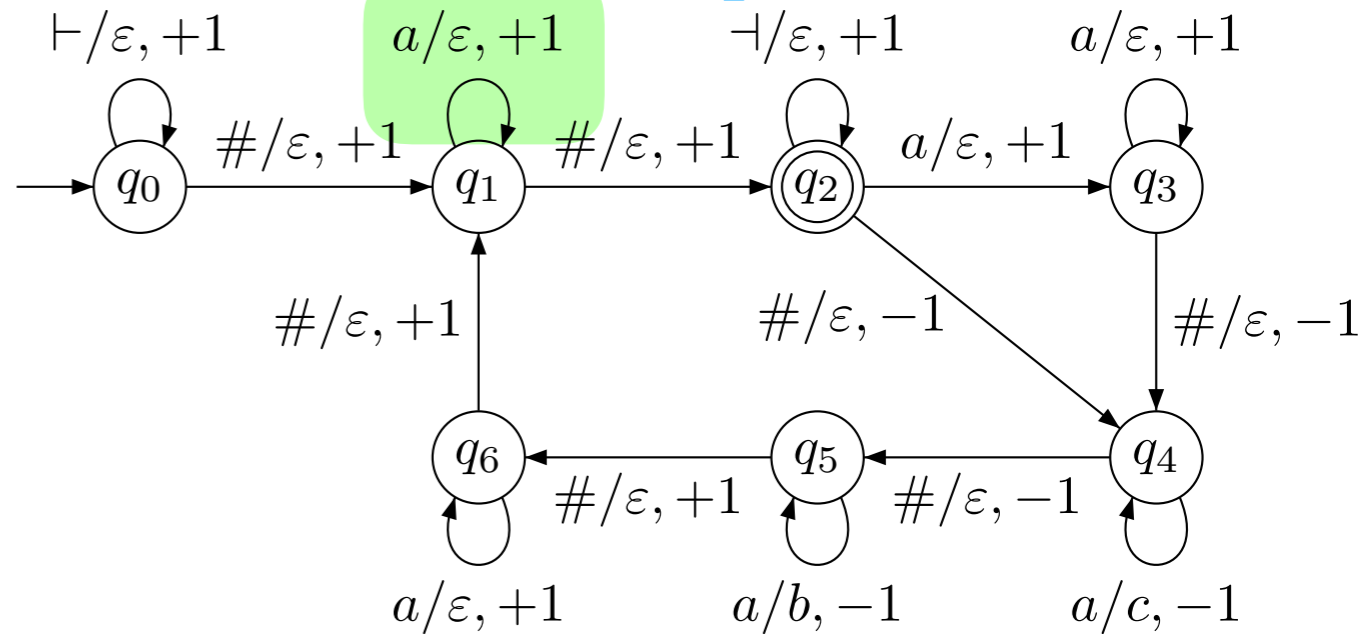
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2DFA to RTE: Kleene-plus

$\text{Tr}: \Sigma^* \rightarrow \text{TrM}$

$$C_{a^+}(q_1, \rightarrow, q_1) = (a? \varepsilon : \perp) \boxplus$$



$$\text{Tr}(a) = \{(q_1, \rightarrow, q_1), \dots, (q_4, \leftarrow, q_4), \dots, (q_5, \leftarrow, q_5), \dots\} = \text{Tr}(a^+)$$

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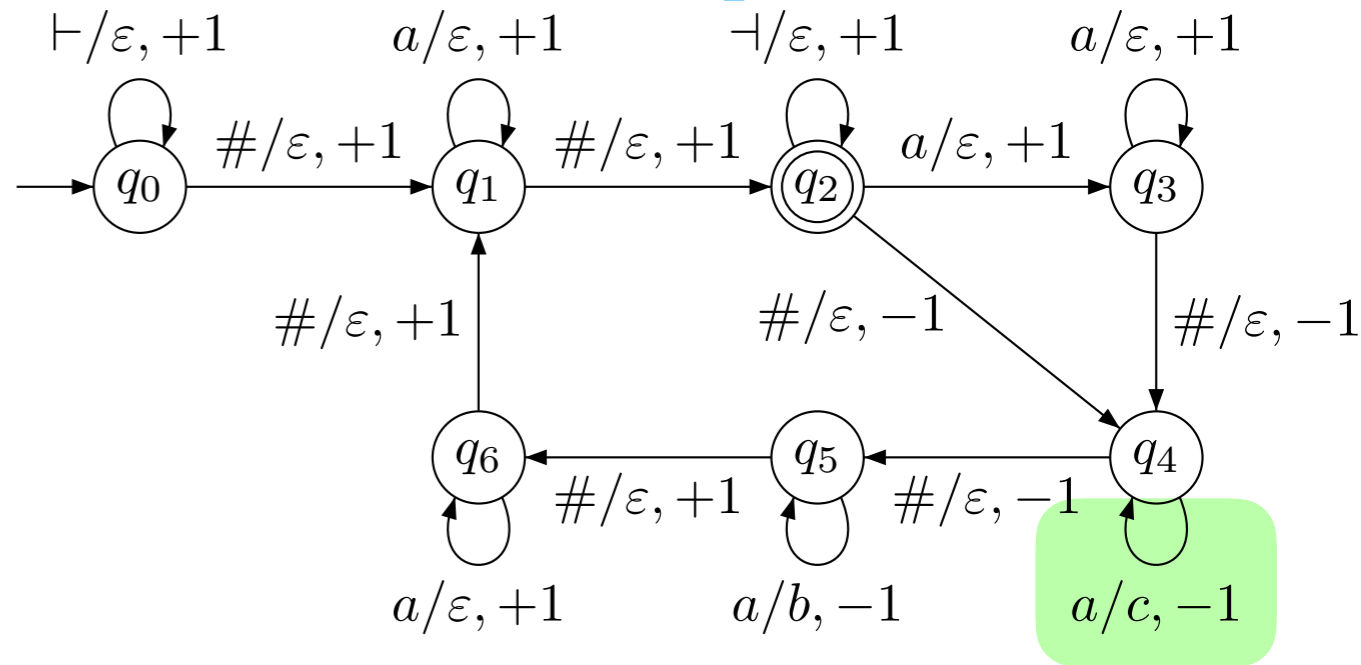
1. $\text{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $\llbracket C_F(x) \rrbracket(u)$ is the output produced by \mathcal{A} when running step x on u (i.e., running \mathcal{A} on u from p to q following direction d).

2DFA to RTE: Kleene-plus

$\text{Tr}: \Sigma^* \rightarrow \text{TrM}$

$$C_{a^+}(q_1, \rightarrow, q_1) = (a? \varepsilon : \perp) \boxplus$$

$$C_{a^+}(q_4, \leftarrow, q_4) = (a? c : \perp) \boxleftarrow$$



$$\text{Tr}(a) = \{(q_1, \rightarrow, q_1), \dots, (q_4, \leftarrow, q_4), \dots, (q_5, \leftarrow, q_5), \dots\} = \text{Tr}(a^+)$$

Main Lemma:

$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

Let F be an ε -free Tr-good rational expression with $\text{Tr}(F) = s_F$. We can construct a map $C_F: s_F \rightarrow \text{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\text{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $\llbracket C_F(x) \rrbracket(u)$ is the output produced by \mathcal{A} when running step x on u (i.e., running \mathcal{A} on u from p to q following direction d).

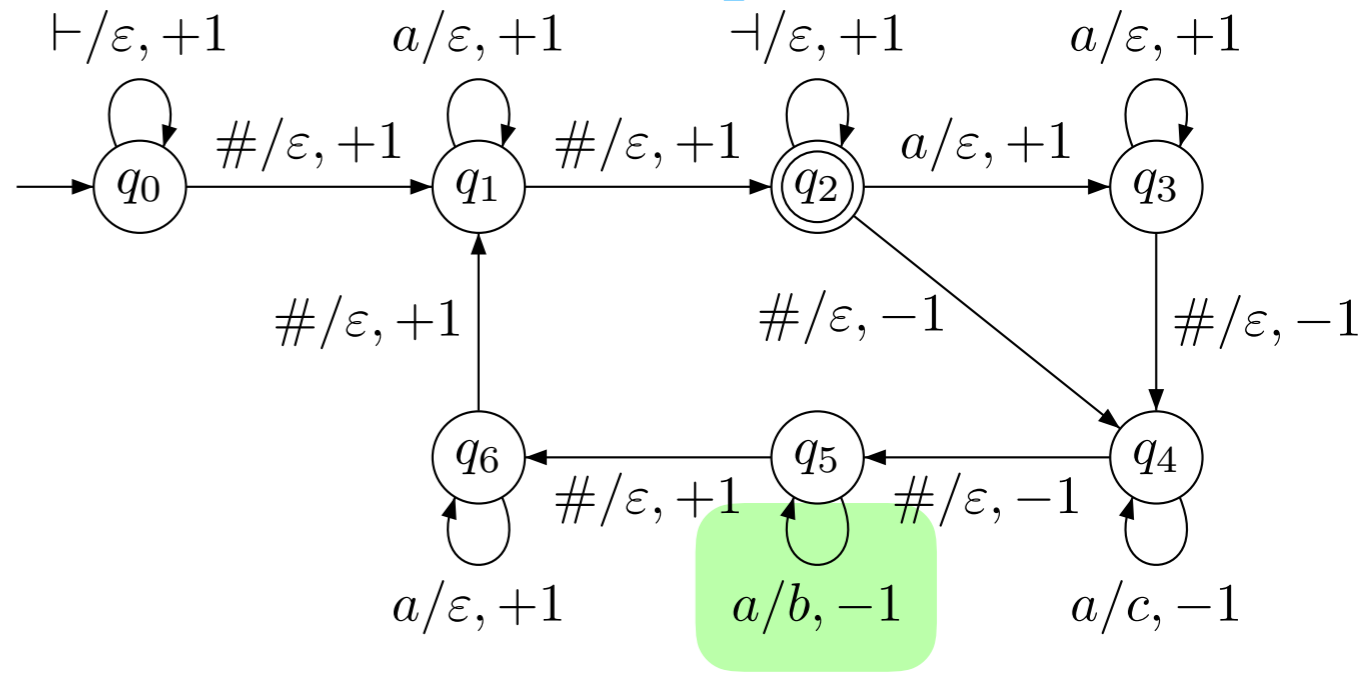
2DFA to RTE: Kleene-plus

$\text{Tr}: \Sigma^* \rightarrow \text{TrM}$

$$C_{a^+}(q_1, \rightarrow, q_1) = (a? \varepsilon : \perp) \begin{array}{c} \boxplus \\ \leftarrow \end{array}$$

$$C_{a^+}(q_4, \leftarrow, q_4) = (a? c : \perp) \begin{array}{c} \boxplus \\ \leftarrow \end{array}$$

$$C_{a^+}(q_5, \leftarrow, q_5) = (a? b : \perp) \begin{array}{c} \boxplus \\ \leftarrow \end{array}$$



$$\text{Tr}(a) = \{(q_1, \rightarrow, q_1), \dots, (q_4, \leftarrow, q_4), \dots, (q_5, \leftarrow, q_5), \dots\} = \text{Tr}(a^+)$$

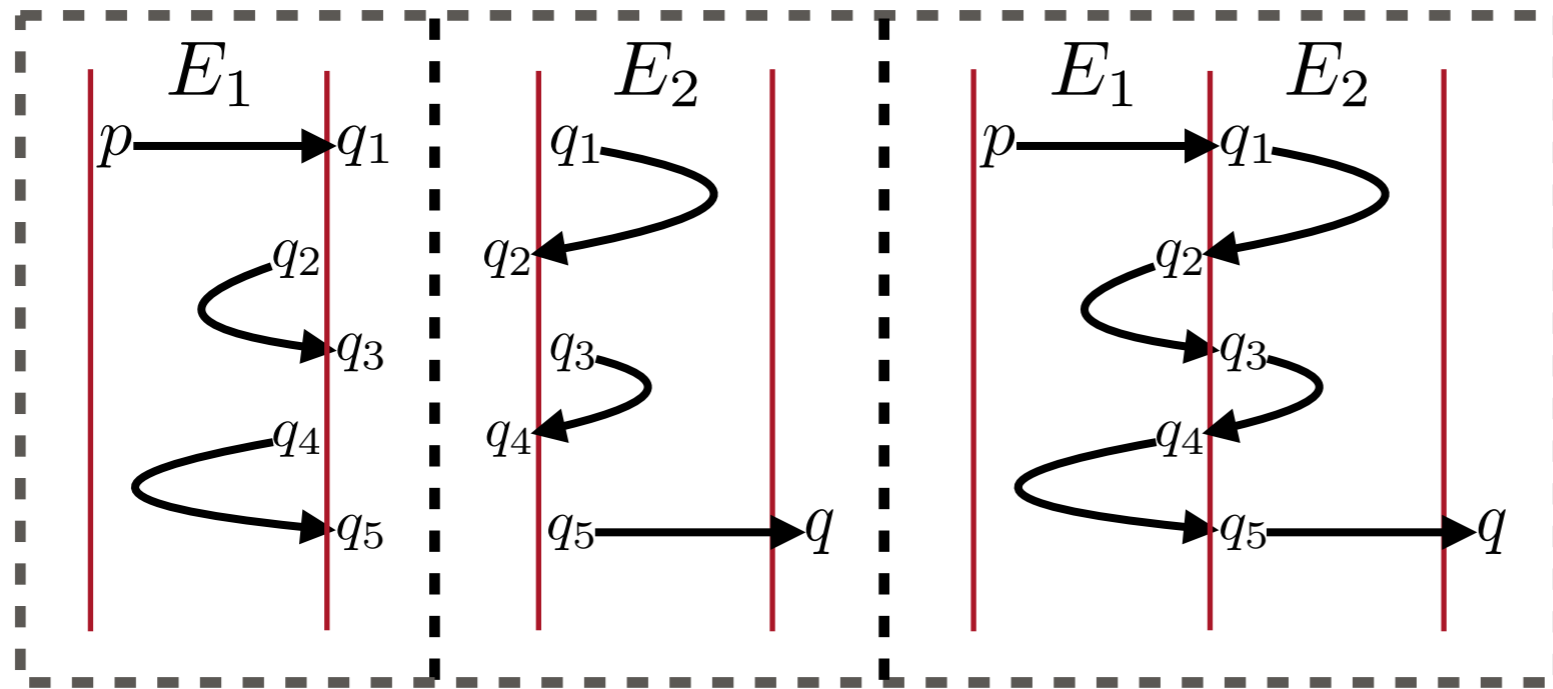
Main Lemma:

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2DFA to RTE: concatenation



$$\begin{aligned}
 x &= (p, \rightarrow, q) \\
 x_1 &= (p, \rightarrow, q_1) \\
 x_2 &= (q_1, \curvearrowright, q_2) \\
 x_3 &= (q_2, \curvearrowleft, q_3) \\
 x_4 &= (q_3, \curvearrowright, q_4) \\
 x_5 &= (q_4, \curvearrowleft, q_5) \\
 x_6 &= (q_5, \rightarrow, q)
 \end{aligned}$$

$$C_{E_1 \cdot E_2}(x) = (C_{E_1}(x_1) \square C_{E_2}(x_2)) \odot (C_{E_1}(x_3) \square C_{E_2}(x_4)) \odot (C_{E_1}(x_5) \square C_{E_2}(x_6))$$

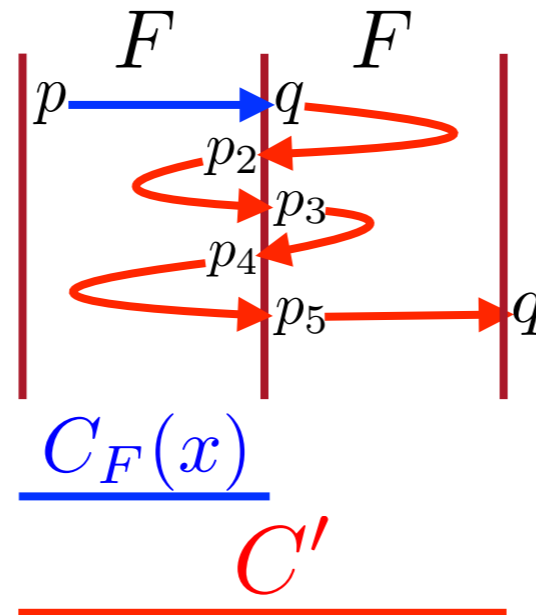
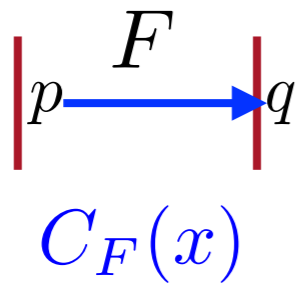
Main Lemma:

$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

Let F be an ε -free Tr-good rational expression with $\text{Tr}(F) = s_F$. We can construct a map $C_F: s_F \rightarrow \text{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\text{dom}(C_F(x)) = \mathcal{L}(F)$,
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2DFA to RTE: Kleene-plus



- $x = (p, \rightarrow, q)$
- $x_2 = (q, \curvearrowright, p_2)$
- $x_3 = (p_2, \curvearrowleft, p_3)$
- $x_4 = (p_3, \curvearrowright, p_4)$
- $x_5 = (p_4, \curvearrowleft, p_5)$
- $x_6 = (p_5, \rightarrow, q)$

$$C_{F^+}(x) = (C_F(x) \square (F^* ? \varepsilon : \perp)) \odot [F, C']^{2\boxplus}$$

$$C' = ((F ? \varepsilon : \perp) \square C_F(x_2)) \odot (C_F(x_3) \square C_F(x_4)) \odot (C_F(x_5) \square C_F(x_6))$$

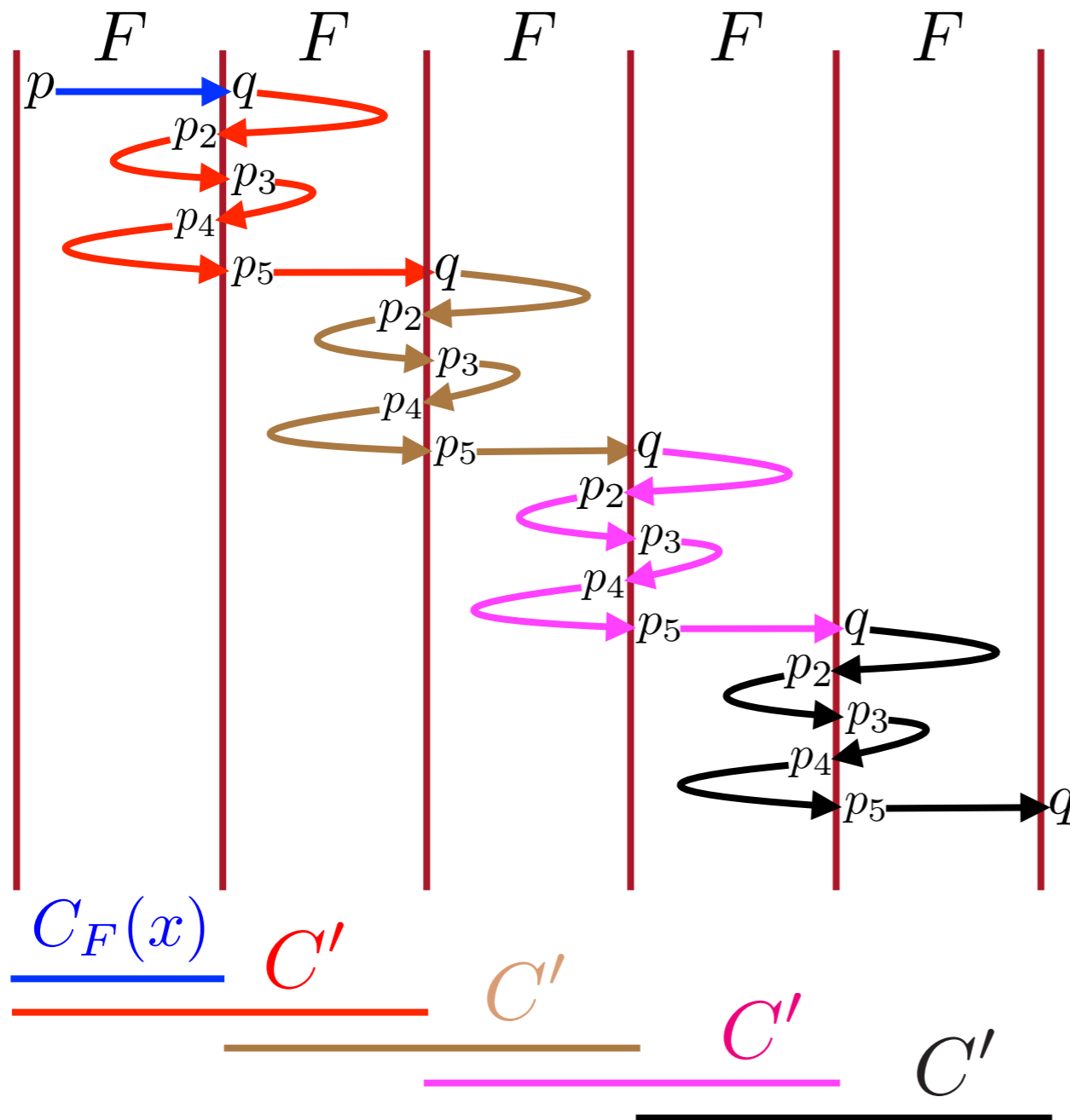
Main Lemma:

$$F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$$

Let F be an ε -free Tr-good rational expression with $\text{Tr}(F) = s_F$. We can construct a map $C_F: s_F \rightarrow \text{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

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2DFA to RTE: Kleene-plus



$$x = (p, \rightarrow, q)$$

$$x_2 = (q, \curvearrowright, p_2)$$

$$x_3 = (p_2, \curvearrowleft, p_3)$$

$$x_4 = (p_3, \curvearrowright, p_4)$$

$$x_5 = (p_4, \curvearrowleft, p_5)$$

$$x_6 = (p_5, \rightarrow, q)$$

$$C_{F^+}(x) = (C_F(x) \square (F^* ? \varepsilon : \perp)) \odot [F, C']^{2\boxplus}$$

$$C' = ((F ? \varepsilon : \perp) \square C_F(x_2)) \odot (C_F(x_3) \square C_F(x_4)) \odot (C_F(x_5) \square C_F(x_6))$$

Conclusion

Programs

Specifications

Regular Transducer Expressions

MSO Transductions

Functional Transductions

Engelfriet & Hoogeboom

Machines

**Finite Transducers
Deterministic, two-way**

New proof technique
Works directly with 2DFT
Unambiguous Forest Factorizations
Extension to infinite words