MODULAR DESCRIPTIONS OF REGULAR FUNCTIONS

PAUL GASTIN
LSV, ENS PARIS-SACLAY

CAI 2019

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  We suffer only 2\% of failures.

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- Reorder arguments (bibtex -> bbl)

  Author = {Engelfriet, Joost and Hoogeboom, Hendrik Jan},

  Year = {2001},

  [EH01] Joost Engelfriet and Hendrik Jan Hoogeboom.

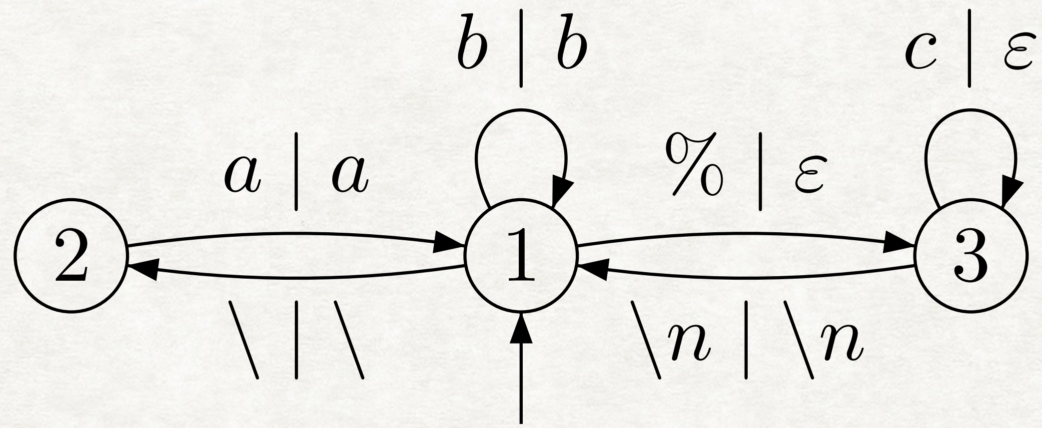
  [EH01] J. Engelfriet and H.J. Hoogeboom.

SUMMARY

- Operational models (transducers)
  - 1DFT = sequential functions
  - f1NFT = rational functions
  - 2DFT = regular functions
  - Transducers with registers
- Modular descriptions
  - Rational expressions
  - Composition

SEQUENTIAL FUNCTIONS - 1DFT

- Deterministic left to right parsing of the input
- Produce output along the way



$a \in \Sigma$

$b \in \Sigma \setminus \{\backslash, \%\}$

$c \in \Sigma \setminus \{\backslash n\}$

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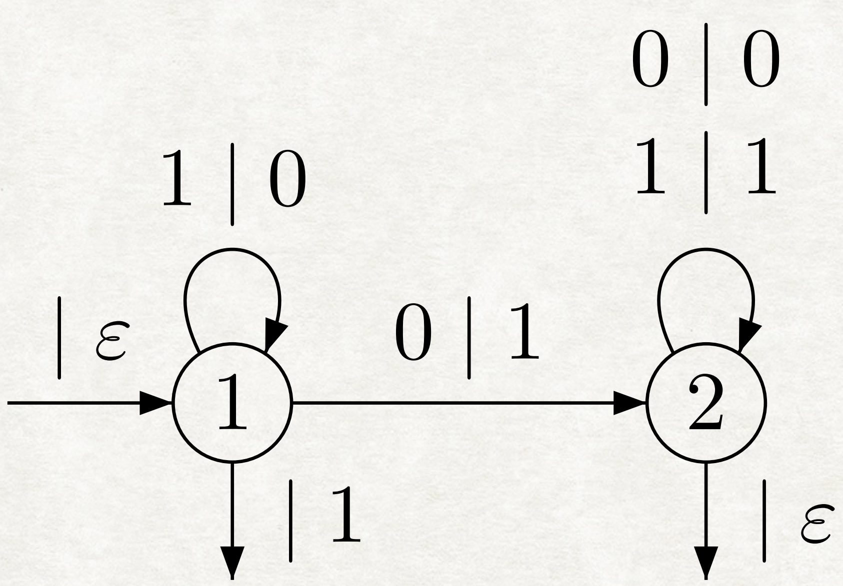
Increment of a number with lsb on the left is sequential.



1111100101101

0000010101101

11111111

000000001

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➡ Equivalence is decidable for sequential transducers

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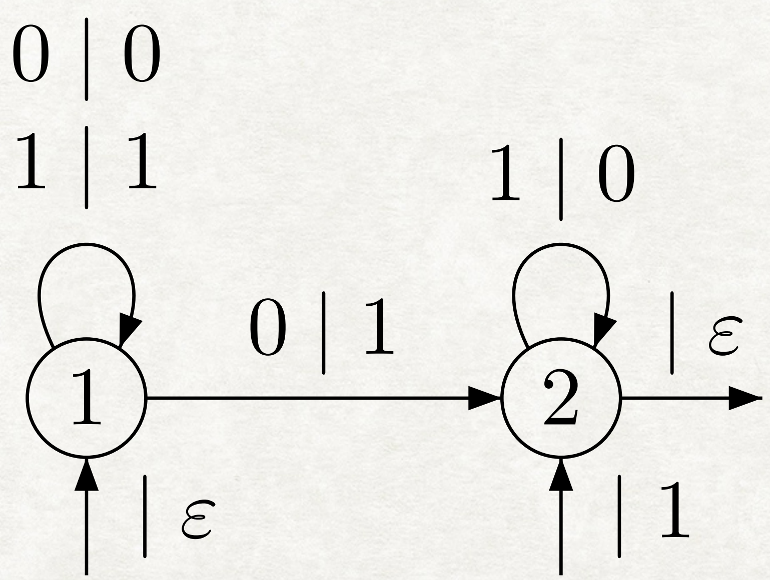
Increment of a number with lsb on the right is a rational function



1011010011111
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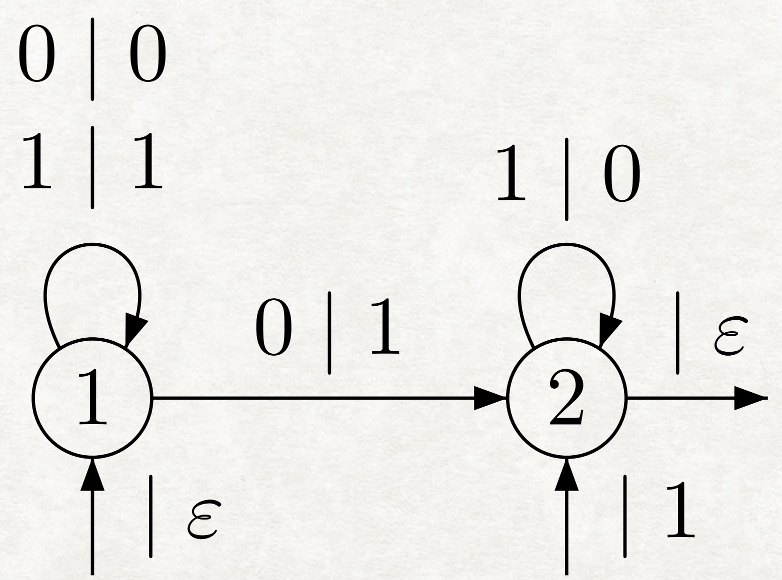

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Lemma Let $\mathcal{A}$ be a 1NFT with m states.
If $\mathcal{A}$ is functional on all words of length $\leq 2m^2$
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Let $w = a_1 a_2 \ldots a_n \in \mathsf{dom}(\mathcal{A})$ with $n > 2m^2$.
Assume $\mathcal{A}$ is functional on all words of length $< n$.

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \cdots p_{n-1} \xrightarrow{a_n} p_n$$
$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \cdots q_{n-1} \xrightarrow{a_n} q_n$$

Let $0 \leq i < j < k \leq n$ with $(p_i, q_i) = (p_j, q_j) = (p_k, q_k)$.

$$p_0 \xrightarrow{a_1 \cdots a_i | x_1} p_i \xrightarrow{a_{i+1} \cdots a_j | x_2} p_j \xrightarrow{a_{j+1} \cdots a_k | x_3} p_k \xrightarrow{a_{k+1} \cdots a_n | x_4} p_n$$
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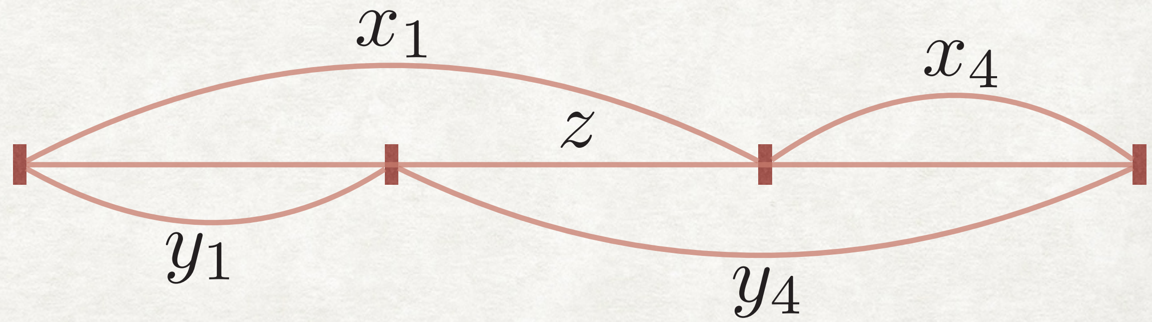
$$x_1 x_4 = y_1 y_4 \implies \begin{aligned} x_1 &= y_1 z \\ z x_4 &= y_4 \end{aligned}$$



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$\mathcal{A}$ is functional on w.

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Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two f1NFT.

Check that $\mathsf{dom}(\mathcal{A}_1) = \mathsf{dom}(\mathcal{A}_2)$.

Check that $\mathcal{A}_1 \uplus \mathcal{A}_2$ is functional.

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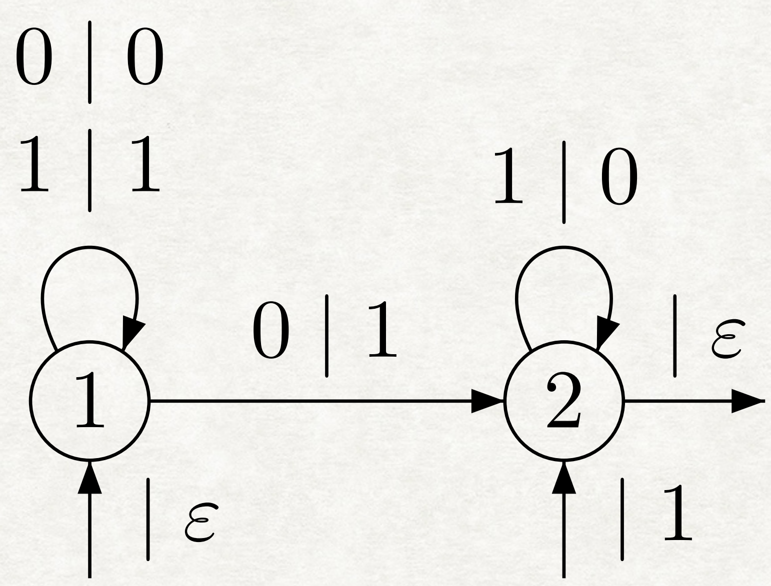

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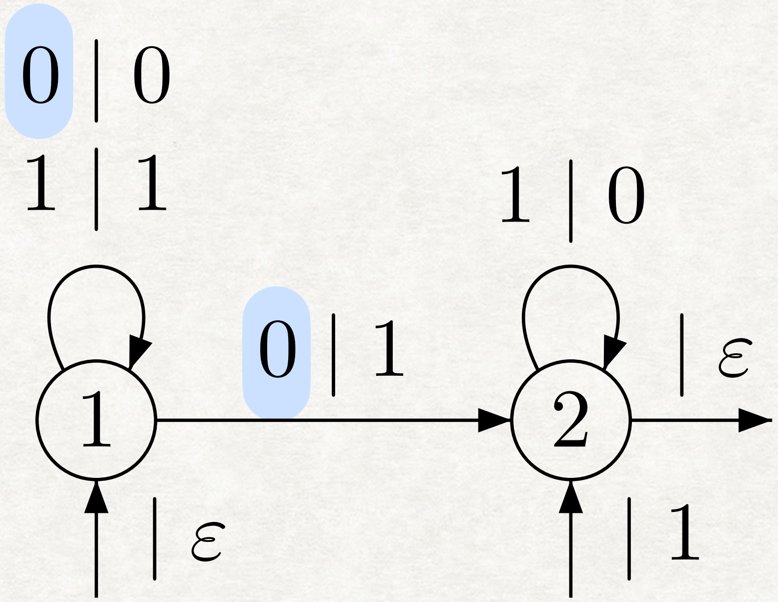

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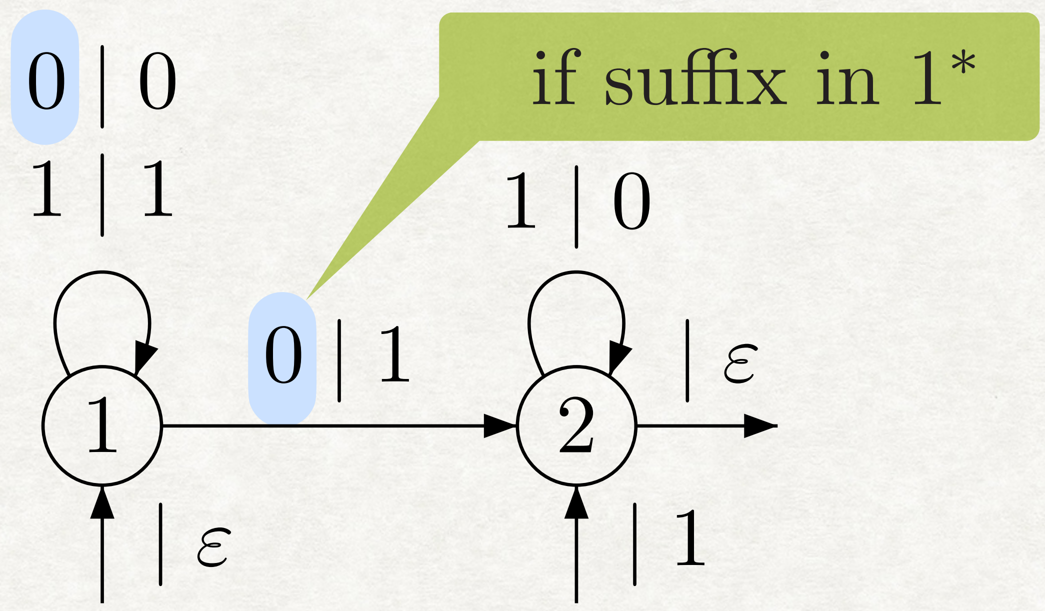

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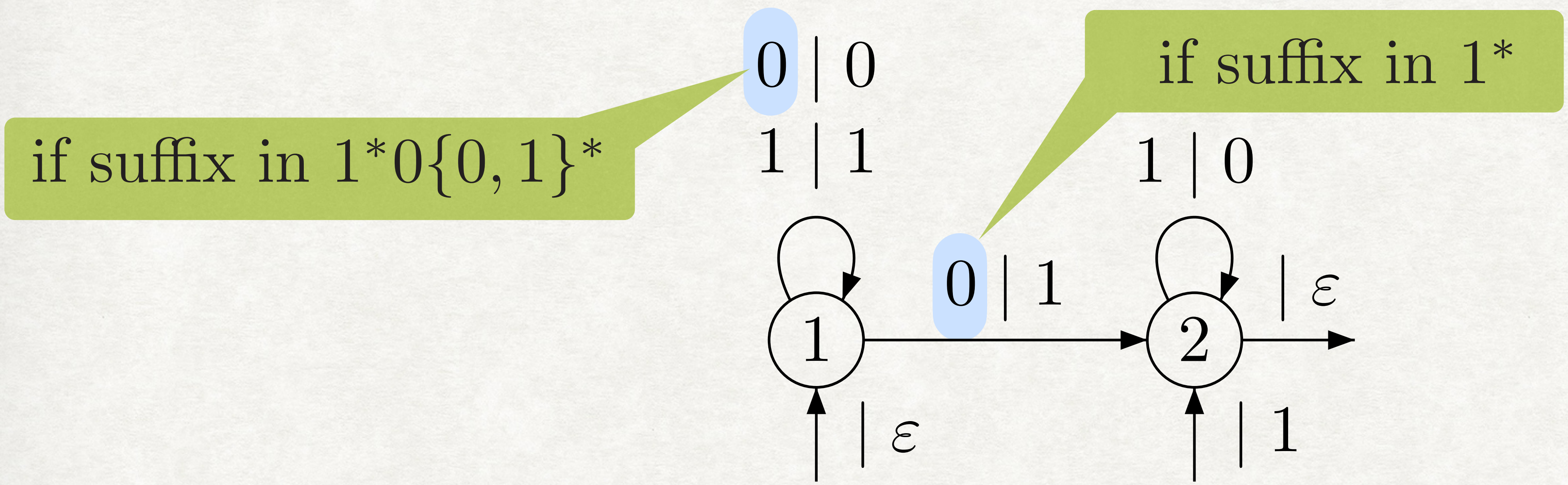

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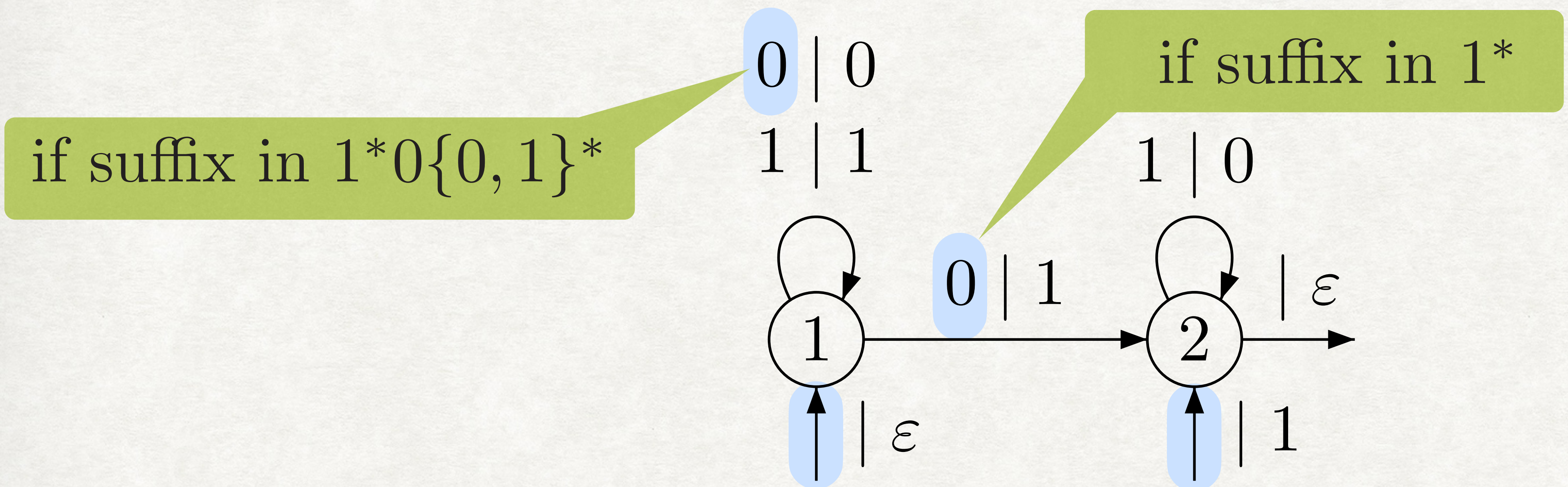

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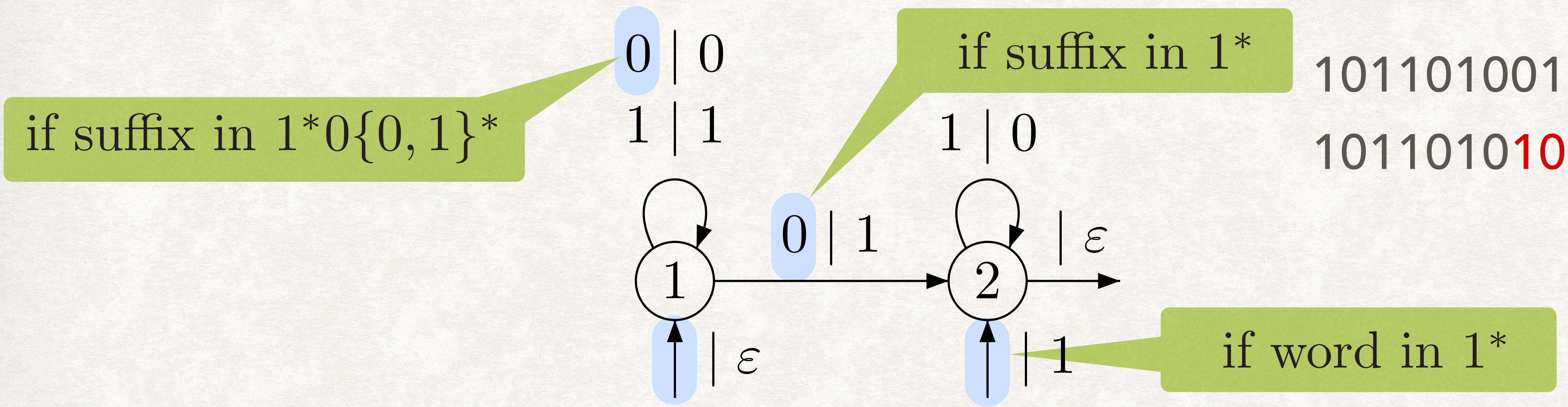

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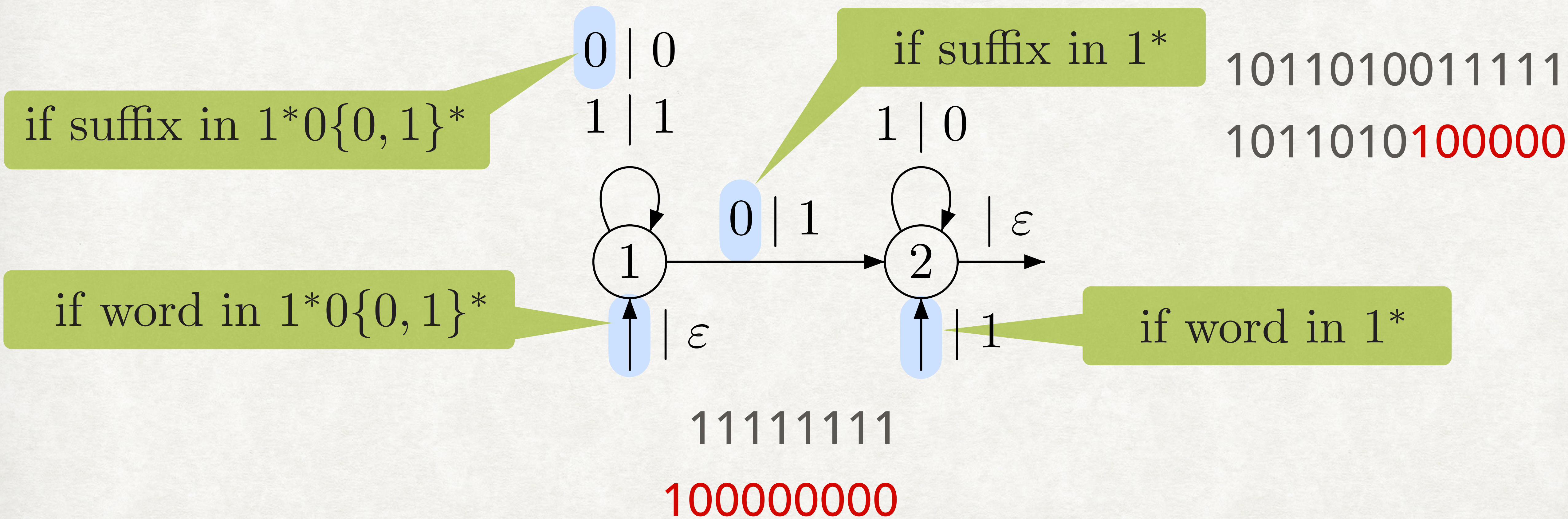

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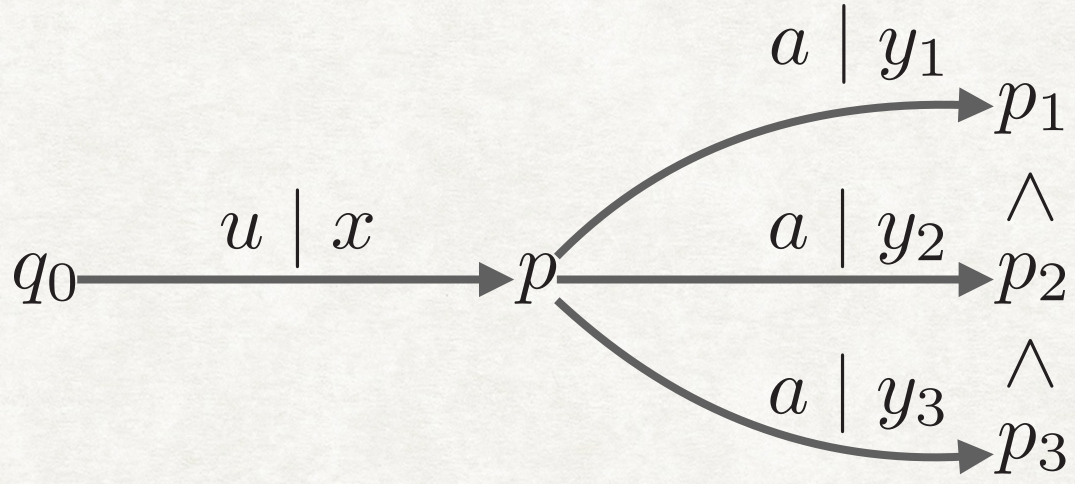
Given $w = uav$, select the least accepting path wrt. lexicographic order.



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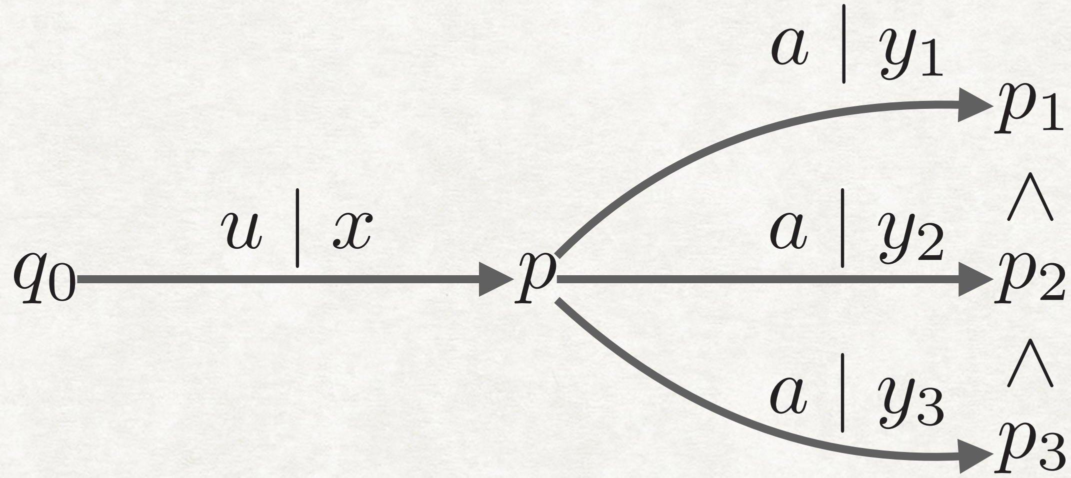

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$v \in \mathsf{dom}(\mathcal{A}, p_2)$

First, deterministic with look-ahead.

Then unambiguous without look-ahead.

[3] Weber & Klemm, Economy of description for single-valued transducers, 1995

RATIONAL FUNCTIONS - f1NFT

- Non-deterministic left to right parsing of the input
- Produce output along the way
- 1UFT: Unambiguous left to right parsing of the input

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SUMMARY

- Operational models (transducers)
  - 1DFT = sequential functions
  - f1NFT = rational functions
  - 2DFT = regular functions
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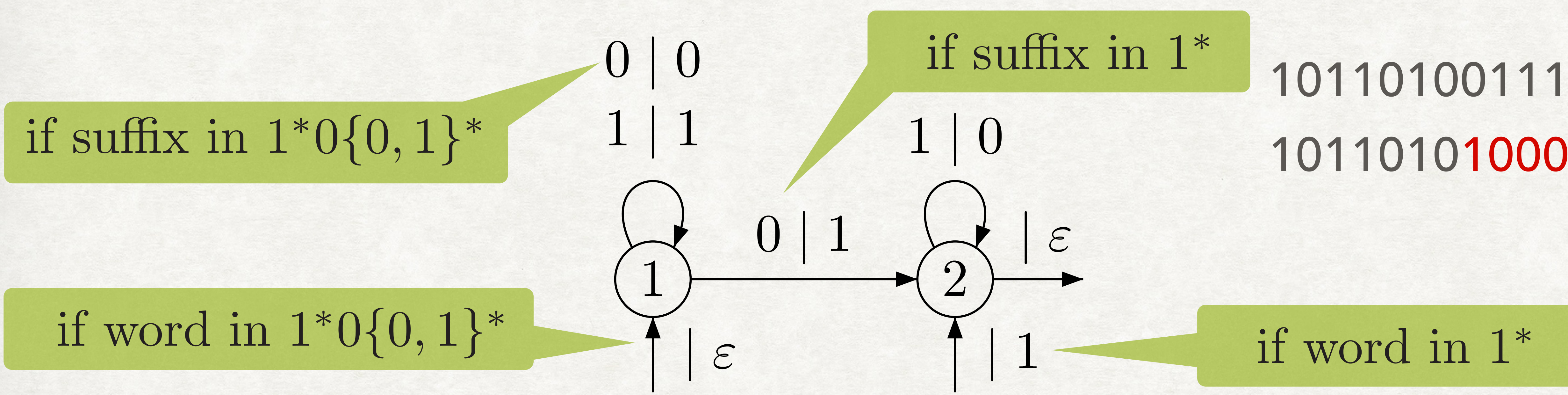
REGULAR FUNCTIONS - 2DFT



10110100011111

10110100100000

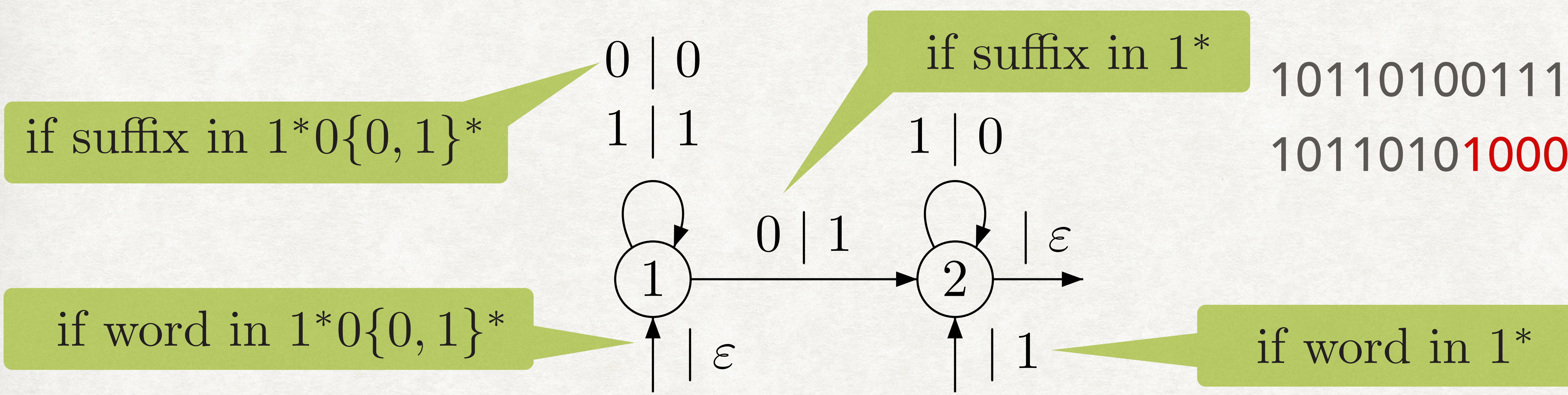
11111111

100000000

REGULAR FUNCTIONS - 2DFT

Increment of a number with lsb on the right

We have a 1DFT with look-ahead: locate the last 0



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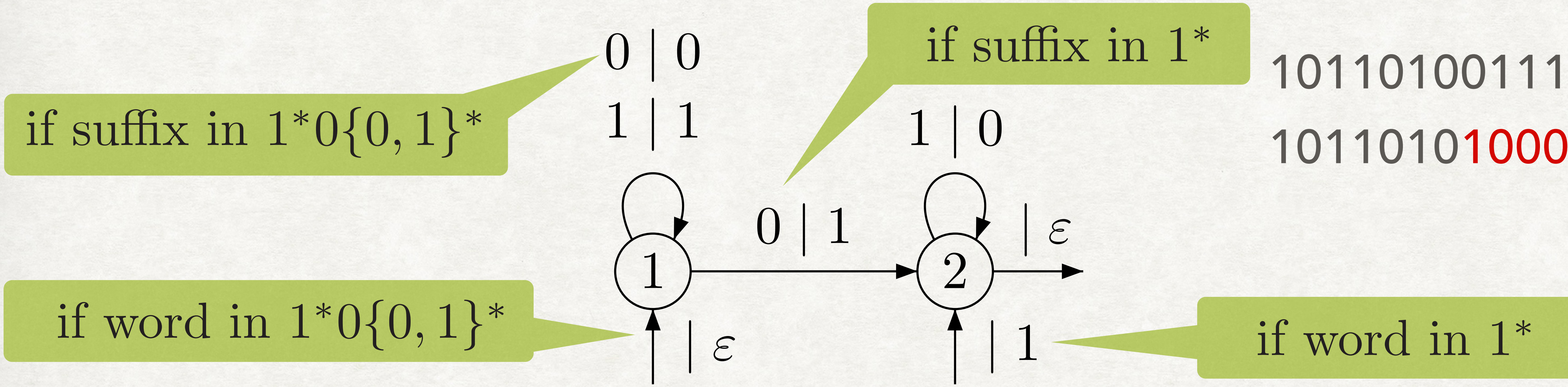
Implement the look-ahead with deterministic 2-way parsing



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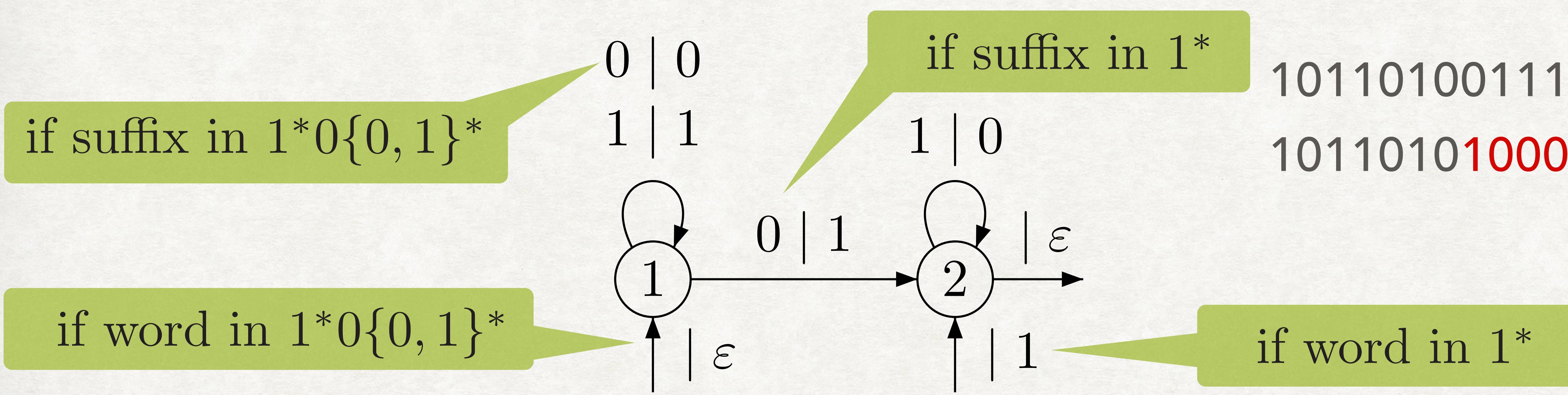

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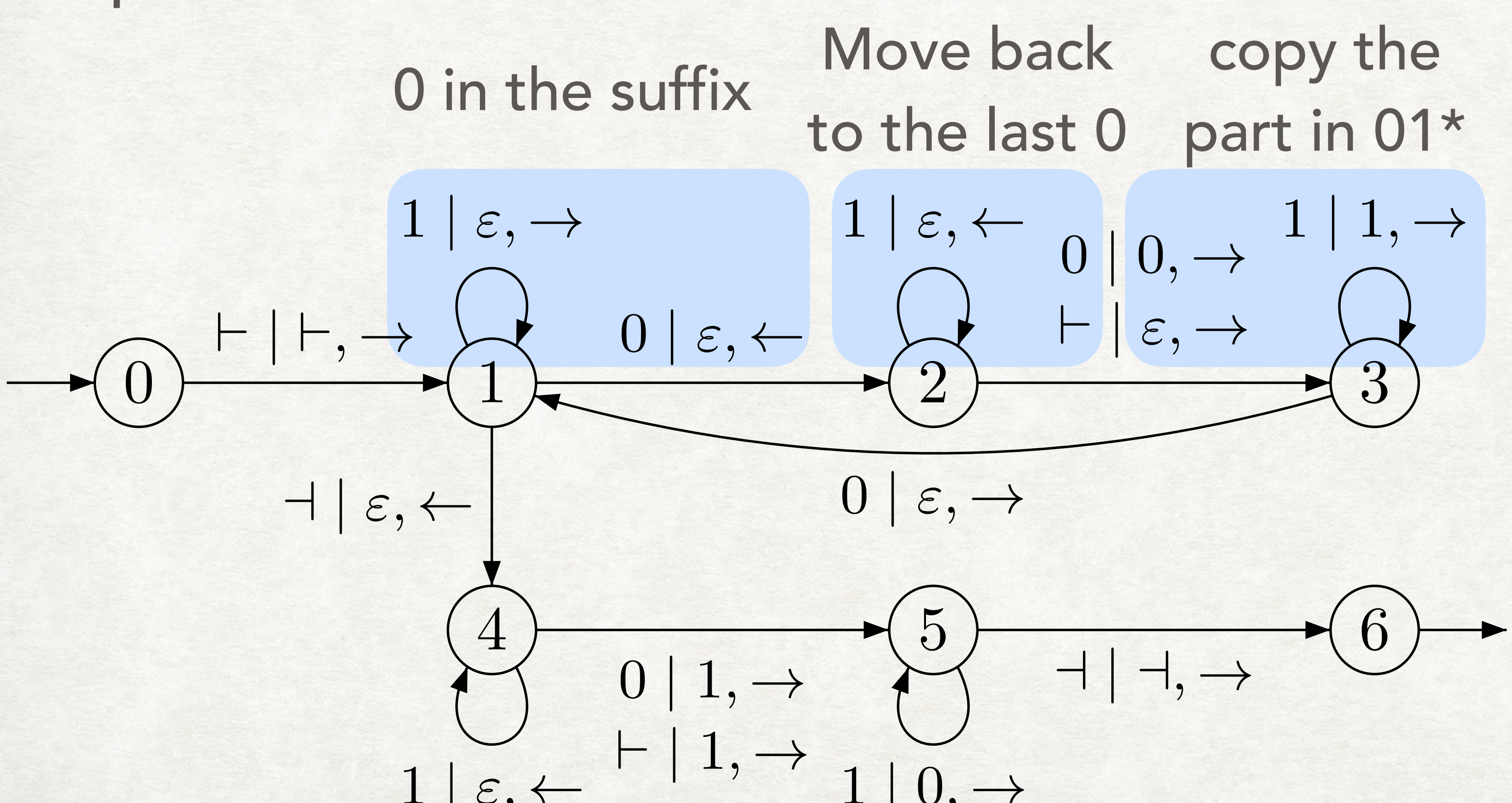

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Increment of a number with lsb on the right with a 2DFT

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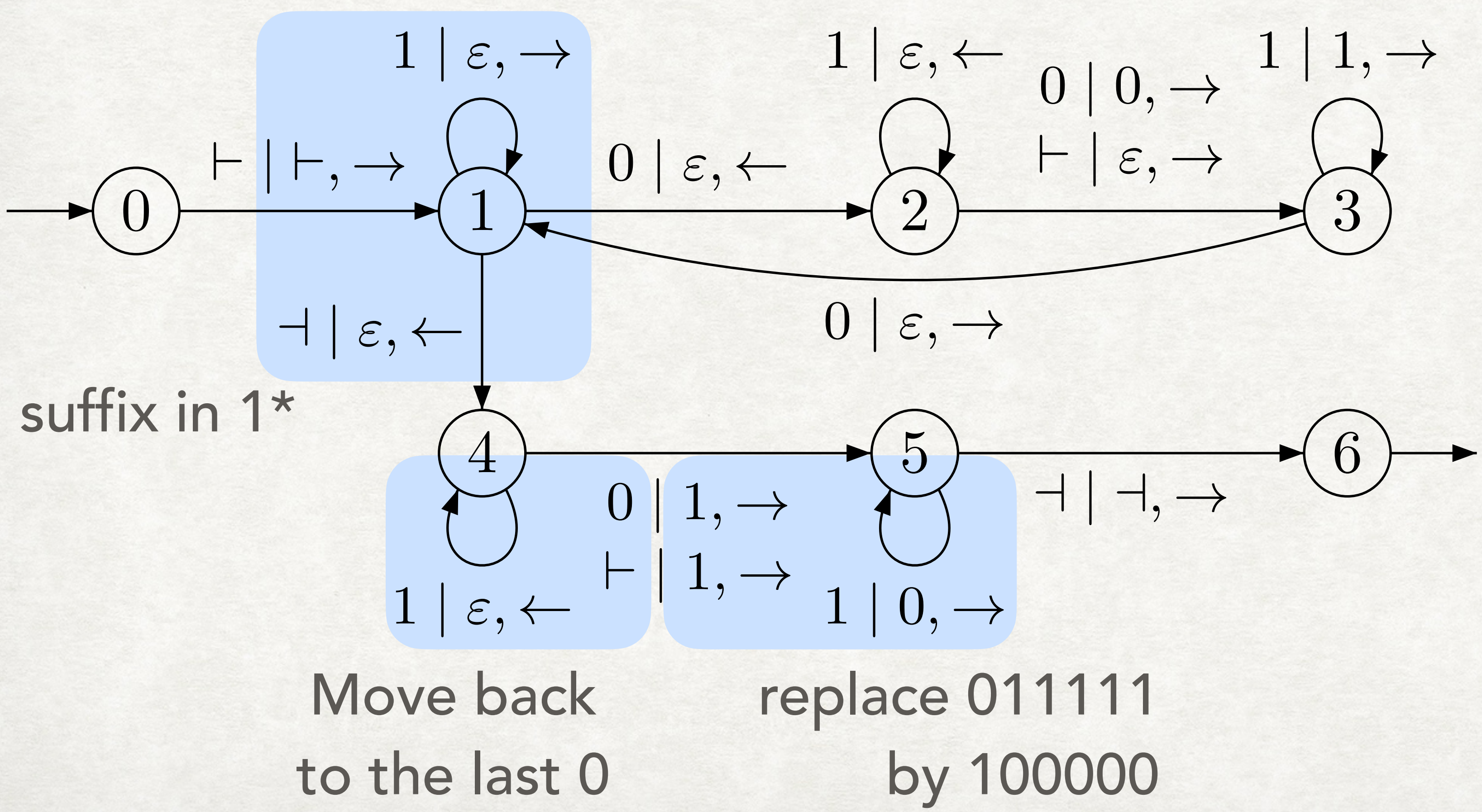
Implement the look-ahead with 2-way parsing



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[9] Culik & Karhumäki, The equivalence of finite valued transducers (on HDT0L languages) is decidable, 1986

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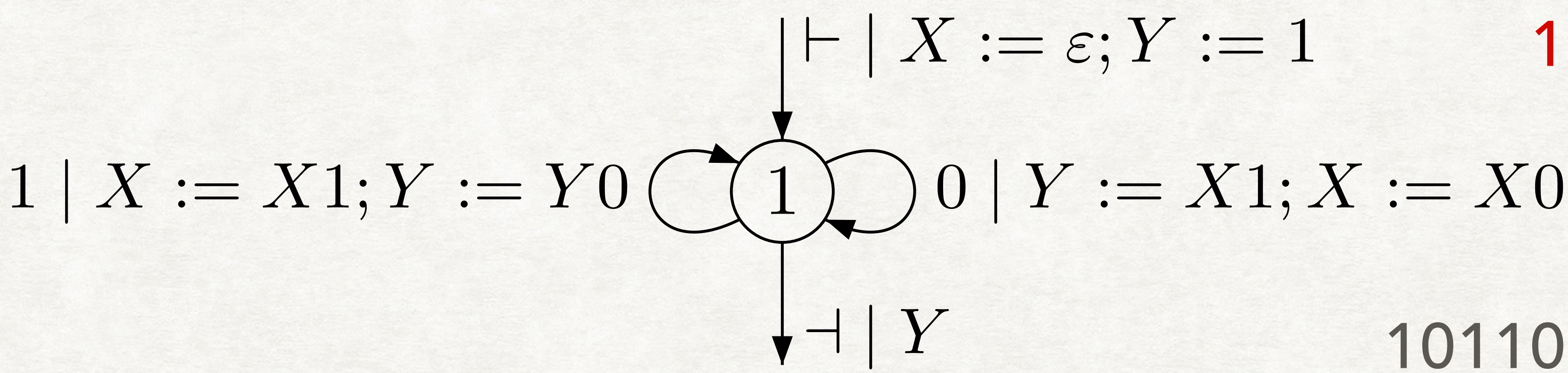

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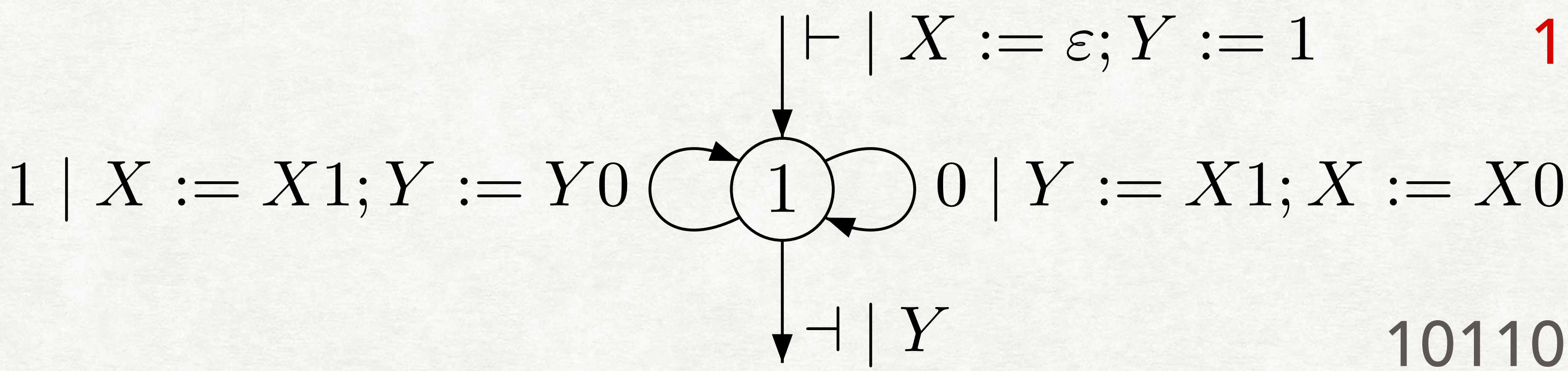

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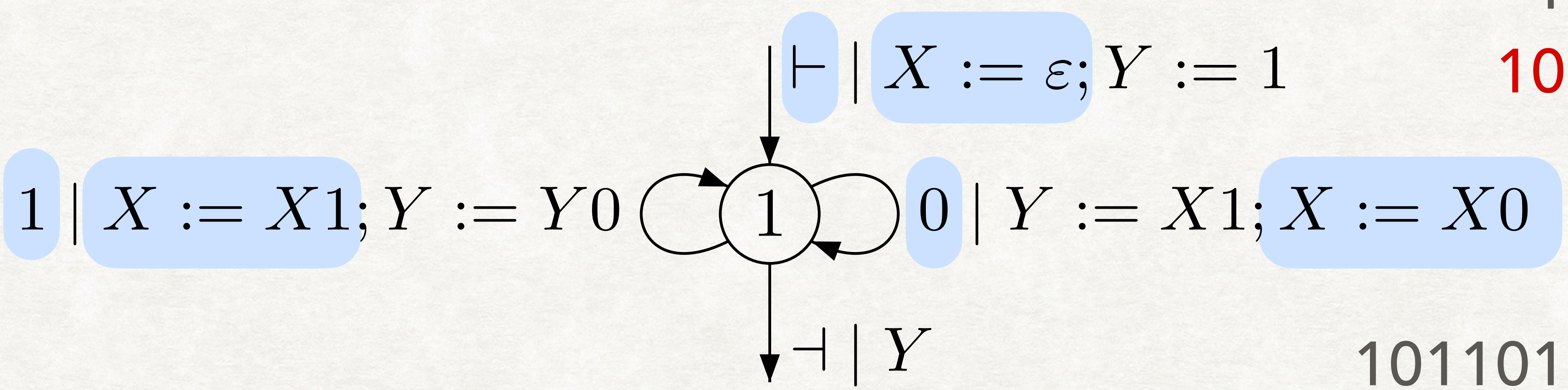

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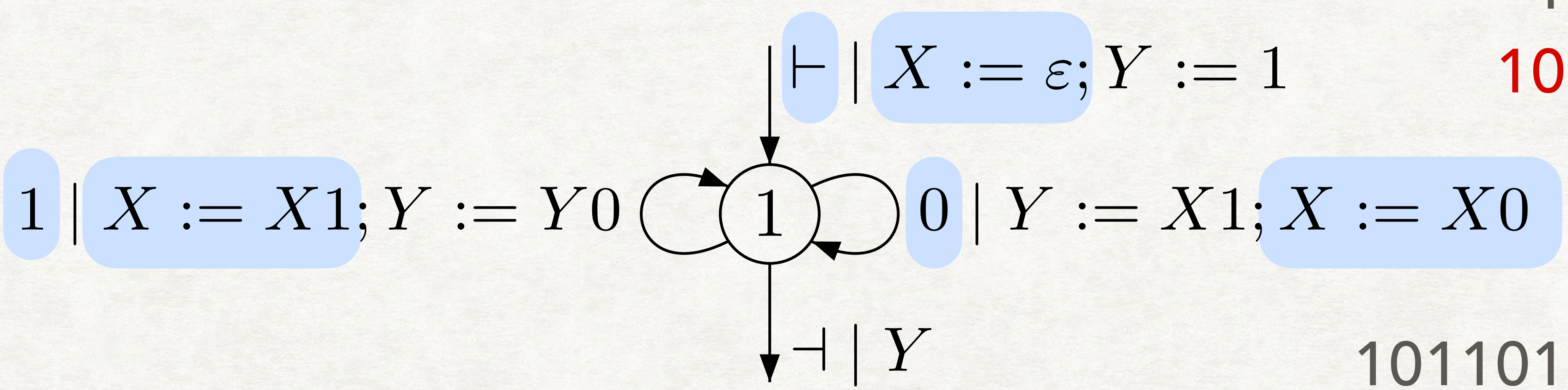

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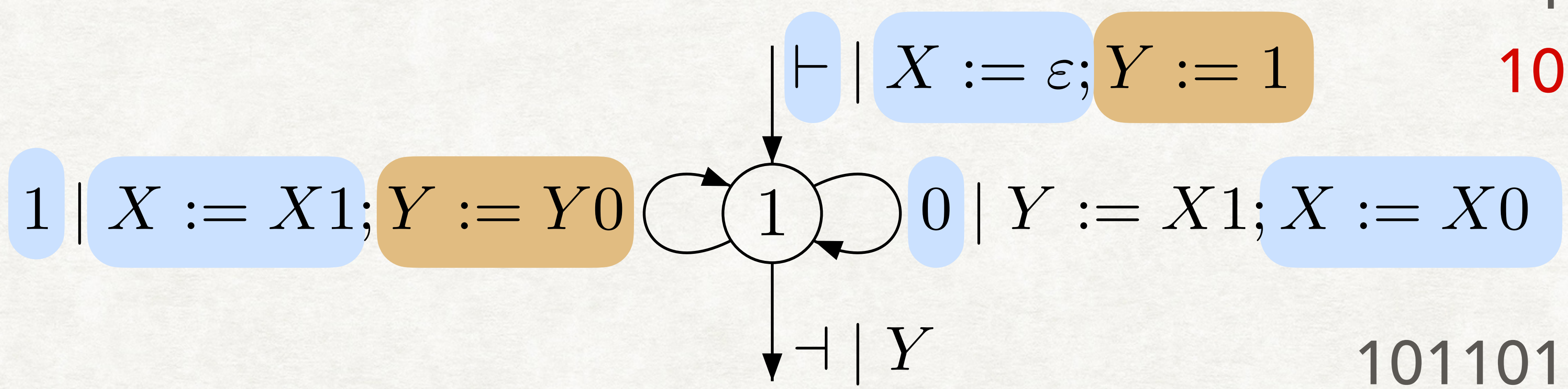

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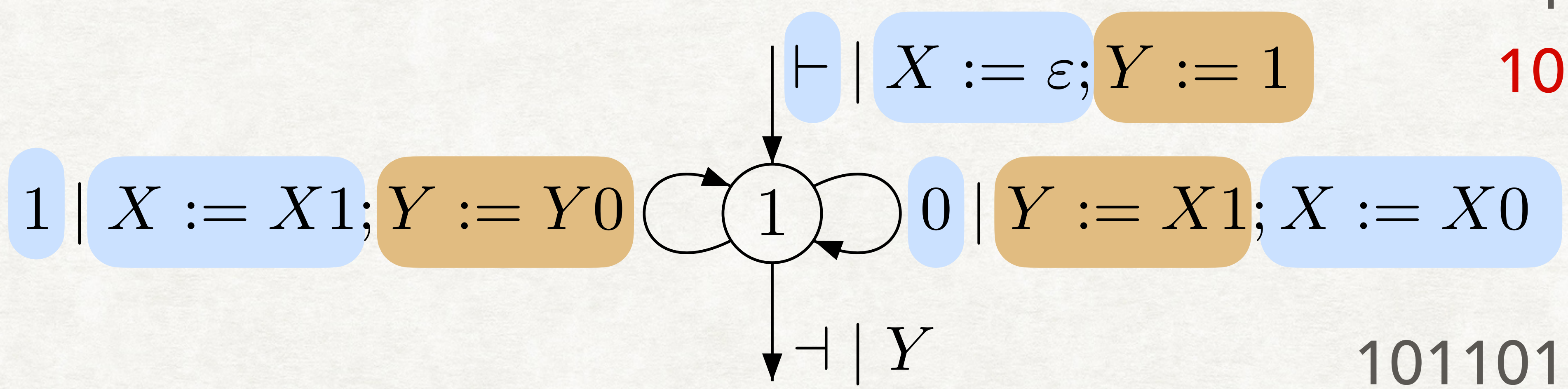

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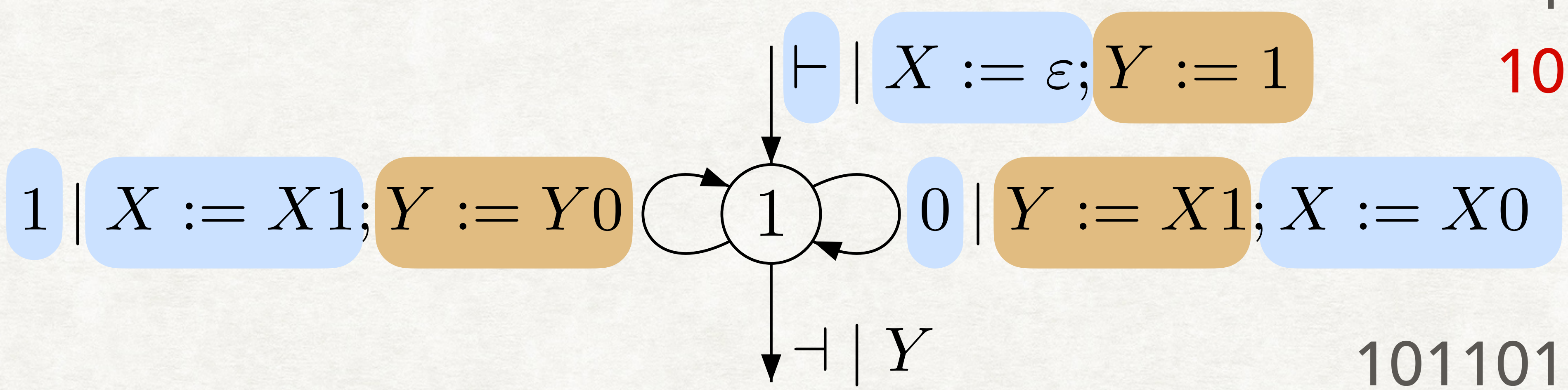

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- X keeps a copy of the input binary number
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- Register updates: X := u | X := Xu | X := Yu (with u finite string)
- 1-way or 2-way
- Simple programs may be composed
- Simple programs = 2DFT

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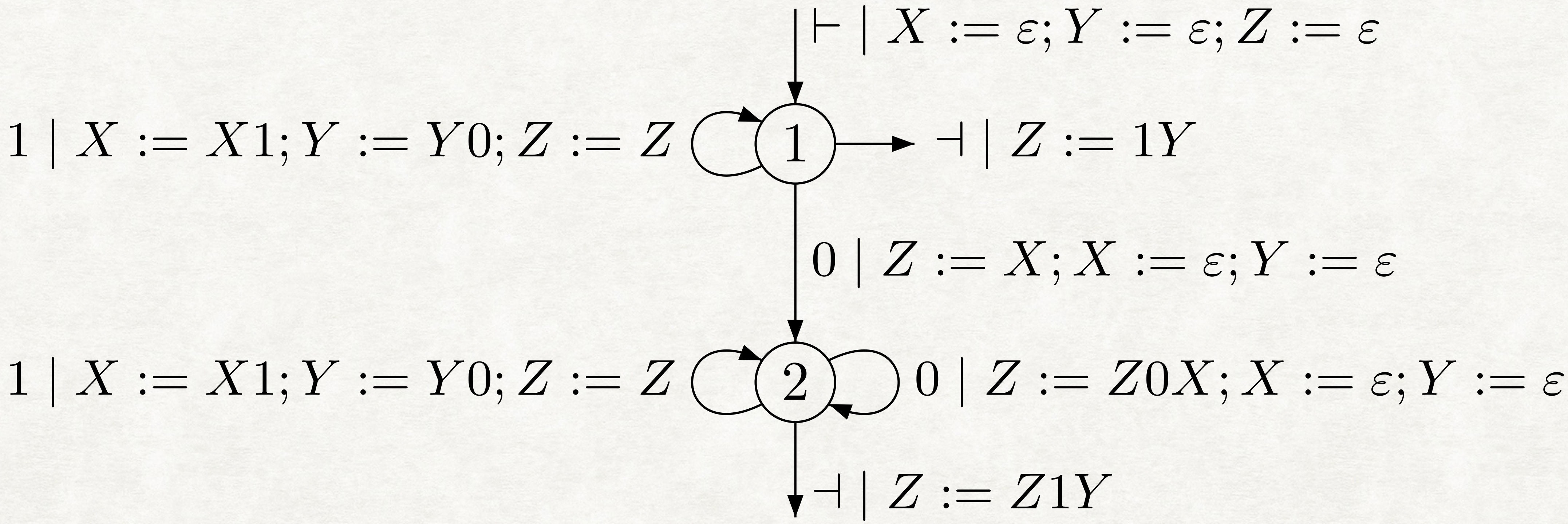
STREAMING STRING TRANSDUCERS



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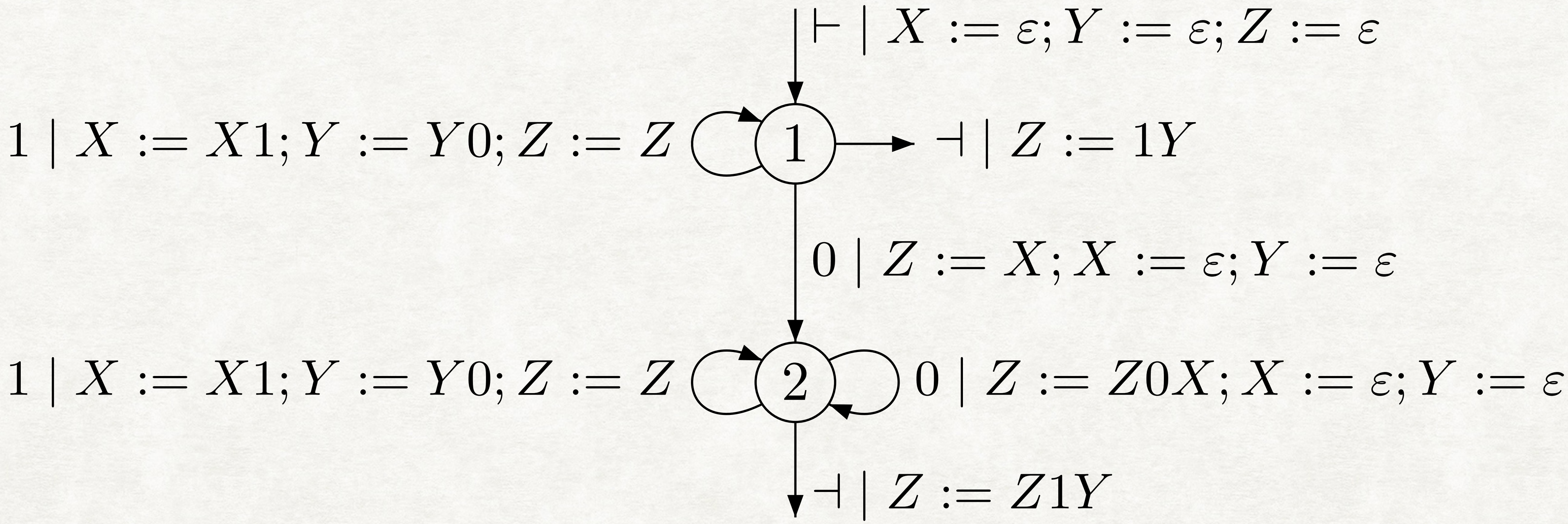

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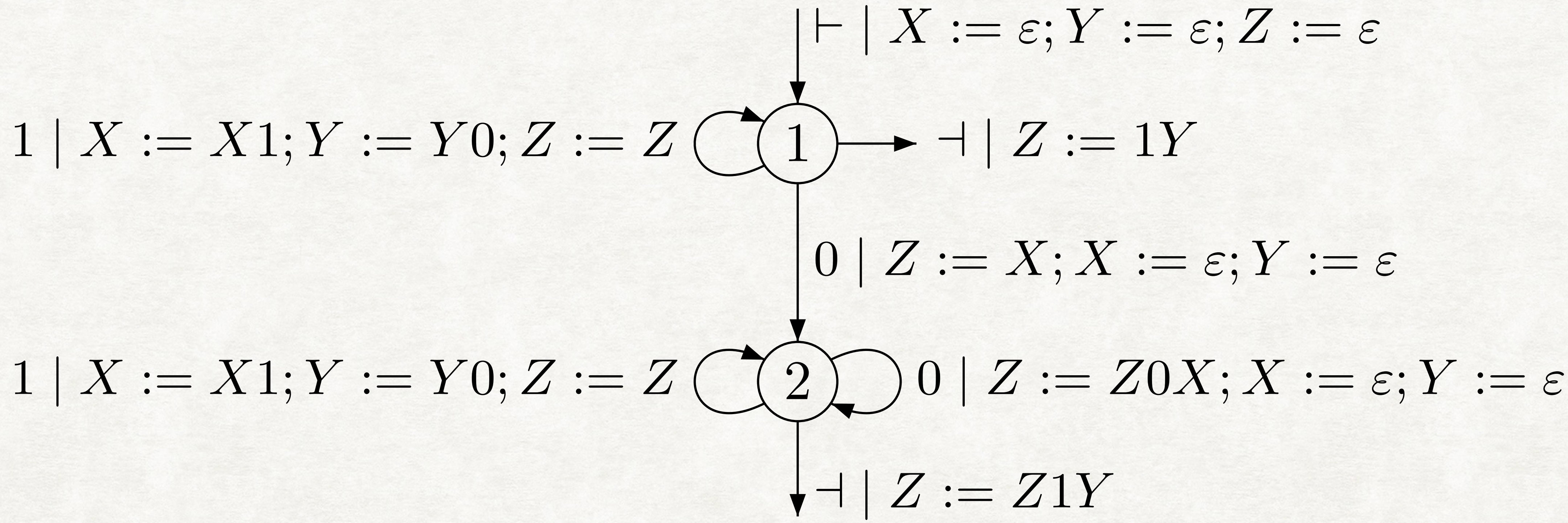

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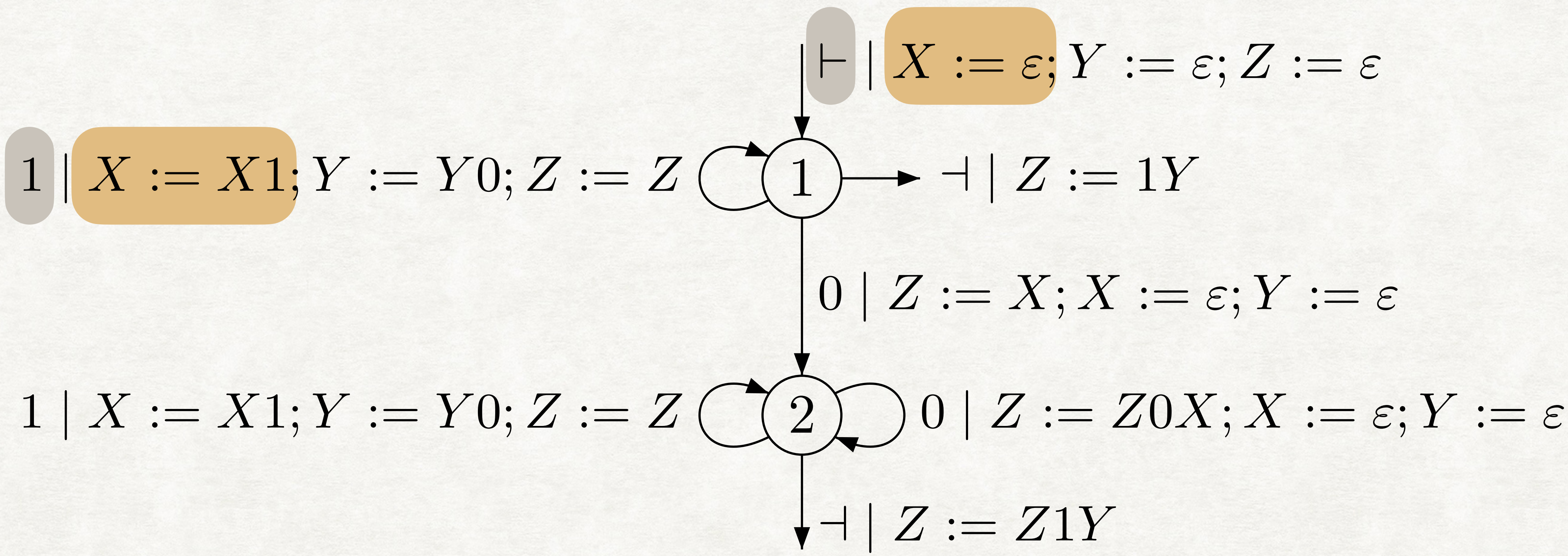

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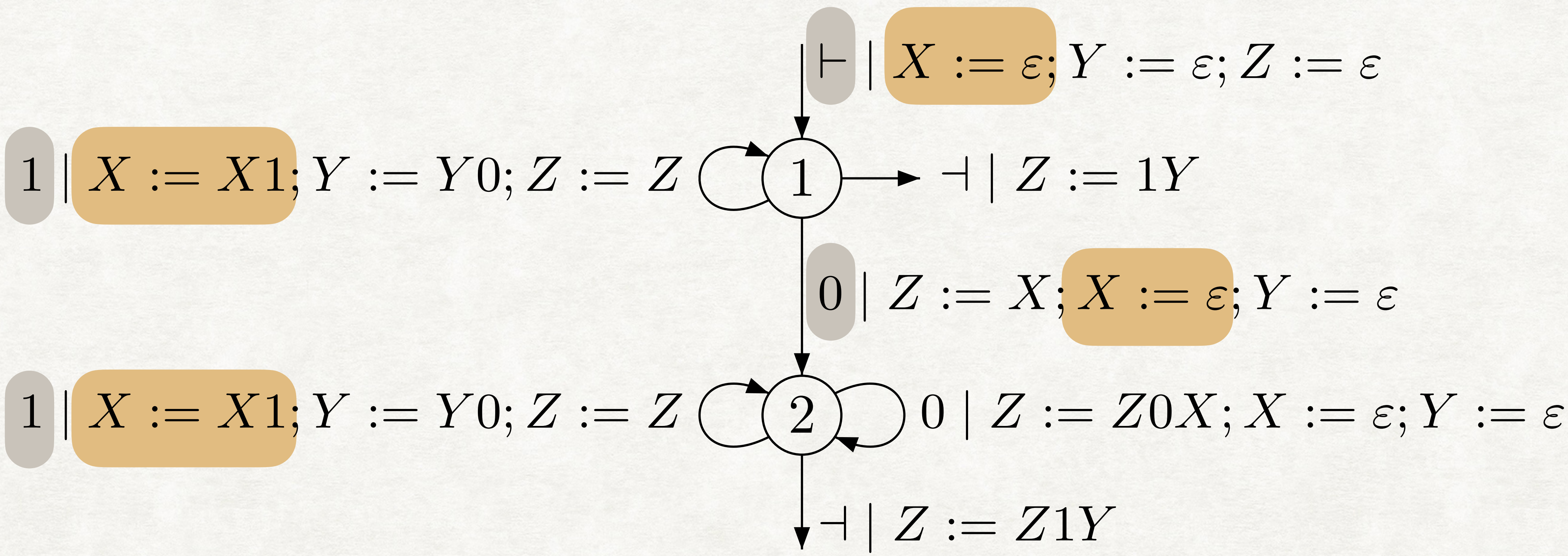

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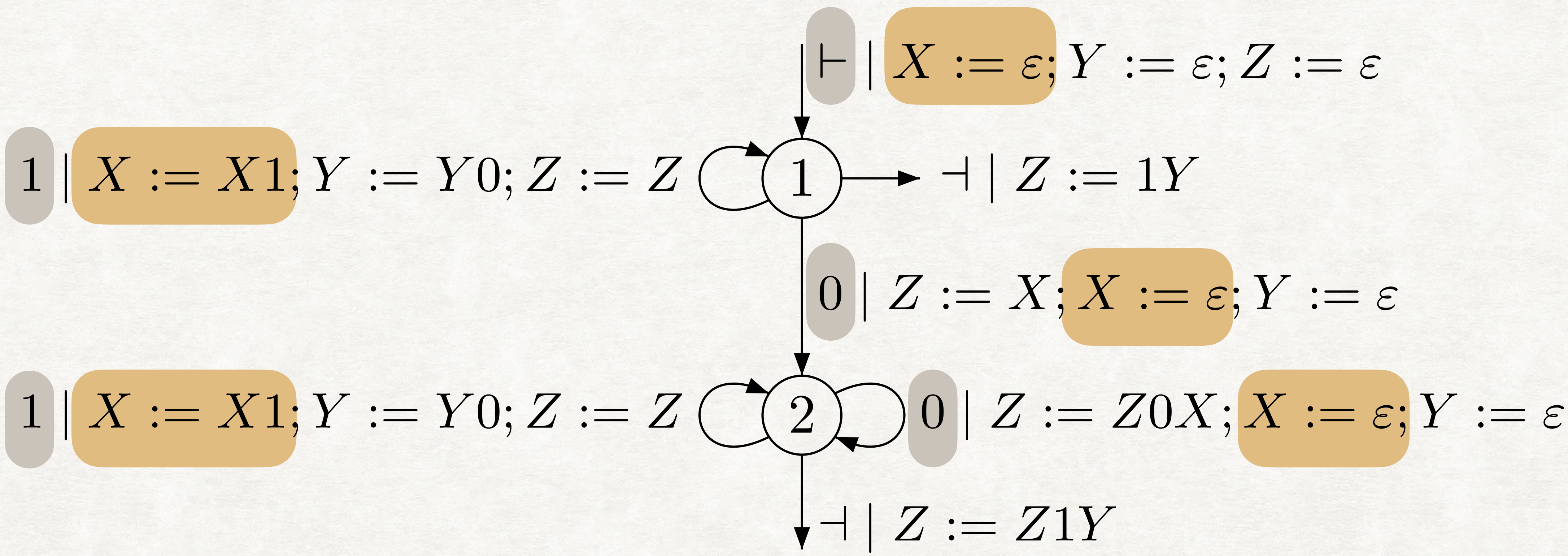

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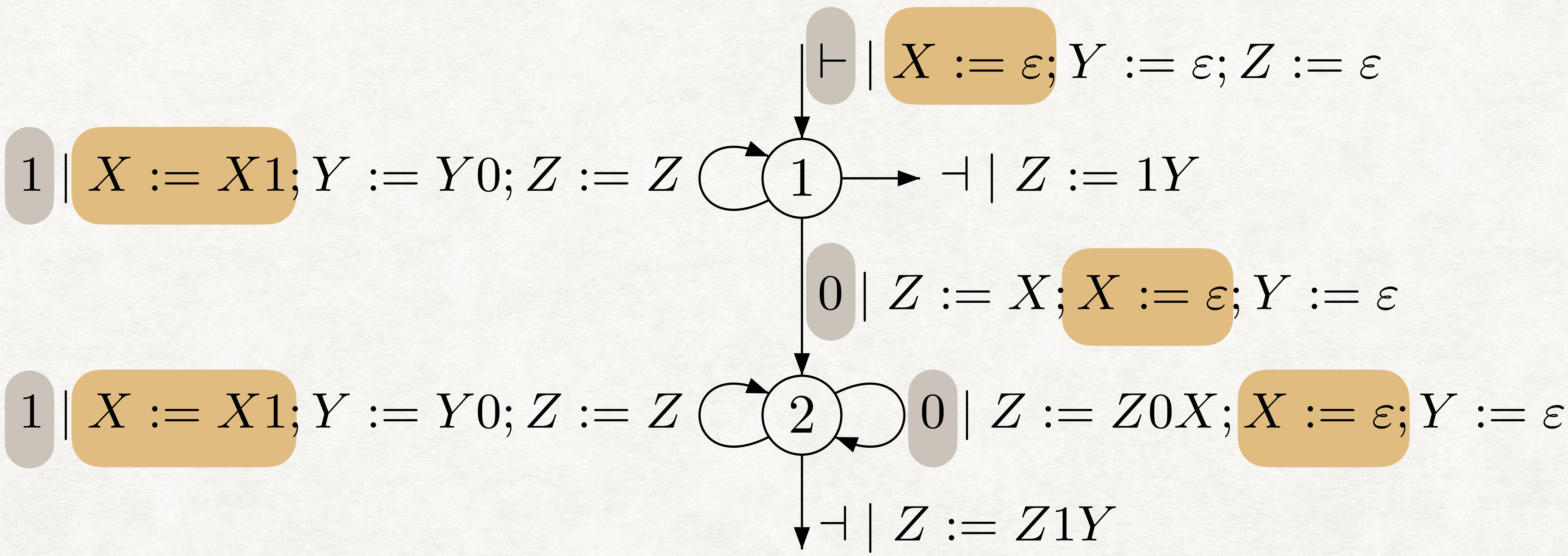

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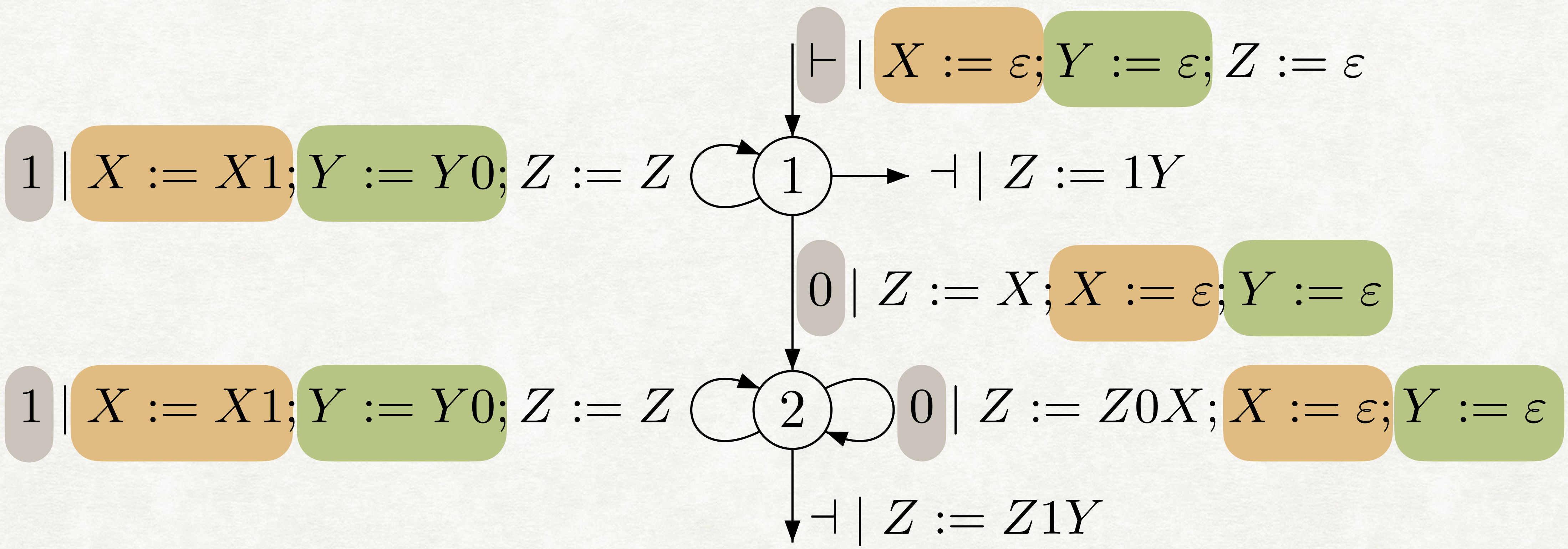

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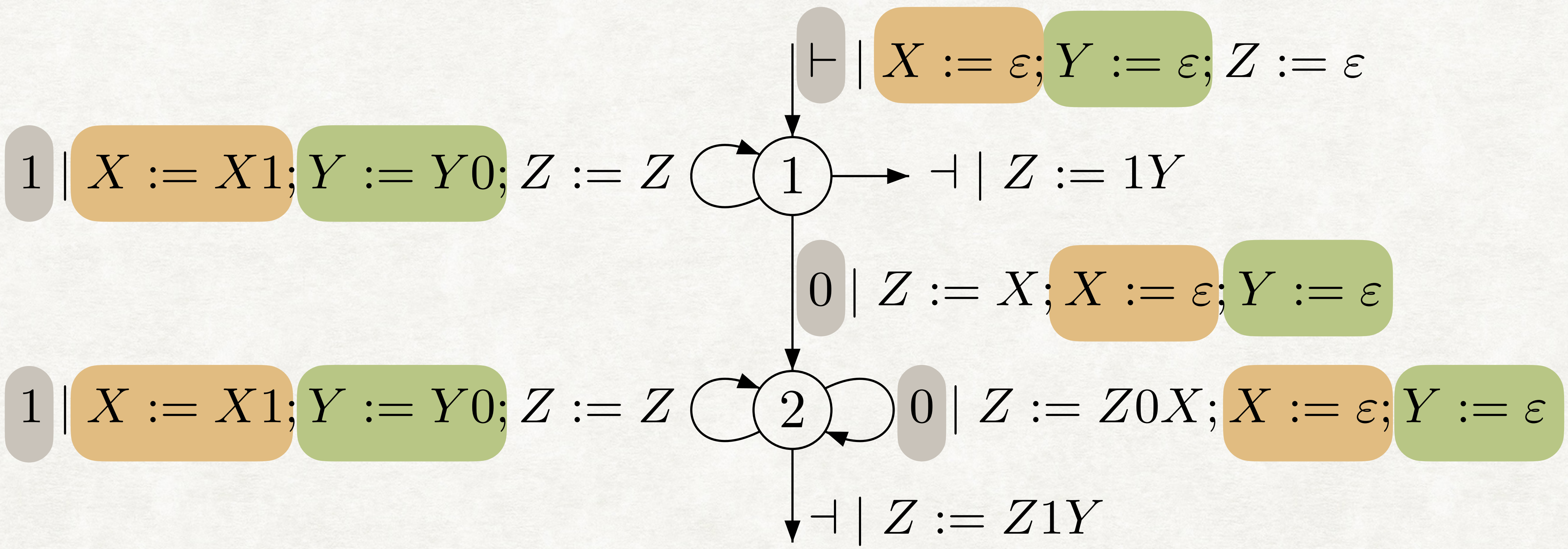

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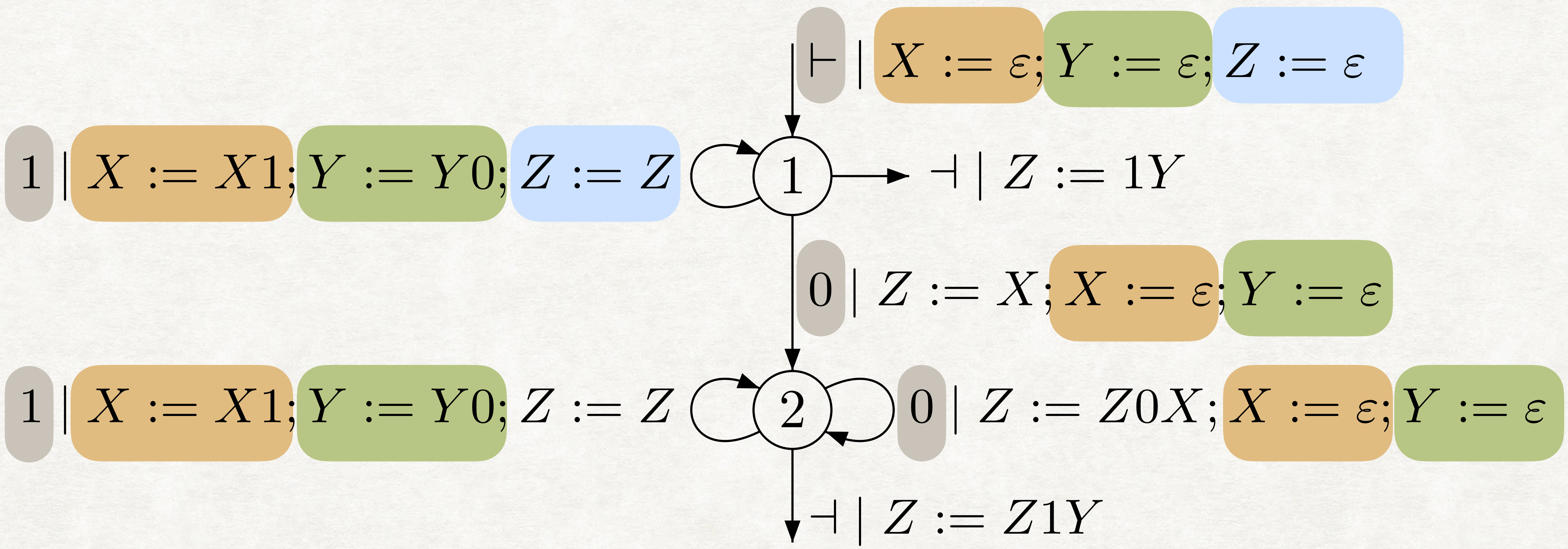

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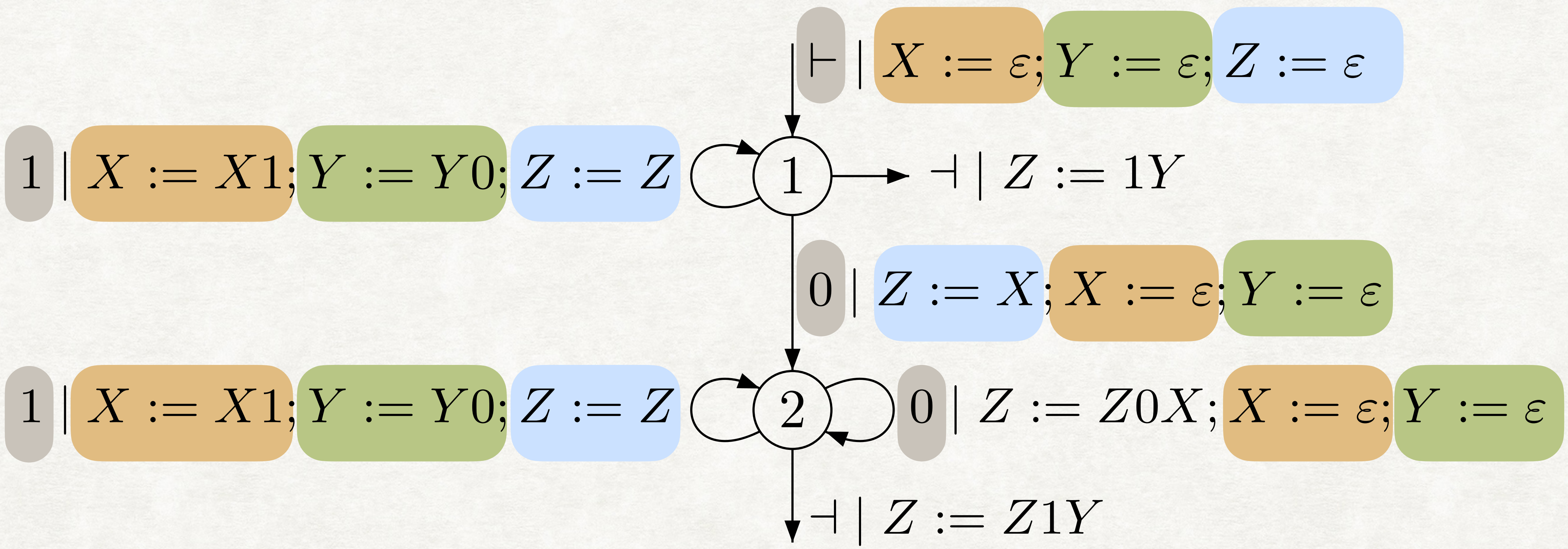

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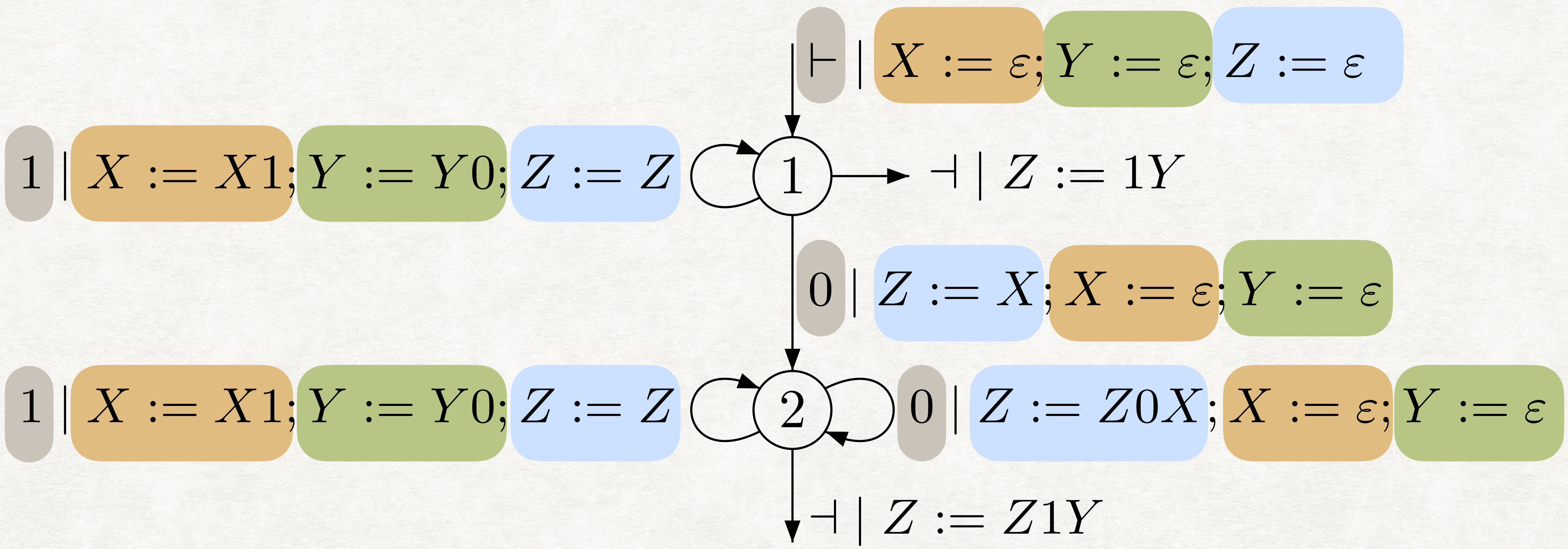

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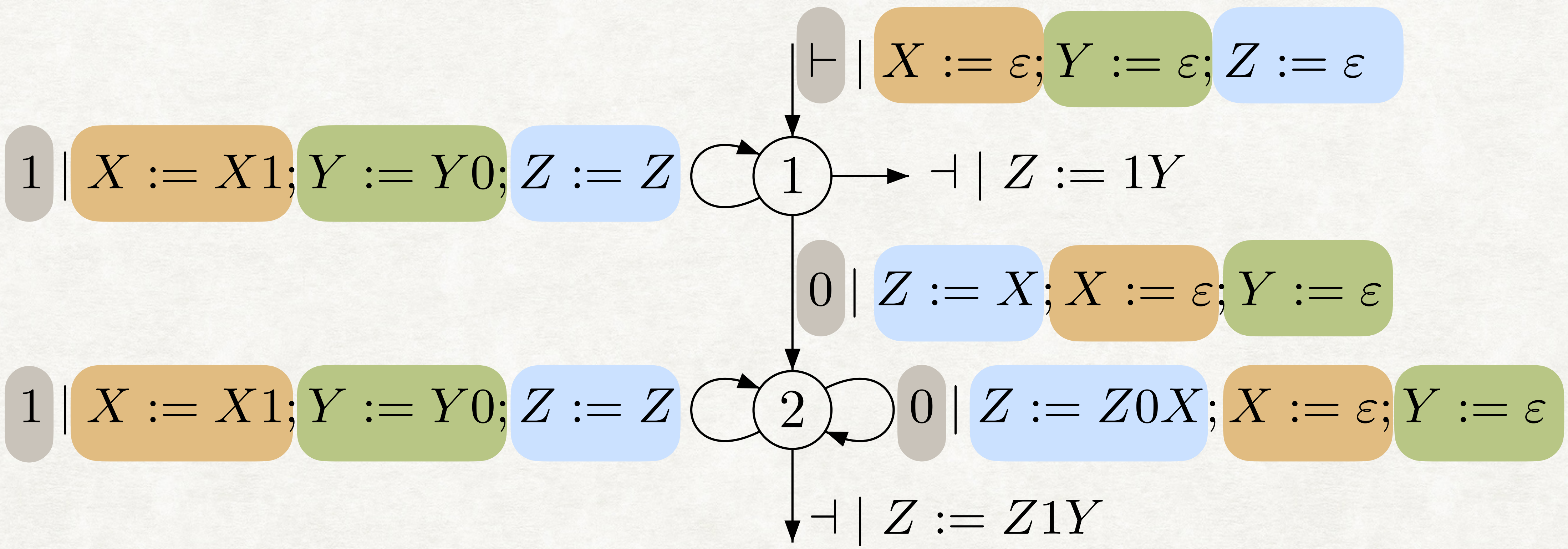

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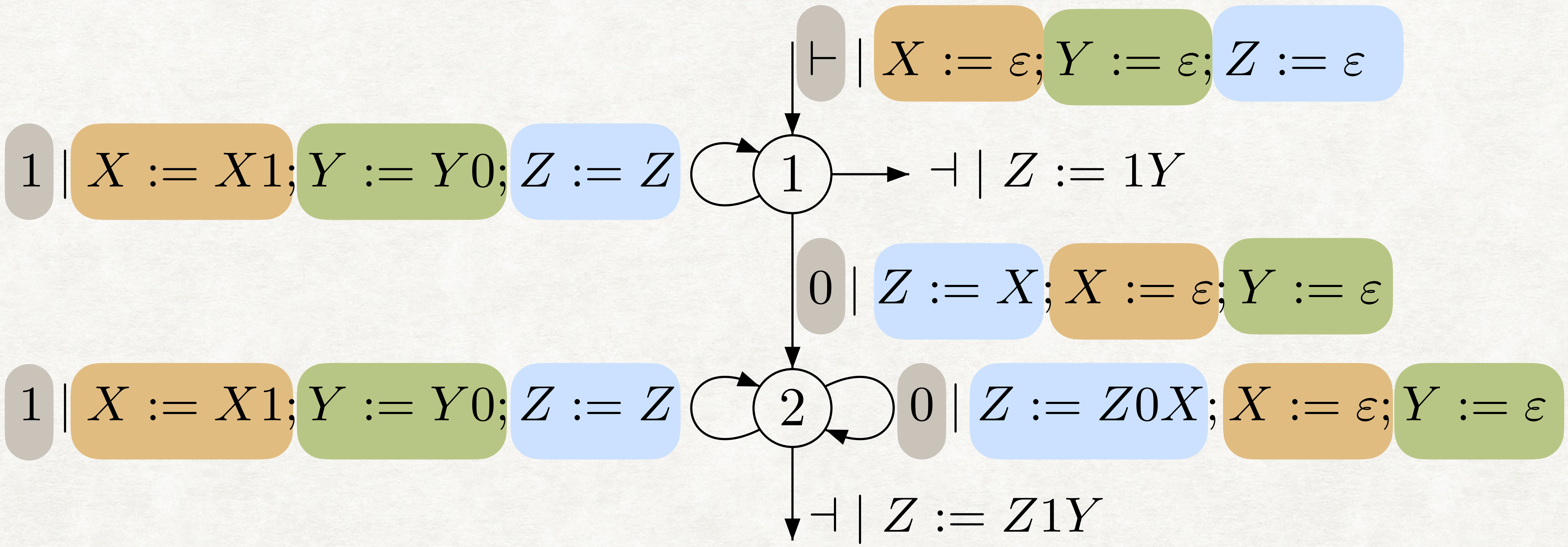

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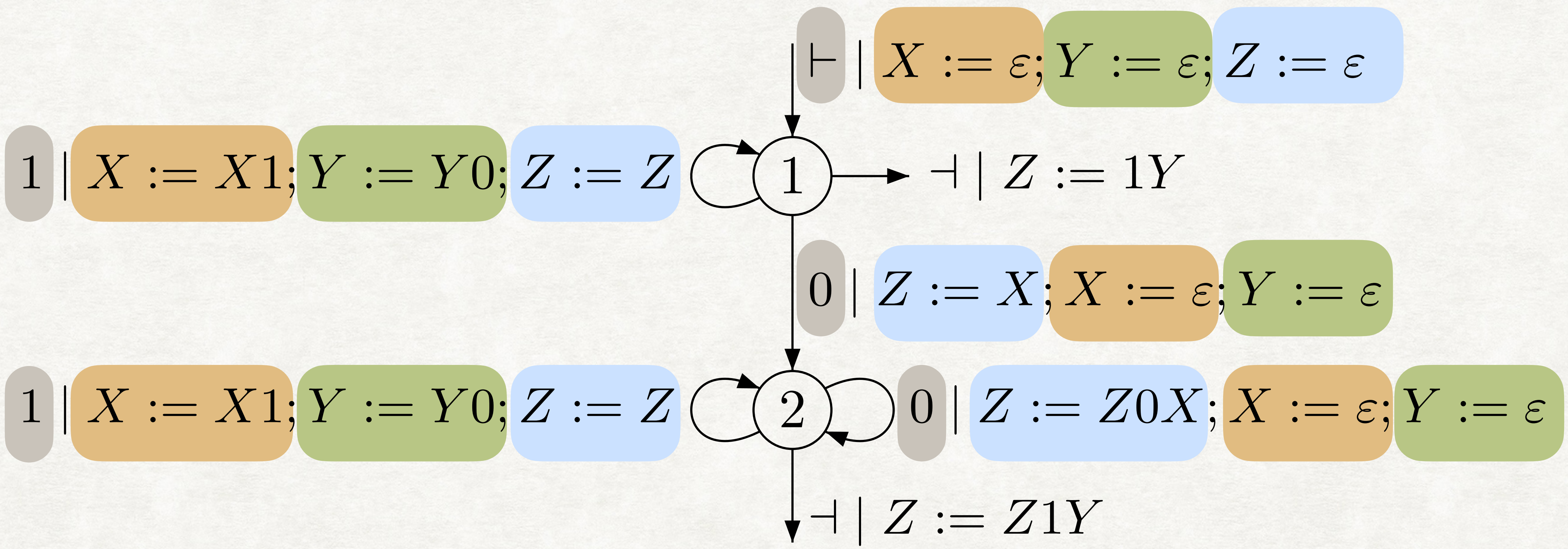

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$\mathsf{inc1} := (\varepsilon \mid 1) \cdot (1 \mid 0)^*$ $\qquad \mathsf{dom}(\mathsf{inc1}) = 1^*$

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$\mathsf{inc0} := \mathsf{copy} \cdot (0 \mid 1) \cdot (1 \mid 0)^*$ $\qquad \mathsf{dom}(\mathsf{inc0}) = (0 + 1)^* 01^*$

$\mathsf{inc1} := (\varepsilon \mid 1) \cdot (1 \mid 0)^*$ $\qquad \mathsf{dom}(\mathsf{inc1}) = 1^*$

$\mathsf{inc} := \mathsf{inc0} + \mathsf{inc1}$ $\qquad \mathsf{dom}(\mathsf{inc}) = (0 + 1)^*$

REGULAR TRANSDUCER EXPRESSIONS

- Compositional and modular
- Unambiguous input parsing with regular expressions

$(u \mid v)$ means "read u and output v"

$(1 \mid 0)^*$ replaces 1^n by 0^n

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$\mathsf{copy} \cdot (1 \mid 0)^*$ is ambiguous

$1011 = 10 \cdot 11 = 101 \cdot 1 = 1011 \cdot \varepsilon$

SIMPLE RTE

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$$f, g ::= (u, v) \mid f + g \mid f \cdot g \mid f^*$$

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- 1-way parsing (no reverse)
- No duplication, no composition

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➡ SRTE = Rational functions (1UFT = 1DFT with look-ahead = f1NFT)

➡ Special case of weighted automata (unambiguous)

[11] Schützenberger, On the definition of a family of automata, 1961

FULL RTE FOR REGULAR FUNCTIONS

- A 2DFT (or 2UFT) may read its input several times

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$$(\mathsf{copy} \cdot (\varepsilon \mid \$)) \odot \mathsf{copy}$$

exchange: $u\#v \mapsto vu$

$$\Big(\mathsf{erase} \cdot (\# \mid \varepsilon) \cdot \mathsf{copy}\Big) \odot \Big(\mathsf{copy} \cdot (\# \mid \varepsilon) \cdot \mathsf{erase}\Big)$$

$\mathsf{copy} := ((0 \mid 0) + (1 \mid 1))^*$ $\mathsf{erase} := ((0 \mid \varepsilon) + (1 \mid \varepsilon))^*$

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➡ Composition $\qquad (f\circ g)(w) = f(g(w))$

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duplicate: $w\mapsto w\$w$ erase: $w\mapsto\varepsilon$ exchange: $u\#v\mapsto vu$

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$$g\circ f\colon u_1\#u_2\#\cdots u_n\# \mapsto u_2u_1\#u_3u_2\#\cdots u_nu_{n-1}\#$$

$$h = g \circ f$$

duplicate: $w \mapsto w\$w$ erase: $w \mapsto \varepsilon$ exchange: $u\#v \mapsto vu$

RTE WITH COMPOSITION AND REGULAR FUNCTIONS

- A 2DFT may produce output while reading its input backwards

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WITH COMPOSITION, HADAMARD PRODUCT AND REVERSE

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$$f \odot g = (f \cdot (\$ \mid \varepsilon) \cdot g) \circ \textsf{duplicate}$$

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WITH COMPOSITION, DUPLICATE AND REVERSE

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CONCLUDING REMARKS

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- Sequential functions: 1DFT
- Rational functions: f1NFT = 1UFT = 1DFT with l-a = simple RTE
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