Dependencies in Strategy Logic*

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Abstract

Strategy Logic (SL) is a very expressive logic for expressing and verifying properties of multi-agent systems: in SL, one can quantify over strategies, assign them to agents, and express properties of the resulting plays. Such a powerful framework has two drawbacks: first, model checking SL has non-elementary complexity; second, the exact semantics of SL is rather intricate, and may not correspond to what is expected. In this paper, we focus on *strategy dependencies* in SL, by tracking how existentially-quantified strategies in a formula may (or may not) depend on other strategies selected in the formula. We study different kinds of dependencies, refining the approach of [Mogavero *et al.*, Reasoning about strategies: On the model-checking problem, 2014], and prove that they give rise to different satisfaction relations. In the setting where strategies may only depend on what they have observed, we identify a large fragment of SL for which we prove model checking can be performed in *2*-EXPTIME.

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1 Introduction

Temporal logics. Since Pnueli's seminal paper [18] in 1977, temporal logics have been widely used in theoretical computer science, especially by the formal-verification community. Temporal logics provide powerful languages for expressing properties of reactive systems, and enjoy efficient algorithms for satisfiability and model checking [8].

Since the early 2000s, new temporal logics have been developed in order to cope with *open* and *multi-agent systems*. While classical temporal logics (e.g. CTL [7, 19] and LTL [18]) could only deal with one or all the executions of the system, ATL [2] expresses properties on (executions generated by) strategies of agents in a game played on a graph. ATL has been extensively studied since then, both w.r.t. expressiveness and verification algorithms [2, 10, 12].

Strategic interactions in ATL. Strategies in ATL are handled in a very limited way, and there is no real *strategic interactions* in that logic (which, in return, enjoys polynomial-time model-checking algorithm). Over the last 10 years, various extensions have been defined and studied in order to allow for more interactions [1, 6, 5, 14, 20]. *Strategy Logic* (SL for short) [6, 14] is such a powerful approach, in which strategies are first-class citizens; formulas can quantify (universally and existentially) over strategies, store them in variables, assign them to players, and express properties of the resulting plays. As a simple example, the existence of a winning strategy for Player A (with objective ϕ_A) against any strategy of Player B would be written as $\exists\sigma_A.\ \forall\sigma_B.\ \mathsf{assign}(A \mapsto \sigma_A; B \mapsto \sigma_B).\ \phi_A$. This makes the logic both expressive and easy to use (at first sight), at the expense of a very high complexity: SL model checking is non-elementary [14, 11].

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Strategy dependencies in SL. It has been noticed in recent works that the nice expressiveness of SL comes with unexpected phenomena. One recently-identified phenomenon [4] is induced by the separation of strategy quantification and strategy assignment: are the events between strategy quantifications and strategy assignments part of the *memory* of the strategy? While both options may make sense, depending on the applications, only one of them makes model checking decidable [4].

A second phenomenon—which is the main focus of the present paper—concerns *strategy dependencies* [14]: in a formula such as $\forall\sigma_A.\ \exists\sigma_B.\ \xi$, the existentially-quantified strategy σ_B may depend on *the whole* strategy σ_A; in other terms, the action returned by strategy σ_B after some finite history ρ may depend on what strategy σ_A would play on any other history ρ'. Again, this may be desirable in some contexts, but it may also make sense to require that strategy σ_B after history ρ can be *computed* based solely on what has been observed along ρ. For some fragments of SL, both approaches have been shown equivalent; moreover, it has been conjectured that for fragments enjoying this equivalence, model checking would be doable in doubly-exponential time [16].

Our contributions. We follow this line of work by performing a more thorough exploration of strategy dependencies in SL. We mainly follow the framework of [16], based on a kind of Skolemization of the formula: for instance, a formula of the form $(\forall x_i \exists y_i)_i.\ \xi$ is satisfied if there exists a *dependence map* θ defining each existentially-quantified strategy y_j based on the universally-quantified strategies $(x_i)_i$. In order to recover the classical semantics of SL, it is only required that the strategy $\theta((x_i)_i)(y_j)$ (i.e. the strategy corresponding to y_j in $\theta((x_i)_i)$) only depends on $(x_i)_{i<j}$.

Based on this definition, other constraints can be imposed on dependence maps, in order to adapt the dependencies of existentially-quantified strategies w.r.t universally-quantified ones. We identify four possible kinds of strategy dependencies: with *local dependencies* (which are always allowed in our setting), existentially-quantified strategies have knowledge about previously-quantified strategies *only along the current history*; with *future* and *side dependencies*, they also get knowledge about previously-quantified strategies along *extensions* and *counterfactual* histories; finally, *timeline dependencies* allow existentially-quantified strategies to depend on what *all* universally-quantified strategies on the prefix of the current history (even those that are quantified deeper in the formula).

As we explain, the classical semantics of SL involves local, future and side dependencies; we prove that restricting those dependencies, and/or allowing timeline dependencies, give rise to different semantics of SL. We also focus on the syntactic negation in our logic: our semantics being defined as the *existence* of a dependence map, the syntactic negation may differ from the semantic negation. Finally, we focus on the setting where only local and timeline dependencies are allowed (which we believe is a relevant setting in many scenarios), and identify a large fragment of SL under which syntactic and semantic negation coincide, and for which we develop a *2*-EXPTIME model-checking algorithm.

By lack of space, only a few sketches of proofs are given in the main body of the paper. The technical appendix contains detailed proofs.

2 Definitions

2.1 Concurrent game structures

For the rest of this paper, we fix a finite set AP of atomic propositions, a finite set $\mathcal{V}$ of variables, and a finite set Agt of agents (or players).

A *concurrent game structure* over a set AP of atomic propositions is a tuple $\mathcal{G} = \langle \mathsf{Act}, \mathsf{Q}, \Delta, \mathsf{lab}\rangle$ where Act is a finite set of actions, Q is a finite set of states, $\Delta\colon \mathsf{Q} \times \mathsf{Act}^{\mathsf{Agt}} \to \mathsf{Q}$ is the transition function, and $\mathsf{lab}\colon \mathsf{Q} \to 2^{\mathsf{AP}}$ is a labelling function. An element of $\mathsf{Act}^{\mathsf{Agt}}$ will be called a *move vector*. For any $q \in \mathsf{Q}$, we let $\mathsf{succ}(q)$ be the set $\{q' \in Q \mid \exists m \in \mathsf{Act}^{\mathsf{Agt}}.\ q' = \Delta(q, m)\}$. For the sake of simplicity, we assume in the sequel that $\mathsf{succ}(q) \neq \varnothing$ for any $q \in Q$. A game $\mathcal{G}$ is said *turn-based* whenever for every state $q \in \mathsf{Q}$, there is a player $\mathsf{own}(q) \in \mathsf{Agt}$ (named the *owner* of q) such that for any two move vectors m_1 and m_2 with $m_1(\mathsf{own}(q)) = m_2(\mathsf{own}(q))$, it holds $\Delta(q, m_1) = \Delta(q, m_2)$. Figure 1 displays an example of a (turn-based) game.

Fix a state $q \in \mathsf{Q}$. A *play* in $\mathcal{G}$ from q is an infinite sequence $\pi = (q_i)_{i\in\mathbb{N}}$ of states in Q such that $q_0 = q$ and $q_i \in \mathsf{succ}(q_{i-1})$ for all $i > 0$. We write $\mathsf{Play}_{\mathcal{G}}(q)$[1] for the set of plays in $\mathcal{G}$ from q. A (strict) prefix of a play π is a finite sequence $\rho = (q_i)_{0\leq i\leq L}$, for some $L \in \mathbb{N}$. We let $\mathsf{Pref}(\pi)$ for the set of strict prefixes of play π. Such finite prefixes are called *histories*, and we let $\mathsf{Hist}_{\mathcal{G}}(q) = \mathsf{Pref}(\mathsf{Play}_{\mathcal{G}}(q))$. We extend the notion of strict prefixes and the notation Pref to histories in the natural way, hence requiring that $\rho \notin \mathsf{Pref}(\rho)$. A (finite) extension of a history ρ is any history ρ' such that $\rho \in \mathsf{Pref}_(\rho')$. Let $\rho = (q_i)_{i\leq L}$ be a history. We define $\mathsf{first}(\rho) = q_0$ and $\mathsf{last}(\rho) = q_L$. Let $\rho' = (q'_j)_{j\leq L'}$ be a history from $\mathsf{last}(\rho)$. The *concatenation* of ρ and ρ' is then defined as the path $\rho \cdot \rho' = (q''_k)_{k\leq L+L'}$ such that $q''_k = q_k$ when $k \leq L$ and $q''_k = q'_{k-L}$ when $L > k$.

A *strategy* from q is a mapping $\delta\colon \mathsf{Hist}_{\mathcal{G}}(q) \to \mathsf{Act}$. We write $\mathsf{Strat}_{\mathcal{G}}(q)$ for the set of strategies in $\mathcal{G}$ from q. Given a strategy $\delta \in \mathsf{Strat}(q)$ and a history ρ from q, the *translation* $\delta_{\overrightarrow{\rho}}$ of δ by ρ is the strategy $\delta_{\overrightarrow{\rho}}$ from $\mathsf{last}(\rho)$ defined by $\delta_{\overrightarrow{\rho}}(\rho') = \delta(\rho \cdot \rho')$ for any $\rho' \in \mathsf{Hist}(\mathsf{last}(\rho))$. A *valuation* from q is a partial function $\chi\colon \mathcal{V} \cup \mathsf{Agt} \to \mathsf{Strat}(q)$. As usual, for any partial function f, we write $\mathsf{dom}(f)$ for the domain of f.

Let $q \in Q$ and χ be a valuation from q. If $\mathsf{Agt} \subseteq \mathsf{dom}(\chi)$, then χ induces a unique play from q, called its *outcome*, and defined as $\mathsf{out}(q, \chi) = (q_i)_{i\in\mathbb{N}}$ such that $q_0 = q$ and for every $i \in \mathbb{N}$, we have $q_{i+1} = \Delta(q_i, m_i)$ with $m_i(A) = \chi(A)((q_j)_{j\leq i})$ for every $A \in \mathsf{Agt}$.

2.2 Strategy Logic with boolean goals

Strategy Logic (SL for short) was introduced in [6], and further extended and studied in [17, 14], as a rich logical formalism for expressing properties of games. SL manipulates strategies as first-order elements, assigns them to players, and expresses LTL properties on the outcomes of the resulting strategic interactions. This results in a very expressive temporal logic, for which satisfiability is undecidable [17] and model checking is TOWER-complete [14, 3]. In this paper, we focus on a restricted fragment of SL, called SL[BG] (where BG stands for *boolean goals*) [14], which we now introduce.

Syntax. Formulas in $\mathsf{SL[BG]}^\flat$ are built along the following grammar[2]:

$$\mathsf{SL[BG]}^\flat \ni \phi ::= \exists x.\ \phi \mid \neg\phi \mid \xi \qquad \xi ::= \neg\xi \mid \xi \vee \xi \mid \beta$$
$$\beta ::= \mathsf{assign}(\sigma).\ \psi \qquad \psi ::= \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi \mid p$$

where x ranges over $\mathcal{V}$, σ ranges over the set $\mathcal{V}^{\mathsf{Agt}}$ of *full assignment*, and p ranges over AP. A *goal* is a formula of the form β in the grammar above; it expresses an LTL property ψ on

[1] In this and all similar notations, we might omit to mention $\mathcal{G}$ when it is clear from the context, and q when we consider the union over all $q \in Q$.

[2] As we explain later, we actually only consider the *flat fragment* $\mathsf{SL[BG]}^\flat$ of SL[BG]. This simplifies the presentation, and our results extend to SL[BG] in a straightforward way.

the outcome of the mapping σ. Formulas in $\mathsf{SL[BG]}^\flat$ are thus made of an initial block of first-order quantifiers (selecting strategies for variables in $\mathcal{V}$), followed by a boolean combination of such goals.

Free variables. With any subformula ζ of some formula $\phi \in \mathsf{SL[BG]}^\flat$, we associate its set of *free agents and variables*, which we write $\mathsf{free}(\zeta)$. It contains the agents and variables that have to be mapped to a strategy in order to univocally evaluate ζ (as will be seen from the definition of the semantics of $\mathsf{SL[BG]}^\flat$ below). The set $\mathsf{free}(\zeta)$ is defined inductively as follows:

$$\begin{aligned} \mathsf{free}(p) &= \varnothing \quad \text{for all } p \in \mathsf{AP} & \mathsf{free}(\mathbf{X}\,\psi) &= \mathsf{Agt} \cup \mathsf{free}(\psi) \\ \mathsf{free}(\neg\alpha) &= \mathsf{free}(\alpha) & \mathsf{free}(\psi_1\,\mathbf{U}\,\psi_2) &= \mathsf{Agt} \cup \mathsf{free}(\psi_1) \cup \mathsf{free}(\psi_2) \\ \mathsf{free}(\alpha_1 \vee \alpha_2) &= \mathsf{free}(\alpha_1) \cup \mathsf{free}(\alpha_2) & \mathsf{free}(\exists x.\ \phi) &= \mathsf{free}(\phi) \setminus \{x\} \end{aligned}$$

$$\mathsf{free}(\mathsf{assign}(\sigma).\ \phi) = \begin{cases} \mathsf{free}(\phi) & \text{if } \mathsf{Agt} \not\subseteq \mathsf{free}(\phi) \\ (\mathsf{free}(\phi) \cup \sigma(\mathsf{Agt})) \setminus \mathsf{Agt} & \text{otherwise} \end{cases}$$

Subformula ζ is said to be *closed* whenever $\mathsf{free}(\zeta) = \varnothing$. We can now comment our choice of considering the flat fragment of $\mathsf{SL[BG]}$: the full fragment, as defined in [14], allows for nesting *closed* $\mathsf{SL[BG]}$ formulas in place of atomic propositions. Since they are required to be closed, those nested subformulas can be handled independently of the rest of the formula.

Semantics. Fix a state $q \in Q$ of $\mathcal{G}$, and a valuation $\chi\colon \mathcal{V} \cup \mathsf{Agt} \to \mathsf{Strat}(q)$. We inductively define the semantics of a subformula α of a formula of $\mathsf{SL[BG]}^\flat$ at q under valuation χ, assuming that $\mathsf{free}(\alpha) \subseteq \mathsf{dom}(\chi)$. We omit the easy cases of boolean combinations and atomic propositions.

Given a mapping $\sigma\colon \mathsf{Agt} \to \mathcal{V}$, the semantics of strategy assignments is defined as follows:

$$\mathcal{G}, q \models_\chi \mathsf{assign}(\sigma).\ \psi \quad\iff\quad \mathcal{G}, q \models_{\chi[A \in \mathsf{Agt} \mapsto \chi(\sigma(A))]} \psi.$$

One may notice that, writing $\chi' = \chi[A \in \mathsf{Agt} \mapsto \chi(\sigma(A))]$, we have $\mathsf{free}(\psi) \subseteq \mathsf{dom}(\chi')$ if $\mathsf{free}(\alpha) \subseteq \mathsf{dom}(\chi)$, so that our inductive definition is sound.

We now consider path formulas $\psi = \mathbf{X}\,\psi_1$ and $\psi = \psi_1\,\mathbf{U}\,\psi_2$. Since $\mathsf{Agt} \subseteq \mathsf{free}(\psi) \subseteq \mathsf{dom}(\chi)$, the valuation χ induces an outcome $\mathsf{out}(q, \chi) = (q_i)_{i\in\mathbb{N}}$ from q. For $n \in \mathbb{N}$, we write $\mathsf{out}_n(q, \chi) = (q_i)_{i \le n}$, and define $\chi_{\overrightarrow{n}}$ as the valuation obtained by shifting all the strategies in the image of χ by $\mathsf{out}_n(q, \chi)$. Under the same conditions, we also define $q_{\overrightarrow{n}} = \mathsf{last}(\mathsf{out}_n(q, \chi))$. We then set

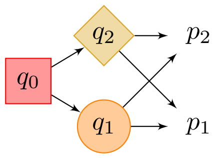
$$\begin{aligned} \mathcal{G}, q \models_\chi \mathbf{X}\,\psi_1 \quad&\iff\quad \mathcal{G}, q_{\overrightarrow{1}} \models_{\chi_{\overrightarrow{1}}} \psi_1 \\ \mathcal{G}, q \models_\chi \psi_1\,\mathbf{U}\,\psi_2 \quad&\iff\quad \exists k \in \mathbb{N}.\ \mathcal{G}, q_{\overrightarrow{k}} \models_{\chi_{\overrightarrow{k}}} \psi_2 \quad\text{and}\quad \forall 0 \le j < k.\ \mathcal{G}, q_{\overrightarrow{j}} \models_{\chi_{\overrightarrow{j}}} \psi_2. \end{aligned}$$

It remains to define the semantics of the strategy quantifiers. This is actually what this paper is all about. We provide here the original semantics, and discuss alternatives in the following sections:

$$\mathcal{G}, q \models_\chi \exists x.\phi \quad\iff\quad \exists \delta \in \mathsf{Strat}(q).\ \mathcal{G}, q \models_{\chi[x \mapsto \delta]} \phi.$$

In the sequel, we heavily use some classical shorthands, such as $\top$ for $p \vee \neg p$ (for any $p \in \mathsf{AP}$), $\alpha_1 \wedge \alpha_2$ for $\neg(\neg\alpha_1 \vee \neg\alpha_2)$, $\forall x.\phi$ for $\neg\exists x.\neg\phi$, $\mathbf{F}\,\psi$ for $\top\,\mathbf{U}\,\psi$, and $\mathbf{G}\,\psi$ for $\neg\mathbf{F}\,\neg\psi$.

▶ **Example 1.** We consider the (turn-based) game $\mathcal{G}$ is depicted on Fig. 1. We name the players after the shape of the state they control. The $\mathsf{SL[BG]}$ formula ϕ to the right of Fig. 1



$$\phi = \forall y.\exists z.\forall x_A.\forall x_B.\ \bigvee \begin{cases} \mathsf{assign}(\square \mapsto x_A; \bigcirc \mapsto y; \Diamond \mapsto z).\ \mathbf{F}\, p_1 \\ \mathsf{assign}(\square \mapsto x_B; \bigcirc \mapsto y; \Diamond \mapsto z).\ \mathbf{F}\, p_2 \end{cases}$$

Figure 1 A game and a SL[BG] formula.

has four quantified variables and two goals. We show that this formula evaluates to true in $\mathcal{G}$: fix any strategy δ_y (to be played by player $\bigcirc$); because $\mathcal{G}$ is turn-based, we identify the actions of the owner of a state with the resulting target state, so that $\delta_y(q_0q_1)$ will be either p_1 or p_2. We then define strategy δ_z (to be played by $\Diamond$) as $\delta_z(q_0q_2) = \delta_y(q_0q_1)$. Then clearly, for any strategy assigned to player $\square$, one of the goals of formula ϕ holds true, so that ϕ itself evaluates to true.

Subclasses of SL[BG]. Several restrictions of SL[BG] have been considered in the literature [13, 15, 16], by adding further restrictions to boolean combinations in the grammar defining the syntax:

- SL[1G] restricts SL[BG] to a unique goal. SL[1G] is then defined from the grammar of SL[BG] by setting $\xi ::= \beta$ in the grammar;
- SL[CG] is the fragment where only *conjunctions* of goals are allowed. It corresponds to formulas defined with $\xi ::= \xi \wedge \xi \mid \beta$;
- similarly, SL[DG] only allows *disjunctions* of goals, i.e. $\xi ::= \xi \vee \xi \mid \beta$;
- finally, SL[AG] mixes conjunctions and disjunctions in a restricted way. Goals in SL[AG] can be combined using the following grammar: $\xi ::= \beta \wedge \xi \mid \beta \vee \xi \mid \beta$.

In the sequel, we write a generic SL[BG] formula ϕ as $(Q_ix_i)_{i\leq l}.\ \xi(\beta_j.\ \psi_j)_{j\leq n}$ where:

- $(Q_ix_i)_{i\leq l}$ is a block of quantifications, with $\{x_i \mid 1 \leq i \leq l\} \subseteq \mathcal{V}$ and $Q_i \in \{\exists, \forall\}$, for every $1 \leq i \leq l$;
- $\xi(g_1, ..., g_n)$ is a boolean combination of its arguments;
- for each $1 \leq j \leq n$, subformula $\beta_j.\ \psi_j$ is a goal: each β_j is a full assignment and ψ_j is an LTL formula.

3 Side and future dependencies

We now follow the framework of [14, 15] and define the semantics of SL[BG] in terms of *dependence maps*. This approach provides a fine way of controlling how *existentially-quantified* strategies depend on previously selected strategies (in a quantifier block). Considering again Example 1, we notice that the value of the existentially-quantified strategy δ_z after history q_0q_2 depends on the value of strategy δ_y on history q_0q_1, i.e. on a *side* history. Using dependence maps, we can limit such dependencies.

Fix a history ρ from some state q. We call *future history* of ρ any extension of ρ, and *side history* of ρ any extension ρ' of a (strict) prefix of ρ that is not ρ nor a future history of ρ. Hence the set $\mathsf{Hist}_{\mathcal{G}}(q)$ is partitioned into four sets: ρ itself, prefixes, future histories and side histories of ρ. Using dependence maps, we will be able to make existentially-quantified strategies only depend on the values of other (previously-quantified) strategies along prefixes and/or future- and/side histories. Figure 2 illustrates such dependencies.

Dependence maps. In order to investigate those dependencies, we extend the concept of *dependence maps*, introduced in [13, 14], to a more general framework.

Consider an $\mathsf{SL[BG]}$ formula $\phi = (Q_i x_i)_{i\leq l}.\ \xi(\beta_j.\ \varphi_j)_{j\leq n}$, assuming w.l.o.g. that $\{x_i \mid 1 \leq i \leq l\} = \mathcal{V}$. We let $\mathcal{V}^\forall = \{x_i \mid Q_i = \forall\} \subseteq \mathcal{V}$ be the set of universally-quantified variables of ϕ. A function $\theta\colon \mathsf{Strat}^{\mathcal{V}^\forall} \to \mathsf{Strat}^{\mathcal{V}}$ is a *ϕ-map* (or *map* when ϕ is clear from the context) if $\theta(w)(x_i)(\rho) = w(x_i)(\rho)$ for any $w \in \mathsf{Strat}^{\mathcal{V}^\forall}$, any $x_i \in \mathcal{V}^\forall$, and any history ρ. In other words, $\theta(w)$ extends w to $\mathcal{V}$.

We can then constrain strategy dependencies by applying adequate restrictions to the maps. In this section, we consider three kinds of dependencies: *local dependencies*, *side dependencies* and *future dependencies*. This gives rise to eight types of maps, denoted by $\mathcal{M}(D)$, with $D \subseteq \{L, S, F\}$. To alleviate notations, we will write e.g. $\mathcal{M}(L, S)$ for $\mathcal{M}(\{L, S\})$. A map in $\mathcal{M}(D)$ has the following additional restriction:

$$\left.\begin{array}{l}\forall w_1, w_2 \in \mathsf{Strat}^{\mathcal{V}^\forall}\\ \forall \rho \in \mathsf{Hist},\ \forall x_i \in \mathcal{V}\end{array}\right\} \Big(\bigwedge_{d\in D} \mathcal{C}(d)\Big) \Rightarrow \Big(\theta(w_1)(x_i)(\rho) = \theta(w_2)(x_i)(\rho)\Big) \tag{1}$$

with

- $\mathcal{C}(L)$: w_1 and w_2 coincide on $\mathcal{V}^\forall \cap [x_1; x_{i-1}]$ and on ρ and on all its prefixes (local)
- $\mathcal{C}(S)$: w_1 and w_2 coincide on $\mathcal{V}^\forall \cap [x_1; x_{i-1}]$ and on side histories of ρ (side)
- $\mathcal{C}(F)$: w_1 and w_2 coincide on $\mathcal{V}^\forall \cap [x_1; x_{i-1}]$ and on future histories of ρ (future)

▶ **Example 2.** Consider the quantifier block $\forall x_1.\exists x_2$, and a map θ in $\mathcal{M}(L)$. Assume that there are two strategies δ_1 and δ_1' such that $\theta(x_1 \mapsto \delta_1)(x_2)$ and $\theta(x_1 \mapsto \delta_1')(x_2)$ differ (on some history ρ). Then condition $\mathcal{C}(L)$ must fail to hold, which means that δ_1 and δ_1' must differ on some prefix of ρ. In other terms, if δ_1 and δ_1' coincide on any prefix of ρ (and on ρ itself), then we must have $\theta(x_1 \mapsto \delta_1)(x_2)(\rho) = \theta(x_1 \mapsto \delta_1')(x_2)(\rho)$. This amounts to saying that $\theta(x_1 \mapsto \delta_1)(x_2)(\rho)$ only depends on the value of δ_1 on ρ and on its prefixes.

On the other hand, if θ were in $\mathcal{M}(L, S, F)$, then for the strategies $\theta(x_1 \mapsto \delta_1)(x_2)$ and $\theta(x_1 \mapsto \delta_1')(x_2)$ to differ on some path ρ, the conjunction of $\mathcal{C}(L)$, $\mathcal{C}(S)$ and $\mathcal{C}(F)$ must fail to hold; this simply implies that δ_1 and δ_1' must differ on some history. Notice in particular that $\mathcal{M}(L) \subseteq \mathcal{M}(L, S, F)$. More precisely, it can be observed that $\mathcal{M}(D) \subseteq \mathcal{M}(D')$ for any $D \subseteq D' \subseteq \{L, S, F\}$.

In the rest of this paper, local dependencies will always be allowed, so that when writing $\mathcal{M}(D)$, we will always have $L \in D$.

Satisfaction relations. We now define four satisfaction relations $(\models^{\mathcal{M}(D)})_{L\in D\subseteq\{L,S,F\}}$, one for the each kind of maps we consider. The definitions of these relations only differ from the definition of $\models$ (as defined in Section 2) over strategy quantification. Pick a formula $\phi = (Q_i x_i)_{i\leq l}.\ \xi\big(\beta_j.\ \varphi_j\big)_{j\leq n}$ in $\mathsf{SL[BG]}$. We define:

$$\mathcal{G}, q \models^{\mathcal{M}(D)} \phi \quad \text{iff} \quad \exists\theta \in \mathcal{M}(D).\ \forall w \in \mathsf{Strat}^{\mathcal{V}^\forall}.\ \mathcal{G}, q \models_{\theta(w)} \xi\big(\beta_j \varphi_j\big)_{j\leq n}$$

In such a case, θ is called an $\mathcal{M}(D)$-*witness* of ϕ. Following [14, Theorem 4.6], the relation $\models^{\mathcal{M}(L,S,F)}$ corresponds to the usual semantics of strategy quantifiers in $\mathsf{SL[BG]}$, as given in Section 2. Also, from the inclusion at the end of Example 2, we have $\models^{\mathcal{M}(D)} \subseteq \models^{\mathcal{M}(D')}$ for any $D \subseteq D'$. This corresponds to the intuition that it is harder to satisfy a $\mathsf{SL[BG]}$ formula when dependencies are more restricted. But it in turn pinpoints the question of what it means to satisfy the (syntactic) negation of a formula in this setting.

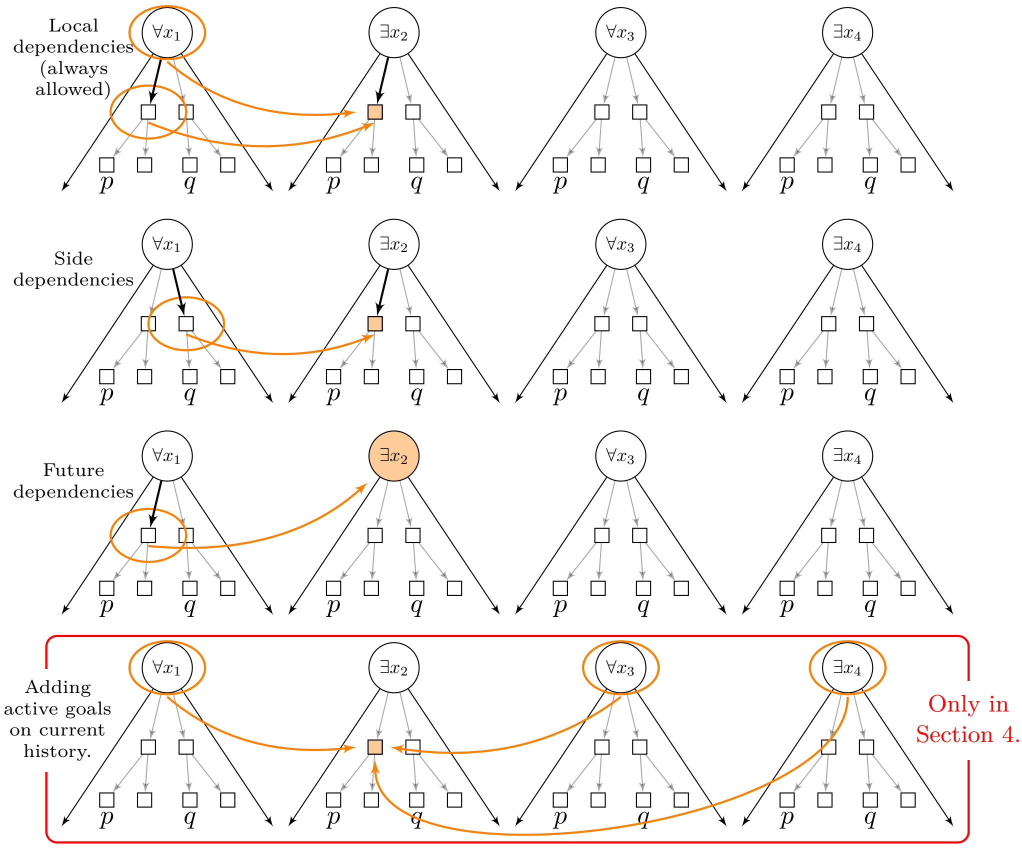


Figure 2 The different kinds of dependencies.

The syntactic negation. If $\phi = (Q_i x_i)_{i\leq l}\xi(\beta_j\varphi_j)_{j\leq n}$ is an SL[BG] formula, its syntactic negation $\neg\phi$ is the formula $(\overline{Q}_i x_i)_{i\leq l}(\neg\xi)(\beta_j\varphi_j)_{j\leq n}$, where $\overline{Q}_i = \exists$ if $Q_i = \forall$ and $\overline{Q}_i = \forall$ if $Q_i = \exists$. Looking at the definition of $\models^{\mathcal{M}(D)}$, it could be the case that $\mathcal{G}, q \models^{\mathcal{M}(D)} \phi$ and $\mathcal{G}, q \models^{\mathcal{M}(D)} \neg\phi$: this only entails the existence of two adequate maps.

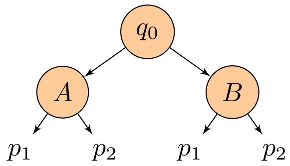
However, since $\models^{\mathcal{M}(L,S,F)}$ and $\models$ coincide, and since $\mathcal{G}, q \models \phi \iff \mathcal{G}, q \not\models \neg\phi$, we get that $\mathcal{G}, q \models^{\mathcal{M}(L,S,F)} \phi \iff \mathcal{G}, q \not\models^{\mathcal{M}(L,S,F)} \neg\phi$. Moreover, for any $D \subseteq \{L, S, F\}$ containing L, since $\mathcal{M}(D) \subseteq \mathcal{M}(L, S, F)$, then we also get that $\mathcal{G}, q \models^{\mathcal{M}(D)} \phi \implies \mathcal{G}, q \not\models^{\mathcal{M}(D)} \neg\phi$. However, as we now show, the converse implication may fail to hold. Notice that this is not specific to our setting, and that the following result already holds (but, to our knowledge, had never been mentioned) with the *elementary semantics* of SL[BG], as defined in [14].

▶ **Proposition 3.** *For any $D \subsetneq \{L, S, F\}$ containing L, there exist a game $\mathcal{G}$ with initial state q_0 and a formula $\phi \in$ SL[BG]$^\flat$ such that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \phi$ and $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \neg\phi$.*

Proof. We prove the result for $\mathcal{M}(L)$; the other two cases are handled in Appendix A. Consider the formula and the one-player game of Fig. 3.

We start by proving that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L)} \phi$. We first notice that $y(q_0)$ must be B for the first conjunct to be satisfied. Now, if $x(q_0) = A$, then we must have $y(q_0 B) = x(q_0 A)$ in order to fulfill the second conjunct. Such dependencies are not allowed in $\mathcal{M}(L)$.

On the other hand, we can also prove that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L)} \neg\phi$. Indeed, following the previous discussion, we easily get that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F)} \phi$, by letting $y(q_0) = B$ and $y(q_0 B) = x(q_0 A)$ if $x(q_0) = A$ and $y(q_0 B) = x(q_0 B)$ if $x(q_0) = B$. As explained above, this entails



$$\phi = \forall x.\exists y \bigwedge \begin{cases} \mathsf{assign}(\bigcirc \mapsto y).\ \mathbf{F}\, B \\ \mathsf{assign}(\bigcirc \mapsto x).\ \mathbf{F}\, p_1 \Leftrightarrow \mathsf{assign}(\bigcirc \mapsto y).\ \mathbf{F}\, p_1 \end{cases}$$

Figure 3 A game $\mathcal{G}$ and an $\mathsf{SL[BG]}^\flat$ formula ϕ such that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L)} \phi$ and $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L)} \neg\phi$.

$\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,S,F)} \neg\phi$. Since $\mathcal{M}(L) \subseteq \mathcal{M}(L,S,F)$, this in turn entails $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L)} \neg\phi$. ◀

When restricting to $\mathsf{SL[1G]}$, we recover the equivalence of the semantic and syntactic negations:

▶ **Proposition 4.** *For any $D \subseteq \{L,S,F\}$ containing L, every game $\mathcal{G}$ with initial state q_0, and every formula $\phi \in$ SL[1G], it holds $\mathcal{G}, q_0 \models^{\mathcal{M}(D)} \phi \iff \mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \neg\phi$.*

Sketch of proof. It suffices to prove the result for $\mathcal{M}(L)$, since $\mathcal{M}(L) \subseteq \mathcal{M}(D)$. The full proof is given in Appendix B. When restricting to the satisfaction relation $\models^{\mathcal{M}(L)}$, the interaction between existential and universal quantifications of the formula can be integrated into the game, replacing each state with a tree-shaped subgame where Player $P_\exists$ selects existentially-quantified actions and Player $P_\forall$ selects universally-quantified ones. The unique goal of the formula is then incorporated into the game via a deterministic parity automaton, yielding a two-player turn-based parity game. We can then show that the $\models^{\mathcal{M}(L)}$-truth of ϕ is equivalent to having a winning strategy in the turn-based parity games for $P_\exists$, while the $\models^{\mathcal{M}(L)}$-truth of $\neg\phi$ corresponds to having a winning strategy for $P_\forall$. Our result then follows from the determinacy of turn-based parity games. ◀

Note that the construction of the parity game gives an effective algorithm for the model-checking problem of $\mathsf{SL[1G]}$, which runs in doubly-exponential time; we recover the result of [14] for that problem.

Comparison of the satisfaction relations. While the sets $\mathcal{M}(L)$, $\mathcal{M}(L,S)$, $\mathcal{M}(L,F)$, and $\mathcal{M}(L,S,F)$ are obviously distinct, it is not clear when side and/or future dependencies make a difference when evaluating a formula. We address this question in this section. First, as an easy consequence of Prop. 4 (also in [14]), we have:

▶ **Proposition 5** ([14]). *For any SL[1G] formula ϕ, and any game $\mathcal{G}$ with initial state q_0, it holds:*

$$\forall D \subseteq \{L,S,F\} \text{ s.t. } L \in D. \qquad \mathcal{G}, q_0 \models^{\mathcal{M}(L)} \phi \iff \mathcal{G}, q_0 \models^{\mathcal{M}(D)} \phi$$

However, when considering larger fragments, the satisfaction relations are mostly distinct. More precisely, we have:

▶ **Proposition 6.** *For every game $\mathcal{G}$ with initial state q_0, for every $\phi \in$ SL[CG],*

$$\mathcal{G}, q \models^{\mathcal{M}(L,S)} \phi \iff \mathcal{G}, q \models^{\mathcal{M}(L)} \phi$$

In all other cases, the satisfaction relations are pairwise distinct over $\mathsf{SL[DG]}$ *and* $\mathsf{SL[CG]}$*.*

Idea of proof. By lack of space, we postpone the proof to Appendix C, and only provide one example showing that $\models^{\mathcal{M}(L,F)}$ and $\models^{\mathcal{M}(L,S,F)}$ differ over $\mathsf{SL[DG]}$.

Consider again the game of Fig. 1 in Section 2. We already proved that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F)} \phi$. Now, pick any $\mathcal{M}(L,F)$-map θ for ϕ. Then for any two valuations w_1 and w_2 for universal variables, we must have $\theta(w_1)(z)(q_0q_1) = \theta(w_2)(z)(q_0q_1)$ if w_1 and w_2 only differ for y on history q_0q_2. The disjunctive goal will then fail to hold for one of w_1 and w_2, which proves that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,F)} \phi$. ◀

▶ Remark. From the proof idea above, we also get that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L)} \phi$, contradicting the claim in [15] that both $\models^{\mathcal{M}(L)}$ and $\models^{\mathcal{M}(L,S,F)}$ coincide on SL[DG] (and SL[CG]). Indeed, in [15], the satisfaction relation for SL[DG] and SL[CG] is encoded into a two-player game in pretty much the same way as we did in the proof of Prop. 4. While this indeed rules out side and future dependencies, it also gives informations to Player $P_\exists$ about the values (over prefixes of the current history) of strategies that are universally-quantified later in the formula. In particular, with this information, Player $P_\exists$ knows which assignment(s) of the formula follow the current history, hence which goal(s) has to potentially be satisfied by (some) extensions of the current history. We investigate this kind of extra information in the next sections.

4 Timeline dependencies

Following the discussion above, we introduce a new type of dependencies between strategies. It allows strategies to also observe *all* other universally-quantified strategies on the prefix of the current history. When considering logics where strategies are attached to players (as opposed to variables) such as $\mathsf{ATL}_{\mathsf{sc}}$, this amounts to augmenting histories with the move vectors played at each step along the history. In particular, it would not make any difference on turn-based games. As we explain below, even in turn-based games, adding such dependencies over SL[BG] gives rise to different satisfaction relations.

We reuse the three conditions $\mathcal{C}(L)$, $\mathcal{C}(S)$ and $\mathcal{C}(F)$ defined in Section 3, and define an additional one:

- $\mathcal{C}(T)$: w_1 and w_2 coincide on $\mathcal{V}^\forall$ and on all prefixes of ρ (timeline)

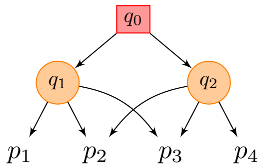
Timeline dependencies are represented at the bottom of Fig. 2. We can then define new sets of maps, in which timeline dependencies are allowed. We now consider $D \subseteq \{L, S, F, T\}$ (still assumed to contain L), and again say that a map is in $\mathcal{M}(D)$ when Equation (1) of page 6 holds. Similarly to Section 3, those new types of maps are associated with new satisfaction relations, still denoted by $\models^{\mathcal{M}(D)}$ for the corresponding sets D.

▶ **Example 7.** Consider again the game of Fig 1 in Section 2. We have seen in the proof of Prop. 6 that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F)} \phi$ in Section 2 and that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,F)} \phi$. With timeline dependencies, we have $\mathcal{G}, q_0 \models^{\mathcal{M}(L,F,T)} \phi$. Indeed, now $z(q_0.q_2)$ may depend on $x_A(q_0)$; if $x_A(q_0) = q_2$ then we must have $z(q_0.q_2) = p_1$, while if $x_A(q_0) = q_1$, then $z(q_0.q_2) = p_2$. This can be encoded in a map θ in $\mathcal{M}(L, F, T)$. Now, in the former case ($x_A(q_0) = q_2$), the first goal is satisfied. In the latter case ($x_A(q_0) = q_1$), depending on $x_B(q_0)$, either the outcome of the second goal visits q_2, in which case it then goes to p_2 and the second goal is satisfied; or both goals visit q_1 and no matter the choice for $y(q_0.q_1)$, one of the two goals must be satisfied.

The syntactic negation. We consider again the link between syntactic and semantic negations in the context of timeline dependencies. We prove that in presence of timeline dependencies, both negations (syntactic and semantic) in general do not coincide.

▶ **Proposition 8.** *For every formula ϕ in SL[BG], for every game $\mathcal{G}$ with initial state q_0,*

- $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,S,F,T)} \phi \implies \mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F,T)} \neg\phi$;
- $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi \implies \mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi$.



$$\phi = \forall x_1.\exists y_1.\exists y_2.\exists x_2.\ \bigwedge \begin{cases} \mathsf{assign}(\square \mapsto y_1; \bigcirc \mapsto x_1)\mathbf{F}\, p_2 \\ \quad \Rightarrow \mathsf{assign}(\square, y_2; \bigcirc \mapsto x_2)\mathbf{F}\, p_1 \\ \mathsf{assign}(\square \mapsto y_1; \bigcirc \mapsto x_1)\mathbf{F}\, p_3 \\ \quad \Rightarrow \mathsf{assign}(\square \mapsto y_2; \bigcirc \mapsto x_2)\mathbf{F}\, p_4 \end{cases}$$

Figure 4 A game $\mathcal{G}$ and a formula ϕ such that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F,T)} \phi$ and $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F,T)} \neg\phi$

Sketch of proof. The first result follows from the fact that at least one of ϕ and $\neg\phi$ holds true for $\models^{\mathcal{M}(L,S,F)}$. Since $\mathcal{M}(L,S,F) \subseteq \mathcal{M}(L,S,F,T)$, at least one also holds for $\models^{\mathcal{M}(L,S,F,T)}$.

We now explain the second implication, the complete proof is in Appendix D. For a contradiction, assume that there exist two maps θ and $\bar{\theta}$ witnessing $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi$ and $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \neg\phi$ resp. (we write $\phi = (Q_i x_i)_{i \leq l} \xi(\beta_j \varphi_j)_{j \leq n}$). Then for any strategy valuations w and $\bar{w}$ for $\mathcal{V}^\forall$ and $\mathcal{V}^\exists$, we have that $\mathcal{G}, q_0 \models_{\theta(w)} \xi(\beta_j \phi_j)_j$ and $\mathcal{G}, q_0 \models_{\bar{\theta}(\bar{w})} \neg\xi(\beta_j \phi_j)_j$. We can then inductively (on histories and on the sequence of quantified variables) build a strategy valuation χ on $\mathcal{V}$ such that $\theta(\chi_{|\mathcal{V}^\forall}) = \bar{\theta}(\chi_{|\mathcal{V}^\exists}) = \chi$. Under valuation χ, both $\xi(\beta_j \phi_j)_j$ and $\neg\xi(\beta_j \phi_j)_j$ hold in q_0, which is impossible. ◀

▶ **Proposition 9.**
- *For any $D \subsetneq \{L,S,F,T\}$ s.t. $\{L,T\} \subseteq D$, there exists a game $\mathcal{G}$ with initial state q_0 and a formula $\phi \in$ SL[BG] such that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \phi$ and $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \neg\phi$;*
- *for any $D \subseteq \{L,S,F,T\}$ s.t. $\{L,T\} \subsetneq D$, there exists a game $\mathcal{G}$ with initial state q_0 and a formula $\phi \in$ SL[BG] such that $\mathcal{G}, q_0 \models^{\mathcal{M}(D)} \phi$ and $\mathcal{G}, q_0 \models^{\mathcal{M}(D)} \neg\phi$.*

Proof. The complete proof is given in Appendix E. We focus on the second item and on the satisfaction relation $\models^{\mathcal{M}(L,S,F,T)}$. Consider the turn-based game $\mathcal{G}$ and the SL[BG] formula ϕ of Fig. 4. Clearly, $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F,T)} \phi$, since at least one of the implication will hold trivially, and the other one can be satisfied by correctly selecting y_2 and x_2. We can also prove that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F,T)} \neg\phi$: since timeline dependencies are allowed, $x_1(q_0q_1)$ and $x_1(q_0q_2)$ may depend on the values of y_1 and y_2 in q_0. We thus consider four cases:
- if $y_1(q_0) = y_2(q_0) = q_1$, then we let $x_1(q_0q_1) = p_3$; then the second conjunct of ϕ is not fulfilled, whatever x_2;
- if $y_1(q_0) = y_2(q_0) = q_2$, then symmetrically, we let $x_1(q_0q_2) = p_2$, so that the first conjunct fails to hold for any x_2;
- if $y_1(q_0) = q_1$ and $y_2(q_0) = q_2$, then we let $x_1(q_0q_1) = p_2$, for which no x_2 make the first conjunct hold;
- if $y_1(q_0) = q_2$ and $y_2(q_0) = q_1$, then we let $x_1(q_0q_2) = p_3$, and again the second conjunct fails to hold independently of x_2. ◀

Comparison of the satisfaction relations. We prove that adding timeline dependencies again mainly yields pairwise distinct satisfaction relations:

▶ **Proposition 10.** *For every game $\mathcal{G}$ with initial state q_0, for every formula $\phi \in$ SL[CG],*

$$\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,T)} \phi \iff \mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi$$

In all other cases (involving timeline dependencies), the satisfaction relations are pairwise distinct for SL[CG], SL[DG], and SL[1G].

Similarly, the satisfaction relations with and without timeline dependencies are distinct for SL[CG] and SL[DG]:

▶ **Proposition 11.** *For any $D \subseteq \{L, S, F\}$ containing L, the satisfaction relations $\models^{\mathcal{M}(D)}$ and $\models^{\mathcal{M}(D\cup\{T\})}$ are distinct for* SL[CG] *and* SL[DG].

Applying techniques similar to those of the proof of Prop. 4 (reduction to a 2-player parity game), we can prove:

▶ **Proposition 12.** *For any game $\mathcal{G}$ with initial state q_0 and any formula $\phi \in$* SL[1G]*, we have $\mathcal{G}, q_0 \models^{\mathcal{M}(L)} \phi \iff \mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi$.*

This implies in particular that $\models^{\mathcal{M}(L,T)}$ behaves smoothly w.r.t. syntactic negation on SL[1G] (by Prop. 4), and that the model-checking problem for this setting is in *2*-EXPTIME. Our aim in the next section is to generalize this positive result to a larger fragment.

5 The fragment SL[EG]

In this section, we focus on the timeline dependencies and on the satisfaction relation $\models^{\mathcal{M}(L,T)}$. We believe that this setting is especially relevant, especially for reactive synthesis, as it corresponds to settings where the informations available to existentially-quantified strategies is exactly what they may have observed along the current history. We exhibit a fragment SL[EG] of SL[BG] for which syntactic and semantic negations coincide, and for which we prove model-checking is *2*-EXPTIME-complete. Formally, SL[EG] satisfies the following theorem:

▶ **Theorem 13.** *For any $\phi \in$* SL[EG]*, any game $\mathcal{G}$ and any state q_0, it holds:*

$$\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi \iff \mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi$$

Moreover, model checking SL[EG] *w.r.t. $\mathcal{M}(L,T)$-maps is 2*-EXPTIME-*complete.*

The SL[EG] fragment. We introduce some notations that will be useful to define our fragment. For $n \in \mathbb{N}$, we let $\{0,1\}^n$ be the set of mappings from $[1,n]$ to $\{0,1\}$. We write $\mathbf{0}^n$ (or $\mathbf{0}$ if the size n is clear) for the function that maps all integers in $[1,n]$ to 0, and $\mathbf{1}^n$ (or $\mathbf{1}$) for the function that maps $[1,n]$ to 1. The *size* of $f \in \{0,1\}^n$ is defined as $|f| = \sum_{1\leq i\leq n} f(i)$. For two elements f and g of $\{0,1\}^n$, we write $f \leq g$ whenever $f(i) = 1$ implies $g(i) = 1$ for all $i \in [1,n]$. Given $B^n \subseteq \{0,1\}^n$, we write $\uparrow B^n = \{g \in \{0,1\}^n \mid \exists f \in B^n,\ f \leq g\}$. A set $F^n \subseteq \{0,1\}^n$ is *upward-closed* if $F^n = \uparrow F^n$. Finally, we also define the following operators:

$$\bar{f} : i \mapsto 1 - f(i) \qquad f \curlywedge g : i \mapsto \min\{f(i), g(i)\} \qquad f \curlyvee g : i \mapsto \max\{f(i), g(i)\}.$$

▶ **Definition 14.** A set $F^n \subseteq \{0,1\}^n$ is *semi-stable* if for any f and g in F^n, it holds that

$$\forall s \in \{0,1\}^n \qquad (f \curlywedge s) \curlyvee (g \curlywedge \bar{s}) \in F^n \text{ or } (g \curlywedge s) \curlyvee (f \curlywedge \bar{s}) \in F^n.$$

▶ **Example 15.** Obviously, $\{0,1\}^n$ is semi-stable, as well as the empty set. For $n = 2$, the set $\{(0,1),(1,0)\}$ is easily seen not to be semi-stable: taking $f = (0,1)$ and $g = (1,0)$ with $s = (1,0)$, we get $(f \curlywedge s) \curlyvee (g \curlywedge \bar{s}) = (0,0)$ and $(g \curlywedge s) \curlyvee (f \curlywedge \bar{s}) = (1,1)$. Similarly, $\{(0,0),(1,1)\}$ is not semi-stable. It can be checked that any other subset of $\{0,1\}^2$ is semi-stable.

We now define (the flat fragment of) SL[EG][3] as follows:

[3] EG stands here for *elementary goal*. This also provides a natural continuation with the fragments SL[AG] [16], SL[BG] [14], and SL[CG], and SL[DG] [15].

$$\mathsf{SL}[\mathsf{EG}]^\flat \ni \phi ::= \forall x.\phi \mid \exists x.\phi \mid \xi \qquad\qquad \xi ::= F^n((\beta_i)_{1\le i\le n})$$
$$\beta ::= \mathsf{assign}(\sigma).\ \psi \qquad\qquad \psi ::= \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi \mid p$$

where F^n ranges over semi-stable subsets of $\{0,1\}^n$, for any $n \in \mathbb{N}$. The semantics of the operator F^n is defined as

$$\mathcal{G}, q \models_\chi F^n((\beta_i)_{i\le n}) \iff \text{letting } f \in \{0,1\}^n \text{ s.t. } f(i) = 1 \text{ iff } \mathcal{G}, q \models_\chi \beta_i, \text{ it holds } f \in F^n.$$

Notice that if F^n would range over all subsets of $\{0,1\}^n$, then this definition would exactly correspond to SL[BG]. Similarly, the case where $F^n = \{\mathbf{1}^n\}$ corresponds to SL[CG], while $F^n = \{0,1\}^n \setminus \{\mathbf{0}^n\}$ gives rise to SL[DG].

▶ **Proposition 16.** *SL[EG] contains SL[AG]. The inclusion is strict (syntactically).*

Proof. Remember that boolean combinations in SL[AG] follow the grammar $\xi ::= \xi \vee \beta \mid \xi \wedge \beta \mid \beta$. In terms of subsets of $\{0,1\}^n$, it corresponds to considering sets defined in one of the following two forms:

$$F^n_\xi = \{f \in \{0,1\}^n \mid f(n) = 1\} \cup \{g \in \{0,1\}^n \mid g_{|[1;n-1]} \in F^{n-1}_{\xi'}\}$$
$$F^n_\xi = \{f \in \{0,1\}^n \mid f(n) = 1 \text{ and } f_{|[1;n-1]} \in F^{n-1}_{\xi'}\}$$

depending whether $\xi(p_j)_j = \xi'(p_j)_j \vee p_n$ or $\xi(p_j)_j = \xi'(p_j)_j \wedge p_n$. Assuming (by induction) that $\mathbf{F}^{n-1}_{\xi'}$ is semi-stable, it can be shown that $\mathbf{F}^{n}_{\xi}$ also is.

That the inclusion is strict is proven by considering the semi-stable set $H^3 = \{\langle 1,1,1\rangle, \langle 1,1,0\rangle, \langle 1,0,1\rangle, \langle 0,1,1\rangle\}$, which cannot be obtained by any boolean combination of goals from SL[AG]. ◀

▶ **Example 17.** Consider the following formula, expressing the existence of a Nash equilibrium for two players with respective LTL objectives ψ_1 and ψ_2:

$$\exists x_1.\exists x_2.\forall y_1.\forall y_2.\ \bigwedge \begin{cases} (\mathsf{assign}(A_1 \mapsto y_1; A_2 \mapsto x_2).\psi_1) \Rightarrow (\mathsf{assign}(A_1 \mapsto x_1; A_2 \mapsto x_2).\psi_1) \\ (\mathsf{assign}(A_1 \mapsto x_1; A_2 \mapsto y_2).\psi_2) \Rightarrow (\mathsf{assign}(A_1 \mapsto x_1; A_2 \mapsto x_2).\psi_2) \end{cases}$$

This formula has four goals, and it corresponds to the set

$$F^4 = \{(1,1,1,1), (0,1,1,1), (1,1,0,1), (0,1,0,1), (0,0,1,1), (1,1,0,0), \\ (0,0,0,1), (0,1,0,0), (0,0,0,0)\}$$

Taking $f = (1,1,0,0)$ and $g = (0,0,1,1)$, with $s = (1,0,1,0)$ we have $(f \curlywedge s) \curlyvee (g \curlywedge \bar{s}) = (1,0,0,1)$ and $(g \curlywedge s) \curlyvee (f \curlywedge \bar{s}) = (0,1,1,0)$, none of which is in F^4. Hence our formula is not (syntactically) in SL[EG].

Transformation into an upward-closed set by bit flipping Fix a vector $b \in \{0,1\}^n$. We define the operation $\mathsf{flip}_b \colon \{0,1\}^n \to \{0,1\}^n$ that maps any vector f to $(f \curlywedge b) \curlyvee (\bar{f} \curlywedge \bar{b})$. In other terms, flip_b flips the i-th bit of its argument if $b_i = 0$, and keeps this bit unchanged if $b_i = 1$. In SL[EG], flipping bits is equivalent to negating the corresponding goals. The first part of the following lemma thus indicates that our definition for SL[EG] is somewhat sound.

▶ **Lemma 18.** *If $F^n \subseteq \{0,1\}^n$ is semi-stable, then so is $\mathsf{flip}_b(F^n)$. Moreover, for any semi-stable set F^n, there exists $B \in \{0,1\}^n$ such that $\mathsf{flip}_B(F^n)$ is upward-closed (i.e. for any $f \in \mathsf{flip}_B(F^n)$ and any $s \in \{0,1\}$, we have $f \curlyvee s \in \mathsf{flip}_B(F^n)$).*

Defining quasi-orders from semi-stable sets. For $F^n \subseteq \{0,1\}^n$, we write $\overline{F^n}$ for the complement of F^n. Fix such a set F^n, and pick $s \in \{0,1\}^n$. For any $h \in \{0,1\}^n$, we define

$$\mathbb{F}^n(h,s) = \{h' \in \{0,1\}^n \mid (h \curlywedge s) \curlyvee (h' \curlywedge \overline{s}) \in F^n\}$$
$$\overline{\mathbb{F}^n}(h,s) = \{h' \in \{0,1\}^n \mid (h \curlywedge s) \curlyvee (h' \curlywedge \overline{s}) \in \overline{F^n}\}$$

Trivially $\mathbb{F}^n(h,s) \cap \overline{\mathbb{F}^n}(h,s) = \emptyset$ and $\mathbb{F}^n(h,s) \cup \overline{\mathbb{F}^n}(h,s) = \{0,1\}^n$. If we assume F^n to be semi-stable, then the family $(\mathbb{F}^n(h,s))_{h\in\{0,1\}^n}$ satisfies the following property:

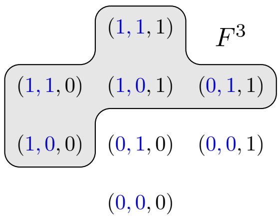
▶ **Lemma 19.** *Fix a semi-stable set F^n and $s \in \{0,1\}^n$. For any $h_1, h_2 \in \{0,1\}^n$, either $\mathbb{F}^n(h_1,s) \subseteq \mathbb{F}^n(h_2,s)$ or $\mathbb{F}^n(h_2,s) \subseteq \mathbb{F}^n(h_1,s)$.*

Given a semi-stable set F^n and $s \in \{0,1\}^n$, we can use the inclusion relation of Lemma 19 to define a relation $\preceq_s^{F^n}$ (written $\preceq_s$ when F^n is clear) over the elements of $\{0,1\}^n$.

▶ **Definition 20.** Fix F^n semi-stable and $s \in \{0,1\}^n$. We define $\preceq_s^{F^n} \subseteq \{0,1\}^n \times \{0,1\}^n$ so that $h_1 \preceq_s h_2$ iff $\mathbb{F}^n(h_1,s) \subseteq \mathbb{F}^n(h_2,s)$.

This relation is a quasi-order: its reflexiveness and transitivity both follow from the reflexiveness and transitivity of the inclusion relation $\subseteq$. By Lemma 19, this quasi-order is total. Intuitively, $\preceq_s$ orders the elements of $\{0,1\}^n$ based on how "easy" it is to complete their restriction to s so that the completion belongs to F^n. In particular, only the indices on which s take value 1 are used to check whether $h_1 \preceq_s h_2$: given $h_1, h_2 \in \{0,1\}^n$ such that $(h_1 \curlywedge s) = (h_2 \curlywedge s)$, we have $\mathbb{F}(h_1,s) = \mathbb{F}(h_2,s)$, and $h_1 =_s h_2$.

▶ **Example 21.** Consider for instance the semi-stable set $F^3 = \{\langle 1,0,0\rangle, \langle 1,1,0\rangle, \langle 1,0,1\rangle, \langle 0,1,1\rangle, \langle 1,1,1\rangle\}$ represented on the figure opposite, and which can be shown to be semi-stable. Fix $s = \langle 1,1,0\rangle$. Then $\mathbb{F}^3(\langle 0,1,\star\rangle, s) = \{0,1\}^2 \times \{1\}$, while $\mathbb{F}^3(\langle 1,1,\star\rangle, s) = \mathbb{F}^3(\langle 1,0,\star\rangle, s) = \{0,1\}^3$ and $\mathbb{F}^3(\langle 0,0,\star\rangle, s) = \emptyset$. It follows that $\langle 0,0,\star\rangle \preceq_s \langle 0,1,\star\rangle \preceq_s \langle 1,0,\star\rangle =_s \langle 1,1,\star\rangle$.



Sketch of proof of Theorem 13. We can now give an intuition for the proof of Theorem 13 stated at the beginning of this section. Consider a SL[EG] formula $\phi = (Q_i x_i)_{i\leq l} F^n(\beta_j \varphi_j)_{j\leq n}$, where we assume F^n is upward-closed. Fix a game $\mathcal{G}$ and an initial state q_0. First, given any history ρ and any valuation χ of *partial strategies* (defined at least on any prefix of ρ), timeline dependencies convey enough information to track of those goals $\beta_j \varphi_j$ for which ρ is a prefix of the outcomes of $\beta_j(\chi)$ (such goals are said to be *active* along ρ under χ). By representing the set of currently-active goals as a vector $s \in \{0,1\}^n$, the quasi-order $\preceq_s$ gives a preference relation between the goals that one wants to keep active when selecting an action for existentially-quantified strategy x_i at ρ.

In order to keep track of which goals are satisfied, we define a set $\{D_{s,h} \mid s,h \in \{0,1\}^n\}$ of deterministic parity automata, each automaton $D_{s,h}$ accepting those words over 2^{AP} satisfying formula $\bigvee_{k\in\{0,1\}^n.\ h\preceq_s k} \bigwedge_{j.\ (k\curlywedge s)(j)=1} \varphi_j$. In other terms, a play is accepted by $D_{s,h}$ if the set of formulas φ_j it satisfies is at least as good as h w.r.t. $\preceq_s$. In particular, $D_{s,\mathbf{0}}$ is universal.

Now, consider a set of goals $\beta_j.\ \varphi_j$, represented by a vector $s \in \{0,1\}^n$. Assume that they are all active along some infinite play ρ. Since the quasi-order $\preceq_s$ is total, there exists some maximal vector k for which ρ is accepted by $D_{s,k}$. Then ρ can serve as a witness for the goals $k \curlywedge s$, and cannot do better. For the goals defined by $\bar{s}$, if another play $\bar{\rho}$ and its maximal vector $\bar{k}$, then formula $F^n(\beta_j\varphi_j)_j$ will be true in the corresponding context if and only if $(k \curlywedge s) \curlyvee (\bar{k} \curlywedge \bar{s}) \in F^n$.

In general, all the goals will not follow a single infinite play ρ; the idea is then to follow the common play for a while, until the goals split and follow different branches. If after some history, the active goals are represented by some vector s, then the vector s is *partitioned* into several vectors s_i, associated with the branches followed after ρ.

Formally, through an induction, we exhibit some specific elements $b_{q,d,s} \in \{0,1\}^n$, where q is a state of $\mathcal{G}$, d represents some knowledge about the history (represented a states of the automata $D_{s,h}$), and s is a set of active goals. An element $b_{q,d,s}$ corresponds to the best we can hope to achieve w.r.t. $\preceq_s$ when we consider a history ρ ending in q, carrying the information d and where s represents the set of active goals on ρ (which are accessible using the timeline dependency). The induction is based on the size of s, The key to determine an element $b_{q,d,s}$ is to know when we should keep the active goals together and when we should split them among different paths. For this, we build a set $(\mathcal{H}_k)_{k\in\{0,1\}^n}$ of two-player parity games, with players $P_\forall$ and $P_\exists$. Player $P_\exists$ wins game $\mathcal{H}_k$ if, and only if, we can achieve a result at least as good as k for $\preceq_s$ from any ρ ending in q with information d and active goals s. The values $b_{q,d,s}$ can be computed by solving each game and choosing the largest k for which $P_\exists$ wins in $\mathcal{H}_k$. In $\mathcal{H}_k$, the parities are chosen using the automata $(D_{s,h})_{s,h}$ to encode the possibility of all goals staying on a common path, and using values $b_{q',d',s'}$ defined at previous step of the induction when the goals split among different paths.

We then produce an (optimal) $\mathcal{M}(L,T)$-map θ. The optimal positional strategies of $P_\exists$ in $\mathcal{H}_{b_{q,d,s}}$ give us the adequate behaviour for θ in $\mathcal{G}$ on histories finishing in q, carrying information d and with active goals represented by s. With a dual reasoning, we produce another $\mathcal{M}(L,T)$-map $\overline{\theta}$ for $\neg\phi$.

By optimality of the values $b_{q,d,s}$ (used to build θ and $\overline{\theta}$), on any history, the choices of θ for the existential strategy and $\overline{\theta}$ for the universal ones are as good as possible. Formally, this translates into the following lemma:

▶ **Lemma 22.** *There exists a valuation χ of domain $\mathcal{V}$ such that $\theta(\chi_{|\mathcal{V}^\forall}) = \chi$ and $\overline{\theta}(\chi_{|\mathcal{V}^\exists}) = \chi$. Moreover, the valuation χ satisfies*

$$\mathcal{G}, q_0 \models_\chi \xi(\beta_j\varphi_j)_{j\le n} \quad\Longrightarrow\quad \forall w \in (\mathsf{Hist}_\mathcal{G} \to \mathsf{Act})^{\mathcal{V}^\forall}\ \mathcal{G}, q_0 \models_{\theta(w)} \xi(\beta_j\varphi_j)_{j\le n}$$
$$\mathcal{G}, q_0 \models_\chi (\neg\xi)(\beta_j\varphi_j)_{j\le n} \quad\Longrightarrow\quad \forall \overline{w} \in (\mathsf{Hist}_\mathcal{G} \to \mathsf{Act})^{\mathcal{V}^\exists}\ \mathcal{G}, q_0 \models_{\overline{\theta}(\overline{w})} (\neg\xi)(\beta_j\varphi_j)_{j\le n}$$

In the first case, θ witnesses the fact that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi$ and (by Prop. 8) $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi$. In the second case, $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \neg\phi$ and (by Prop. 8) $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \phi$.

In order to build θ and $\overline{\theta}$ and to determine which case holds, we can build 2^n automata of size $2^{2^{|\phi|}}$ and index $2^{|\phi|}$, then build 2^n parity games of size $2^{2^{|\phi|}}$ and index $2^{|\phi|}$, and finally solve each of these game. This gives us the expected *2*-EXPTIME algorithm, and concludes the proof of Theorem 13. ◀

Finally, we prove that the fragment SL[EG] is, in a sense, maximal for the first property of Theorem 13:

▶ **Proposition 23.** *For any non-semi-stable boolean set $F^n \subseteq \{0,1\}^n$, there exists a SL[BG] formula ϕ built on F^n, a game $\mathcal{G}$ and a state q_0 such that*

$$\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi \qquad\qquad \mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \phi$$

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Appendix

A Proof of Proposition 3

▶ **Proposition 3.** *For any $D \subsetneq \{L, S, F\}$ containing L, there exist a game $\mathcal{G}$ with initial state q_0 and a formula $\phi \in$ SL[BG]$^\flat$ such that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \phi$ and $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \neg\phi$.*

Proof. In the paper, we proved the result for $\models^{\mathcal{M}(L)}$. The remaining two cases can be handled similarly. The counter-example for $\models^{\mathcal{M}(L,F)}$ uses the game on the left of Fig. 5 and the following formula

$$\phi = \forall x.\ \exists y.\ \bigwedge \begin{cases} \mathsf{assign}(\bigcirc \mapsto x).\ \mathbf{F}\, p_1 \Leftrightarrow \mathsf{assign}(\bigcirc \mapsto y).\ \mathbf{F}\, p_1 \\ \mathsf{assign}(\bigcirc \mapsto x).\ \mathbf{F}\, p_2 \Leftrightarrow \mathsf{assign}(\bigcirc \mapsto y).\ \mathbf{F}\, p_2 \end{cases}$$

Checking that both ϕ and $\neg\phi$ fail to hold follows similar arguments to the ones used in the paper for $\models^{\mathcal{M}(L)}$. The counter-example for $\models^{\mathcal{M}(L,S)}$ uses the game to the right of Fig. 5 and the formula $\phi = \forall x.\ \exists y.\ \big(\mathsf{assign}(\bigcirc \mapsto x).\ \mathbf{F}\, p_1 \Leftrightarrow \mathsf{assign}(\bigcirc \mapsto y).\ \mathbf{F}\, p_2\big)$. ◀

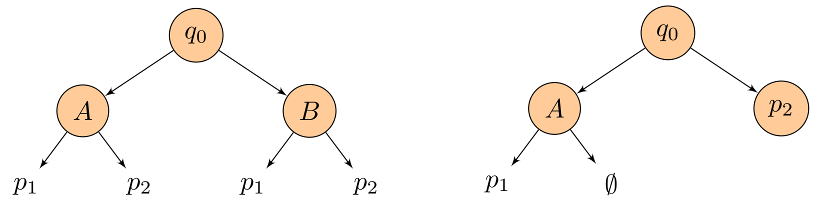


Figure 5 Two games for the proof of Prop. 3.

B Proof of Proposition 4

▶ **Proposition 4.** *For any $D \subseteq \{L, S, F\}$ containing L, every game $\mathcal{G}$ with initial state q_0, and every formula $\phi \in$ SL[1G], it holds $\mathcal{G}, q_0 \models^{\mathcal{M}(D)} \phi \iff \mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \neg\phi$.*

Proof. It suffices to prove the result for $\mathcal{M}(L)$, since $\mathcal{M}(L) \subseteq \mathcal{M}(D)$. When restricting to the satisfaction relation $\models^{\mathcal{M}(L)}$, the interaction between existential and universal quantifications of the formula can be integrated into the game, replacing each state with a tree-shaped subgame where Player $P_\exists$ selects existentially-quantified actions and Player $P_\forall$ selects universally-quantified ones. The unique goal of the formula is then incorporated into the game via a deterministic parity automaton, yielding a two-player turn-based parity game. We can then show that the $\models^{\mathcal{M}(L)}$-satisfaction of ϕ is equivalent to having a winning strategy in the turn-based parity games for $P_\exists$, while the $\models^{\mathcal{M}(L)}$-satisfaction of $\neg\phi$ corresponds to having a winning strategy for $P_\forall$.

For the rest of the proof, we fix a game $\mathcal{G}$ and a SL[1G] formula $\phi := (Q_i x_i)_{i \leq l} \beta\varphi$.

Specifying a turn-based parity game $\mathcal{H}$. We start by defining some specific turn-based arenas to "flatten" the game and formula. Figure 6 illustrates the construction. First, relatively to $\mathcal{G}$ and ϕ we define the following turn-based tree-like arena, which we call *cluster*:

- there are two players $P_\exists$ and $P_\forall$.
- the set of states is $S_{cluster} := \{\mathfrak{m} \in \mathsf{Act}^* \mid 0 \leq |\mathfrak{m}| \leq l\}$, thereby defining a tree of depth $l + 1$ with directions Act. A state $\mathfrak{m}$ in $S_{cluster}$ with $|\mathfrak{m}| < l$ belongs to $P_\exists$ if and only if $Q_{|\mathfrak{m}|+1} = \exists$. To whom the states with $|\mathfrak{m}| = l$ belong does not matter.

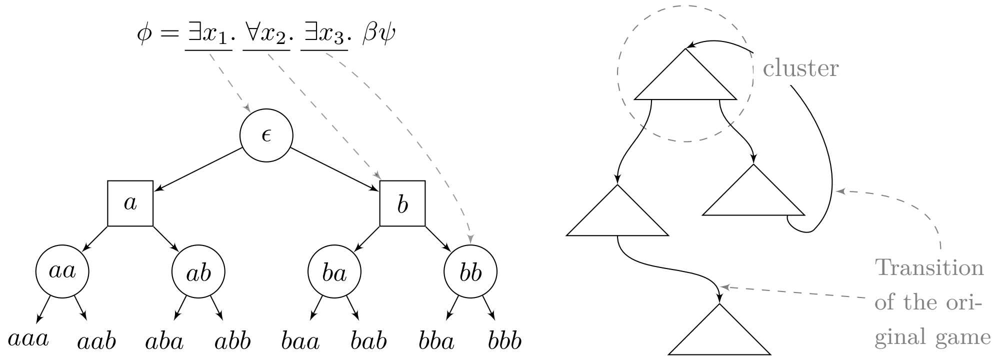


Figure 6 On the left: representation of a cluster. On the right: the overall shape of $\mathcal{H}$.

- there is a transition from each $\mathfrak{m}$ of size strictly less than l to all $\mathfrak{m} \cdot a$ for all $a \in \mathsf{Act}$. In particular, the empty word $\epsilon \in S_{cluster}$ is the starting node of the cluster, and it has no incoming transitions, while all words of length l have no outgoing transitions;

A leaf in such a cluster represents a move vector of domain $\mathcal{V} = \{x_i \mid 1 \leq i \leq l\}$: the leaf $\mathfrak{m}$ represents the move vector m where $m(x_i) = \mathfrak{m}(i)$.

We denote by D the deterministic parity automata over 2^{AP} associated with φ. We also write d_0 for the initial state of D. Using the notion of clusters, we define a turn-based parity game $\mathcal{H}$.

- the players are the same as before: $P_\exists, P_\forall$.
- for each state q of $\mathcal{G}$ and each state d of D, $\mathcal{H}$ contains a copy of a cluster which we call the (q, d) *cluster*. For any $\mathfrak{m} \in \mathsf{Act}^*$ with $|\mathfrak{m}| \leq l$, we refer to the state $\mathfrak{m}$ of the (q, d) cluster as the $(q, d, \mathfrak{m})$ state.
- the transitions in $\mathcal{H}$ are of two types:
  - internal transitions in the cluster are preserved;
  - consider a state $(q, d, \mathfrak{m})$ where $\mathfrak{m}$ is a leaf. If there exists a state q' such that $q' = T(q, m_\beta)$ where $m_\beta : \mathsf{Agt} \to \mathsf{Act}$ is the move vector over Agt defined by $m_\beta(A) = \mathfrak{m}(i-1)$ where $x_i = \beta(A)$ (i.e. applying the choices of $\mathfrak{m}$ according to β in $\mathcal{G}$ leads from q to q'), then we add a transition from $(q, d, \mathfrak{m})$ to (q', d', ϵ) where $d' = \mathsf{succ}(d, q')$.
- the set of priorities are the same as in D and each $(q, d, \mathfrak{m})$ state has the same priority as d.

Correspondence between $\mathcal{G}$ and $\mathcal{H}$. There is not a clear one-to-one correspondence between the histories in $\mathcal{H}$ and the ones in $\mathcal{G}$, however there exists nevertheless some degree of connection. We introduce the notion of *lanes* before going into details.

▶ **Definition 24.** A *lane* in $\mathcal{G}$ is a tuple (ρ, u, i, t) made of a history $\rho := (q_j)_{j \leq a}$ (for some integer a); a function $u : (\mathsf{Pref}_{<\rho} \to \mathsf{Act})^{\mathcal{V}}$; an integer $i \in [0; l]$; a function $t : \mathcal{V} \cap [x_1; x_i] \to \mathsf{Act}$ (with the convention that $t := \emptyset$ if $i = 0$); and such that

$$\forall j < a \quad T(q_j, \beta(m_j)) = q_{j+1} \qquad \begin{cases} \text{where } m_j : \mathcal{V} \to \mathsf{Act} \text{ is the vector move over } \mathcal{V} \\ \text{with } m_j(x) := u(x)(\rho_{\leq j}) \end{cases}$$

We can build a one-to-one application $HtoG_{pth}$ between histories in $\mathcal{H}$ and lanes in $\mathcal{G}$. On a history π in $\mathcal{H}$, of shape

$$\pi \ := \ \Pi_{j<a}(\Pi_{0 \leq i \leq l}(q_j, d_j, \mathfrak{m}_{j,i})).\Pi_{0 \leq i \leq b}(q_a, d_a, \mathfrak{m}_{a,i})$$

of length $a.(l+1)+b$ with $0 \leq b < l+1$, we define $HtoG_{pth}(\pi)$ by

$$HtoG_{pth}(\pi) := ((q_j)_{j\leq a}, u, b, t)$$

with $u : \mathcal{V} \times \mathsf{Pref}_{<\rho} \to \mathsf{Act}$ $\quad$ $\forall a' < a \quad x_i, (q_j)_{j\leq a'} \to \mathfrak{m}_{j,i}$ $\qquad$ $t : \mathcal{V} \cap [x_1; x_{b-1}] \to \mathsf{Act}$ $\quad$ $\forall i \leq b \quad x_i \to \mathfrak{m}_{a,i}$

$HtoG_{pth}$ is clearly injective (two different histories will correspond to two different lanes), but also surjective. To prove it, we build the reciprocal function $GtoH_{pth}$: from a lane $((q_j)_{j\leq a}, u, i, t)$, we set $GtoH_{pth}((q_j)_{j\leq a}, u, i, t) := \pi$ where π is a history in $\mathcal{H}$ of length $a.(l+1)+|\mathsf{dom}(t)|+1$ of shape

$$\pi := \Pi_{j<a}\big(\Pi_{0\leq i\leq l}\big(q_j, d_j, u(x_i, (q_{j'})_{j'\leq j})\big)\big).\Pi_{0\leq i\leq b}\big(q_a, d_a, t(x_i, (q_j)_{j\leq a})\big)$$

where d_j is the vector of states reachable through $(q_{j'})_{j'\leq j}$

Because of the coherence condition imposed on lanes (see their definition), we get that the transition between clusters of $GtoH_{pth}((q_j)_{j\leq a}, u, i, t)$ are valid relatively to $\mathcal{H}$ transition table. $GtoH_{pth}((q_j)_{j\leq a}, u, i, t)$ is therefore a valid history in $\mathcal{H}$ and $GtoH_{pth}$ is well defined. From the definitions, one can easily check that

$$GtoH_{pth}(HtoG_{pth}(\pi)) = \pi$$

and deduce that $GtoH_{pth}$ is the inverse function of $HtoG_{pth}$; therefore

▶ **Lemma 25.** *The application $HtoG_{pth}$ is a bijection between lanes of $\mathcal{G}$ and histories in $\mathcal{H}$ and $GtoH_{pth}$ is its inverse function.*

Extending the correspondence. We can use the $HtoG_{pth}$ correspondence to describe another correspondence between positional strategies for $P_\exists$ in $\mathcal{H}$ and $\mathcal{M}(L)$-maps in $\mathcal{G}$. We recall that a map θ is a function $(\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}^\forall} \to (\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}}$ taking three inputs: a function $w : (\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}^\forall}$, a variable x_i and a history π. We also recall that if $Q_j = \forall$, then $\theta(w)(x_i)(\rho) = w(x_i)(\rho)$, hence we will only define $HtoG$ for the existentially quantified variables. Moreover, by the nature of $\mathcal{M}(L)$ maps, if we consider the variable x_i for the second entry, the first entry can be simplified to a function of shape $w : \mathsf{Pref}_{\leq\rho} \times \mathcal{V}^\forall \cap [x_1; x_i] \to \mathsf{Act}$. This simplifies the definition of θ.

Formally, the application $HtoG$ take as input a positional strategy δ for player $P_\exists$ in $\mathcal{H}$ and returns a $\mathcal{M}(L)$-map. The application is built by a double induction, first on the length of the potential history and second on the variables of ϕ.

- **initial step (empty history):**
  - **initial step (x_1 assuming $x_1 \in \mathcal{V}^\exists$):** We set $HtoG(\delta)(w)(x_1)(\epsilon) = \delta(\epsilon)$.
  - **induction step (x_i assuming $x_i \in \mathcal{V}^\exists$):** As the history and the variable are fixed, the only variable part of the input for the map we are building is a function $w : \{\epsilon\} \times \mathcal{V}^\forall \cap [x_1; x_{i-1}] \to \mathsf{Act}$. From x_i and w we create a function $t_{i,w} : \mathcal{V} \cap [x_1; x_{i-1}] \to \mathsf{Act}$ where $t_{i,w} : [x_1; x_{i-1}] \to \mathsf{Act}$ associate to $x \in \mathcal{V}^\forall$ the action $w(x)(\epsilon)$ and to $x \in \mathcal{V}^\exists$ the action $\theta(w)(x)(\epsilon)$. We can then create the lane $lane_{i,w} = (\epsilon, \emptyset, i, t)$ and define

$$HtoG(\delta)(w)(x_i)(\epsilon) := \delta(GtoH_{pth}(lane_{i,w}))$$

- **induction step (non empty history):** we work on a history π assuming we have define $HtoG(\delta)$ on prefixes of π. Like before, the history and the variable are fixed and the only changing part of the input is a function w of form $\mathsf{Pref}_{\leq\rho} \times \mathcal{V}^\forall \cap [x_1; x_{i-1}] \to \mathsf{Act}$.

- **initial step (x_1 assuming $x_1 \in \mathcal{V}^\exists$):** We set $HtoG(\delta)(w)(x_1)(\pi) = \delta(\mathsf{q})$ where $\mathsf{q} = (\mathsf{last}(\pi), d_\pi, \epsilon)$ is the state of $\mathcal{H}$ with d_π the state reachable by π in D (the parity automaton associate with the goal of ϕ).
- **induction step (x_j assuming $x_j \in \mathcal{V}^\exists$):** From x_i and w, we create a function $t_{i,w} : \mathcal{V} \cap [x_1; x_{i-1}] \to \mathsf{Act}$ where $t_{i,w}$ associate to $x \in \mathcal{V}^\forall$ the action $w(x)$ and to $x \in \mathcal{V}^\exists$ the action $\theta(w)(x)(\pi)$. We can then create the lane $lane_{i,w} = (\pi, \emptyset, i, t)$ and define

  $$HtoG(\delta)(w)(x_i)(\epsilon) := \delta(GtoH_{pth}(lane_{i,w}))$$

  Because δ is positional, feeding the empty function in $lane_{i,w}$ for the second component of the input is without consequence. Indeed, the second component of $lane_{i,w}$ in $GtoH_{pth}(lane_{i,w})$ only describe the actions played on π until the last last, now because δ is positional, theses actions have no influences.

At the end of the induction, we get a map that by construction has no side nor future dependency. Figure 7 illustrates the construction.

Concluding the proof. The winner of the parity game $\mathcal{H}$ gives us informations about $\mathcal{G}$.

▶ **Lemma 26.** *Assume that $P_\exists$ is winning in $\mathcal{H}$ and let δ be its positional winning strategy, then the $\mathcal{M}(L)$ map $HtoG(\delta)$ is a witness that $\mathcal{G}, q_0 \models^{\mathcal{M}(L)} \phi$.*

Similarly, assume that $P_\forall$ is winning in $\mathcal{H}$ and call $\overline{\delta}$ its positional winning strategy. Then the $\mathcal{M}(L)$ map $HtoG(\overline{\delta})$ is a witness that $\mathcal{G}, q_0 \models^{\mathcal{M}(L)} \neg\phi$.

Proof. We prove the first point, the second one follows the same reasoning. Assume that $P_\exists$ is winning in $\mathcal{H}$. Toward a contradiction, assume further that $HtoG(\delta)$ is not a witness of $\mathcal{G}, q_0 \models^{\mathcal{M}(L)} \phi$, then

$$\text{There is } w_0 : (\mathsf{Hist}_\mathcal{G} \to \mathsf{Act})^{\mathcal{V}^\forall} \text{ such that } \mathcal{G}, q_0 \not\models^{\mathcal{M}(L)}_{HtoG(\delta)(w_0)} \beta\varphi \tag{2}$$

We use w_0 to build a strategy $\overline{\delta}$ for $P_\forall$ in $\mathcal{H}$. Given a history ρ in $\mathcal{H}$ of form

$$\rho \;:=\; \Pi_{j<a}(\Pi_{0\le i\le l}(q_j, d_j, \mathfrak{m}_{j,i})).\Pi_{0\le i\le b}(q_a, d_a, \mathfrak{m}_{a,i})$$

we define $\pi = \Pi_{j\le a}(q_j)$ and set

$$\overline{\delta}(\rho) \;:=\; \theta(w)(x_b)(\eta) \quad \text{with} \begin{cases} w & : \mathsf{Pref}_{\le\pi} \times \mathcal{V}^\forall \cap [x_1; x_b] \to \mathsf{Act} \\ & \text{defined by } w(\pi)(x_i) = \rho(|\pi|.l + i) \\ \eta & := \Pi_{j<a}(\Pi_{0\le i\le l}(q_j, d_j, \mathfrak{m}_{j,i})) \end{cases}$$

Write $\eta = (q_j)_{j\in\mathbb{N}}$ for the outcome of $\beta(\theta(w_0))$ in $\mathcal{G}$. Then, by construction of $\overline{\delta}$, the outcome of δ and $\overline{\delta}$ in $\mathcal{H}$ will pass through the clusters $(q_j, d_j)_{j\in\mathbb{N}}$, with d_j the state accessible by $(q_{j'})_{j'\le j}$ in the automaton D associate with the LTL formula φ. Now, because of Formula (2), we get that η does not satisfy φ and therefore the outcome of δ and $\overline{\delta}$ does not satisfy the parity condition. This is in contradiction with δ being the winning strategy of $P_\exists$. In the end, $HtoG(\delta)$ must be a witness that $\mathcal{G}, q_0 \models^{\mathcal{M}(L)} \phi$. ◀

The determinacy of parity games and Prop 26 immediately implies that at least one of ϕ or $\neg\phi$ must hold on $\mathcal{G}$ for $\models^{\mathcal{M}(L)}$. The fact that at most one holds follows from the equivalence between $\models^{\mathcal{M}(L,S,F)}$ and $\models$ [14, Theorem 4.6], and the inclusion $\mathcal{M}(L) \subseteq \mathcal{M}(L, S, F)$. ◀

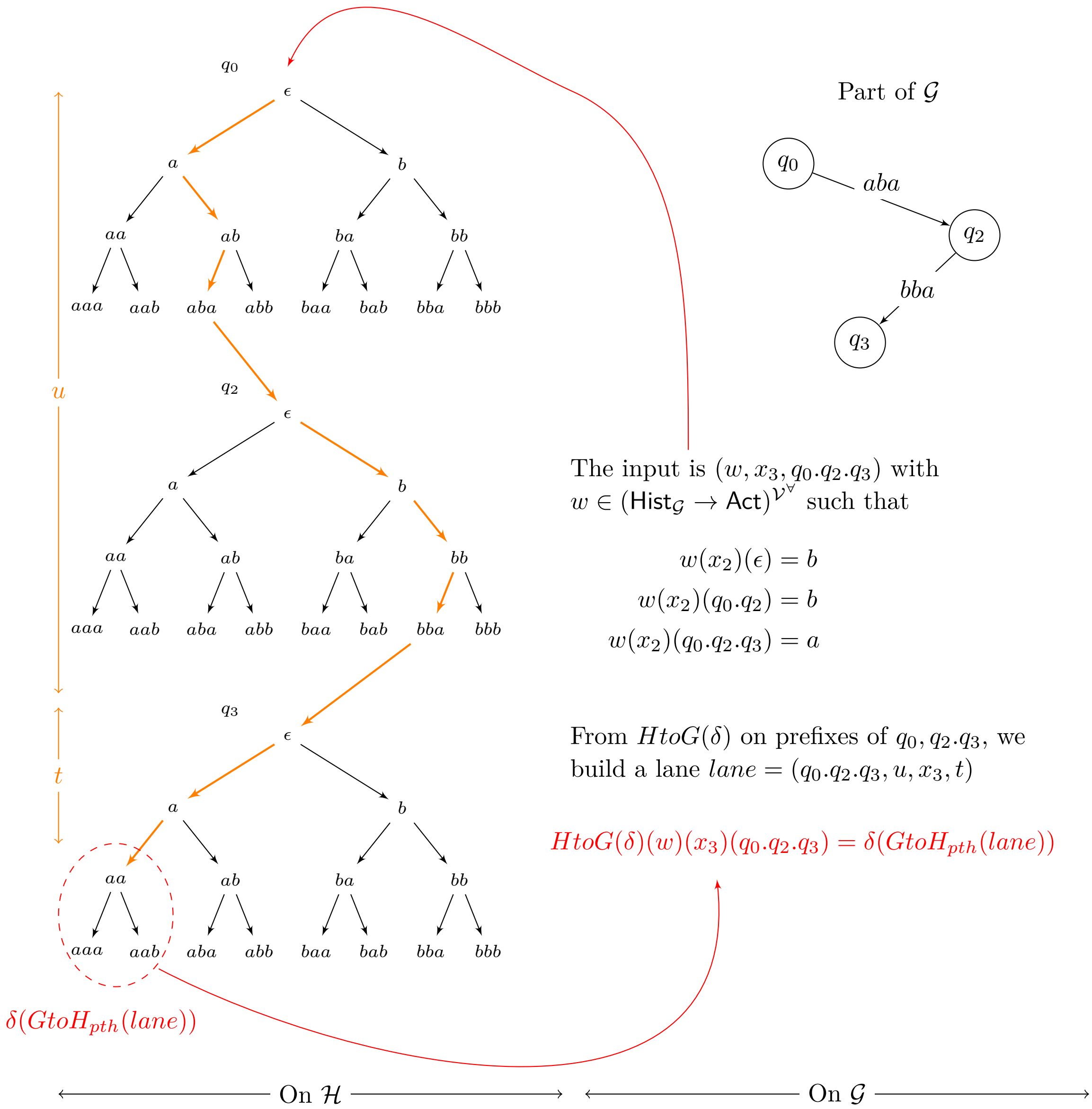


Figure 7 From $\mathcal{H}$ to $\mathcal{G}$ on the formula $\exists x_1.\ \forall x_2.\ \exists x_3.\ \mathsf{assign}(A_1, x_1; A_2, x_2; A_3, x_3)\ \varphi$.

C Proof of Proposition 6

▶ **Proposition 6.** *For every game $\mathcal{G}$ with initial state q_0, for every $\phi \in$ SL[CG],*

$$\mathcal{G}, q \models^{\mathcal{M}(L,S)} \phi \iff \mathcal{G}, q \models^{\mathcal{M}(L)} \phi$$

In all other cases, the satisfaction relations are pairwise distinct over SL[DG] and SL[CG].

Proof. We start by proving the first part of the proposition. Fix a game $\mathcal{G}$, one of its state q and a SL[CG] formula $\phi = (Q_i x_i)_{i \leq l} \bigwedge_{j \leq n} \beta_j \varphi_j$. Assuming $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S)} \phi$, then there must exist Δ a $\mathcal{M}(L,S)$-witness of ϕ holding on $\mathcal{G}$ from q, i.e.

$$\forall w : (\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}^{\forall}} \qquad \mathcal{G}, q \models_{\Delta(w)} \bigwedge_{j \le n} \beta_j \varphi_j \tag{3}$$

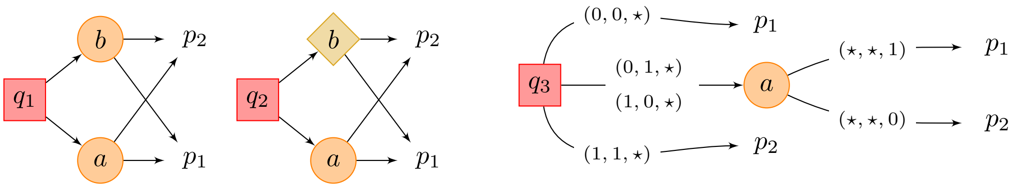
We create a $\mathcal{M}(L)$-map Δ' and prove it to be a $\mathcal{M}(L)$-witness of ϕ on $\mathcal{G}$. For this, choose any fixed function $w_0 : (\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}^{\forall}}$. For a function $w' \in (\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}^{\forall}}$ and a history ρ, we write w'_0 for the function equal to w' on ρ and its prefixes and equal to w_0 elsewhere. We then set, for any variable $x_i \in \mathcal{V}$,

$$\Delta'(w')(x_i)(\rho) = \Delta(w'_0)(x_i)(\rho)$$

Δ' is indeed a $\mathcal{M}(L)$-map: on a history ρ, we fixed universal choices on side histories of ρ to be "as in" w_0 before using Δ, hence we don't import the side dependencies from Δ.

It remains to prove that Δ' is a witness for $\mathcal{G}, q \models^{\mathcal{M}(L)} \phi$. Toward a contradiction, assume this is not the case. There exists some $w_1 \in (\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}^{\forall}}$ such that $\mathcal{G}, q \not\models_{\Delta'(w_1)} \bigwedge_{j \le n} \beta_j \varphi_j$. In particular there is some $j_0 \in [1, \ldots n]$ such that $\mathcal{G}, q \not\models_{\Delta'(w_1)} \beta_{j_0} \varphi_{j_0}$. Let $\pi_{j_0} = \mathsf{out}(\beta_{j_0}(\Delta'(w_1)), q)$ be the outcome of $\Delta'(w_1)$ for the assignments in β_{j_0}. We build another function $w_2 \in (\mathsf{Hist}_{\mathcal{G}} \to \mathsf{Act})^{\mathcal{V}^{\forall}}$ equal to w_1 on π_{j_0} and its prefixes, and equal to w_0 on any other history. By construction, for any prefix $\pi^p_{j_0}$ of π_{j_0} we have that $\Delta'(w_1)(x_i)(\pi^p_{j_0}) = \Delta(w_2)(x_i)(\pi^p_{j_0})$ hence π_{j_0} is also the outcome of $\Delta(w_2)$ for the assignments β_{j_0}. This implies that $\mathcal{G}, q \not\models_{\Delta(w_2)} \beta_{j_0} \varphi_{j_0}$ and $\mathcal{G}, q \not\models_{\Delta(w_2)} \bigwedge_{j \le n} \beta_j \varphi_j$, which is a contradiction with Formula (3). So, Δ' is a $\mathcal{M}(L)$-witness for ϕ on $\mathcal{G}$, it holds that $\mathcal{G}, q \models^{\mathcal{M}(L)} \phi$, and the result follows.

We then prove that in all other cases, the relations are pairwise distinct. For each case,



(a) $\mathcal{G}_1$. **(b)** $\mathcal{G}_2$. **(c)** $\mathcal{G}_3$ where $\mathsf{Agt} = \{\boxed{A}, \boxed{B}, \bigcirc\}$ and $\mathsf{Act} = \{0, 1\}$.

Figure 8 Three games for the proof of Prop. 6.

we only give a counter-example, and leave the verification to the reader. Consider the game on Figure 8a with two agents $\bigcirc$ and $\square$, and the $\mathsf{SL[DG]}$ formula

$$\phi_1 = \forall y.\ \exists x_A.\ \forall x_B.\ \bigvee \begin{cases} \mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_A).\ \mathbf{F}\, p_1 \\ \mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_B).\ \mathbf{F}\, p_2 \end{cases}$$

We have $\mathcal{G}, q_1 \models^{\mathcal{M}(L,F)} \phi_1$ but $\mathcal{G}, q_1 \not\models^{\mathcal{M}(L)} \phi_1$, and $\mathcal{G}, q_1 \models^{\mathcal{M}(L,S)} \phi_1$ but $\mathcal{G}, q_1 \models^{\mathcal{M}(L,S,F)} \phi_1$. Therefore $\models^{\mathcal{M}(L,F)}$ and $\models^{\mathcal{M}(L)}$ are distinct on $\mathsf{SL[DG]}$. Similarly, $\models^{\mathcal{M}(L,S)}$ and $\models^{\mathcal{M}(L,S,F)}$ are also distinct in $\mathsf{SL[DG]}$.

To show that $\models^{\mathcal{M}(L,S)}$ and $\models^{\mathcal{M}(L,S,F)}$ but also $\models^{\mathcal{M}(L,F)}$ $\models^{\mathcal{M}(L,S,F)}$ are distinct on $\mathsf{SL[CG]}$, use the game on Figure 8b and the formula

$$\phi_2 = \forall y.\ \exists z.\ \exists x_A.\ \exists x_B.\ \bigwedge \begin{cases} \mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_A; \diamond \mapsto z).\ \mathbf{F}\, p_1 \\ \mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_B; \diamond \mapsto z).\ \mathbf{F}\, p_2 \end{cases}$$

To show that $\models^{\mathcal{M}(L)}$ and $\models^{\mathcal{M}(L,S)}$ but also $\models^{\mathcal{M}(L,F)}$ $\models^{\mathcal{M}(L,S,F)}$ are distinct on SL[DG], use the game of Figure 8b and the formula

$$\phi_4 = \forall y.\ \exists z.\ \forall x_A.\ \forall x_B.\ \bigvee \begin{cases} \mathsf{assign}(\square \mapsto x_A; \bigcirc \mapsto y; \diamond \mapsto z)\mathbf{F}\, p_1 \\ \mathsf{assign}(\square \mapsto x_B; \bigcirc \mapsto y; \diamond \mapsto z)\mathbf{F}\, p_2 \end{cases}$$

To show that $\models^{\mathcal{M}(L)}$ and $\models^{\mathcal{M}(L,F)}$ are distinct on SL[CG], use the game on Figure 8c and formula

$$\phi_3 = \forall y.\ \exists x^A.\ \exists x_1^B.\ \exists x_2^B.\ \bigwedge \begin{cases} \mathsf{assign}(\boxed{A} \mapsto x^A; \boxed{B} \mapsto x_1^B; \bigcirc \mapsto y).\ \mathbf{F}\, p_1 \\ \mathsf{assign}(\boxed{A} \mapsto x^A; \boxed{B} \mapsto x_2^B; \bigcirc \mapsto y).\ \mathbf{F}\, p_2 \end{cases}$$

◀

D Proof of Proposition 8

▶ **Proposition 8.** *For every formula ϕ in SL[BG], for every game $\mathcal{G}$ with initial state q_0,*

- $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,S,F,T)} \phi \implies \mathcal{G}, q_0 \models^{\mathcal{M}(L,S,F,T)} \neg\phi$*;*
- $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi \implies \mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi$*.*

Proof. The first result follows from the fact that at least one of ϕ and $\neg\phi$ holds true for $\models^{\mathcal{M}(L,S,F)}$. Since $\mathcal{M}(L,S,F) \subseteq \mathcal{M}(L,S,F,T)$, at least one also holds for $\models^{\mathcal{M}(L,S,F,T)}$.

We now prove the second implication. For a contradiction, assume that there exist two maps θ and $\overline{\theta}$ witnessing $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi$ and $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \neg\phi$ resp. There must exist two $\mathcal{M}(L,T)$-maps θ and $\overline{\theta}$ such that

$$\forall w \in (\mathsf{Hist} \to \mathsf{Act})^{\mathcal{V}^\forall} \qquad \mathcal{G}, q_0 \models_{\theta(w)} \xi(\beta_j \varphi_j)_{j \le n} \tag{4}$$

$$\forall \overline{w} \in (\mathsf{Hist} \to \mathsf{Act})^{\mathcal{V}^\exists} \qquad \mathcal{G}, q_0 \models_{\overline{\theta}(\overline{w})} \neg\xi(\beta_j \varphi_j)_{j \le n} \tag{5}$$

We can then inductively (on histories and on the sequence of quantified variables) build a strategy valuation χ on $\mathcal{V}$ such that $\theta(\chi_{|\mathcal{V}^\forall}) = \overline{\theta}(\chi_{|\mathcal{V}^\exists}) = \chi$ (Figure 9 gives an intuition):

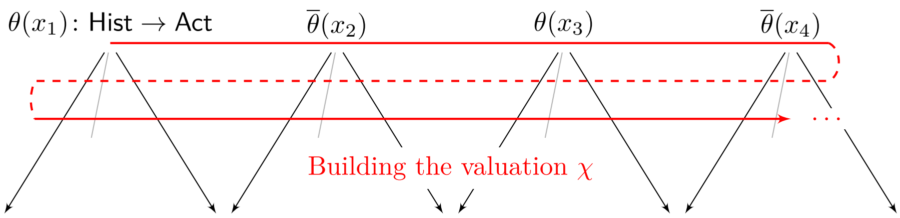


Figure 9 How to confront two $\mathcal{M}(L,T)$ maps θ and $\overline{\theta}$.

- On the empty history ϵ, for any $x_i \in \mathcal{V}$ and having build χ on ϵ for any $x_{i'}$ where $i' < i$, then
  - if $x_i \in \mathcal{V}^\exists$, we set $\chi(x_i)(\epsilon) = \theta(\bigcup_{i'<i} \chi_{|\epsilon\ x_{i'}})(x_i)(\epsilon)$
  - if $x_i \in \mathcal{V}^\forall$, we set $\chi(x_i)(\epsilon) = \overline{\theta}(\bigcup_{i'<i} \chi_{|\epsilon\ x_{i'}})(x_i)(\epsilon)$

  Note that, because θ and $\overline{\theta}$ are $\mathcal{M}(L,T)$ maps, feeding them $\bigcup_{i'<i} \chi_{|\epsilon\ x_{i'}}$ is sufficient.
- On a history ρ and with χ defined on any prefix of ρ, for any $x \in \mathcal{V}$ and any prefix ρ' of ρ For a variable $x_i \in \mathcal{V}$, having build χ on ρ for any $x_{i'}$ where $i' < i$, then

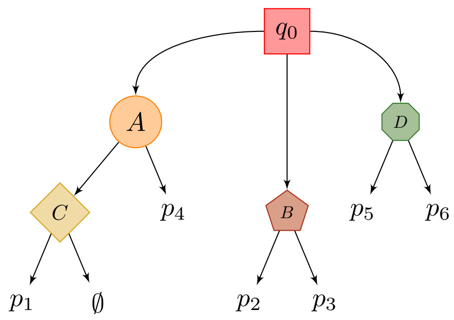


Figure 10 The game $\mathcal{G}$ for the last case of the proof of Prop. 9

- if $x_i \in \mathcal{V}^{\exists}$, we then set $\chi(x_i)(\rho) = \theta(\chi_{|\mathsf{Pref}_{<\rho}} \cup \bigcup_{i'<i} \chi_{|\rho\ x_{i'}})(x_i)(\rho)$
- if $x_i \in \mathcal{V}^{\forall}$, we then set $\chi(x_i)(\rho) = \overline{\theta}(\chi_{|\mathsf{Pref}_{<\rho}} \cup \bigcup_{i'<i} \chi_{|\rho\ x_{i'}})(x_i)(\rho)$

Again, because θ and $\overline{\theta}$ are $\mathcal{M}(L,T)$ maps, feeding them a partial first entry is sufficient.

Now, by construction, $\theta(\chi_{|\mathcal{V}^{\forall}}) = \overline{\theta}(\chi_{|\mathcal{V}^{\exists}}) = \chi$. Then, by Formulas (4) and (5), we have both $\mathcal{G}, q_0 \models_\chi \xi(\beta_j\varphi_j)_{j\leq n}$ and $\mathcal{G}, q_0 \models_\chi \neg\xi(\beta_j\varphi_j)_{j\leq n}$, which is impossible. ◀

E Proof of Proposition 9

▶ **Proposition 9.**
- *For any $D \subsetneq \{L,S,F,T\}$ s.t. $\{L,T\} \subseteq D$, there exists a game $\mathcal{G}$ with initial state q_0 and a formula $\phi \in$ SL[BG] such that $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \phi$ and $\mathcal{G}, q_0 \not\models^{\mathcal{M}(D)} \neg\phi$;*
- *for any $D \subseteq \{L,S,F,T\}$ s.t. $\{L,T\} \subsetneq D$, there exists a game $\mathcal{G}$ with initial state q_0 and a formula $\phi \in$ SL[BG] such that $\mathcal{G}, q_0 \models^{\mathcal{M}(D)} \phi$ and $\mathcal{G}, q_0 \models^{\mathcal{M}(D)} \neg\phi$.*

Proof. The first part of the proposition is easily proven by considering the same games and formulas as in the proof of Prop. 3.

We now focus on the second part of the proposition. The case $D = \{L,S,F,T\}$ was handled in the paper. For the case $D = \{L,F,T\}$, one can take the game 8a of Appendix C and the formula below. We leave the verification to the reader.

$$\phi_2 = \forall y.\ \exists x_A.\ \forall x_B.\ \big(\mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_A).\ \mathbf{F}\, p_1 \vee \mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_B).\ \mathbf{F}\, p_2\big).$$

The last case $\{L,S,T\}$ is more involved. Consider the turn based game of Fig. 10 and the state q_0 as initial state. The game has five agents $\square$, $\bigcirc$, $\Diamond$, ⬠ and ⯃ , and six atomic propositions p_1, p_2, p_3, p_4, p_5, and p_6. Consider further the formula ϕ below:

$$\phi = \forall x_2.\ \forall x_4.\ \forall x_5.\ \exists x_1.\ \exists x_3.\ \exists x_6.\ \exists z_1.\ \exists y_1.\ \forall v_5.\ \exists v_6.\ \forall w_2.\ \exists w_3.\ \forall z_4.\ \phi'$$

$$\text{with } \phi' = \begin{cases} \psi_1 \iff \psi_5 \\ \vee \\ \psi_5 \iff \psi_6 \wedge \psi_2 \iff \psi_3 \wedge \begin{cases} \psi_1 \iff \psi_4 \\ \vee \\ \psi_3 \iff \psi_4 \end{cases} \end{cases}$$

and

$$\psi_1 = \mathsf{assign}(\square \mapsto x_1; \bigcirc \mapsto z_1; \Diamond \mapsto y_1; ⬠ \mapsto w_2; ⯃ \mapsto v_5) F p_1$$
$$\psi_2 = \mathsf{assign}(\square \mapsto x_2; \bigcirc \mapsto z_1; \Diamond \mapsto y_1; ⬠ \mapsto w_2; ⯃ \mapsto v_5) F p_2$$

$$\psi_3 = \mathsf{assign}(\square \mapsto x_3; \bigcirc \mapsto z_1; \diamond \mapsto y_1; \pentagon \mapsto w_3; \octagon \mapsto v_5) F p_3$$
$$\psi_4 = \mathsf{assign}(\square \mapsto x_4; \bigcirc \mapsto z_4; \diamond \mapsto y_1; \pentagon \mapsto w_2; \octagon \mapsto v_5) F p_4$$
$$\psi_5 = \mathsf{assign}(\square \mapsto x_5; \bigcirc \mapsto z_1; \diamond \mapsto y_1; \pentagon \mapsto w_2; \octagon \mapsto v_5) F p_5$$
$$\psi_6 = \mathsf{assign}(\square \mapsto x_6; \bigcirc \mapsto z_1; \diamond \mapsto y_1; \pentagon \mapsto w_2; \octagon \mapsto v_6) F p_6$$

A witness that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,T)} (Q_i x_i)_{i \le l}\ \xi(\beta_j \varphi_j)_{j \le n}$ can be derived from Table 1 (at the end of this appendix) and a witness that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,T)} (\overline{Q_i} x_i)_{i \le l}\ \neg\xi(\beta_j \varphi_j)_{j \le n}$ can be derived from Table 2. We have also made the different dependencies appear in both tables. ◀

F Proof of Proposition 10

▶ **Proposition 10.** *For every game $\mathcal{G}$ with initial state q_0, for every formula $\phi \in$ SL[CG],*

$$\mathcal{G}, q_0 \models^{\mathcal{M}(L,S,T)} \phi \iff \mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi$$

In all other cases (involving timeline dependencies), the satisfaction relations are pairwise distinct for SL[CG], SL[DG], and SL[1G].

Proof. The first part can be handle similarly to the first part of Prop 6 (Section 3 and Annex C). However, the differentiation are more complex than the ones of Section 3.

To separate $\models^{\mathcal{M}(L,T)}$ from $\models^{\mathcal{M}(L,S,T)}$, consider the game $\mathcal{G}$ of Fig. 11 with 6 agents $\square$, $\pentagon_1$, $\pentagon_2$, $\diamond$, ① and ②. Each agent can only influence the state represented in its name, for example ① and ② are the only agents having an influence on the state c. To ease the reading we only represented the actions of the active agents on the transitions. Consider further the following SL[DG] formula

$$\exists x^{\square}_{p1}.\ \exists x^{\square}_{p2}.\ \exists x^{\bigcirc}_1.\ \forall x^{\bigcirc}_2.\ \exists x^{\pentagon}_1.\ \forall x^{\pentagon}_2.\ \exists x^{\diamond}.$$
$$\bigvee \begin{cases} \mathsf{assign}(\square \mapsto x^{\square}_{p1}; \pentagon_1 \mapsto x^{\pentagon}_1; \pentagon_2 \mapsto x^{\pentagon}_2; \diamond \mapsto x^{\diamond}; ① \mapsto x^{\bigcirc}_1; ② \mapsto x^{\bigcirc}_2).\mathbf{F}\, p_1 \\ \mathsf{assign}(\square \mapsto x^{\square}_{p2}; \pentagon_1 \mapsto x^{\pentagon}_1; \pentagon_2 \mapsto x^{\pentagon}_2; \diamond \mapsto x^{\diamond}; ① \mapsto x^{\bigcirc}_1; ② \mapsto x^{\bigcirc}_2).\mathbf{F}\, p_2 \end{cases}$$

Table 3a lists the choices of the existential strategies in function of the universal ones and induces a $\mathcal{M}(L,S,T)$-witness that $\mathcal{G}, q \models^{\mathcal{M}(L,S,T)} \phi$. On the other hand, Table 3b proves through a case decomposition that $\mathcal{G}, q \not\models^{\mathcal{M}(L,T)} \phi$. Both tables can be found in Table 3 at the end of this appendix.

The separations for SL[1G] between $\models^{\mathcal{M}(L,T)}$ and $\models^{\mathcal{M}(L,F,T)}$, between $\models^{\mathcal{M}(L,S,T)}$ and $\models^{\mathcal{M}(L,S,F,T)}$, and between $\models^{\mathcal{M}(L,F,T)}$ and $\models^{\mathcal{M}(L,S,F,T)}$ all follow arguments similar to the ones used in Table 3 to separate $\models^{\mathcal{M}(L,T)}$ from $\models^{\mathcal{M}(L,S,T)}$. ◀

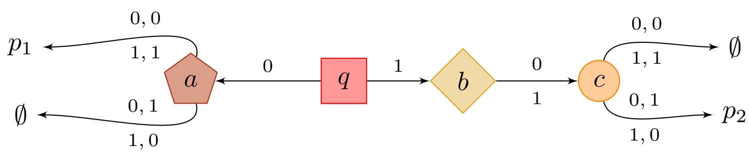


Figure 11 A concurrent game $\mathcal{G}$ for the proof of Prop. 10

G Proof of Proposition 11

▶ **Proposition 11.** *For any $D \subseteq \{L, S, F\}$ containing L, the satisfaction relations $\models^{\mathcal{M}(D)}$ and $\models^{\mathcal{M}(D\cup\{T\})}$ are distinct for SL[CG] and SL[DG].*

Proof. For the cases with SL[CG], consider the game of Fig. 8a (Annex C) and the formula ϕ_C below. For the cases with SL[DG], take the game of Fig. 8b (Annex C) and the formula ϕ_D below. The verification is similar to many result done before.

$$\phi_C = \exists y.\ \forall x_A.\ \exists x_B.\ \bigwedge \begin{cases} \mathsf{assign}(\bigcirc \mapsto y;\ \square \mapsto x_A).\ \mathbf{F}\, p_1 \\ \mathsf{assign}(\bigcirc \mapsto y;\ \square \mapsto x_B).\ \mathbf{F}\, p_2 \end{cases}$$

$$\phi_D = \exists y.\ \forall x_A.\ \forall x_B.\ \forall z.\ \bigvee \begin{cases} \mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_A; \diamond \mapsto z).\ \mathbf{F}\, p_1 \\ \mathsf{assign}(\bigcirc \mapsto y; \square \mapsto x_B; \diamond \mapsto z).\ \mathbf{F}\, p_2 \end{cases}$$

◀

H Proof of Lemma 19

▶ **Lemma 27.** *Fix a semi-stable set F^n and $s \in \{0,1\}^n$. For any $h_1, h_2 \in \{0,1\}^n$, either $\mathbb{F}^n(h_1, s) \subseteq \mathbb{F}^n(h_2, s)$ or $\mathbb{F}^n(h_2, s) \subseteq \mathbb{F}^n(h_1, s)$.*

Proof. Assume otherwise, there is $h_1' \in \mathbb{F}^n(h_1, s)\backslash\mathbb{F}^n(h_2, s)$ and $h_2' \in \mathbb{F}^n(h_2, s)\backslash\mathbb{F}^n(h_1, s)$. We then have:

$$\begin{array}{ll} (h_1 \curlywedge s) \curlyvee (h_1' \curlywedge \overline{s}) \in F^n & (h_2 \curlywedge s) \curlyvee (h_1' \curlywedge \overline{s}) \notin F^n \\ (h_2 \curlywedge s) \curlyvee (h_2' \curlywedge \overline{s}) \in F^n & (h_1 \curlywedge s) \curlyvee (h_2 \curlywedge \overline{s}) \notin F^n \end{array}$$

Now consider $(h_1 \curlywedge s) \curlyvee (h_1' \curlywedge \overline{s})$, $(h_2 \curlywedge s) \curlyvee (h_2' \curlywedge \overline{s})$ and s. As F^n is semi-stable, one of the two following vector is in F^n :

$$\big((h_1 \curlywedge s) \curlyvee (h_1' \curlywedge \overline{s}) \curlywedge s\big) \curlyvee \big((h_2 \curlywedge s) \curlyvee (h_2' \curlywedge \overline{s}) \curlywedge \overline{s}\big)$$
$$\big((h_2 \curlywedge s) \curlyvee (h_2' \curlywedge \overline{s}) \curlywedge s\big) \curlyvee \big((h_1 \curlywedge s) \curlyvee (h_1' \curlywedge \overline{s}) \curlywedge \overline{s}\big)$$

The first vector is equal to $(h_1 \curlywedge s) \curlyvee (h_2' \curlywedge \overline{s})$ and the second to $(h_2 \curlywedge s) \curlyvee (h_1' \curlywedge \overline{s})$ and both are supposed to be in $\overline{F^n}$, we get a contradiction. ◀

I Proof of Theorem 13

▶ **Theorem 13.** *For any $\phi \in$ SL[EG], any game $\mathcal{G}$ and any state q_0, it holds:*

$$\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi \iff \mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi$$

*Moreover, model checking SL[EG] w.r.t. $\mathcal{M}(L, T)$-maps is 2-*EXPTIME*-complete.*

Proof. The proof is quite technical. We start with some preliminary results showing that we may assume F^n upward-close.

Closure under bit flipping and transformation into an upward closed set

Fix a vector $b \in \{0,1\}^n$. We define the operation $\mathsf{flip}_b \colon \{0,1\}^n \to \{0,1\}^n$ that maps any vector f to $(f \curlywedge b) \curlyvee (\bar{f} \curlywedge \bar{b})$. In other terms, flip_b flips the i-th bit of its argument if $b_i = 0$, and keeps this bit unchanged if $b_i = 1$. Notice that flip_b is a permutation of $\{0,1\}^n$. Notice also that $\mathsf{flip}_{\mathbf{0}}(f) = \bar{f}$ and $\mathsf{flip}_f(f) = \mathbf{1}$ for all $f \in \{0,1\}^n$.

The following lemma shows that flipping bits preserves semi-stability. This is a natural property for our logic, since flipping bits corresponds to negating goals. More precisely, for $b \in \{0,1\}^n$, the open formulas $F^n((\beta_i.\ \varphi_i)_{1 \le i \le n})$ and $\mathsf{flip}_b(F^n)((\beta_i.\ \varphi'_i)_{1 \le i \le n})$, where $\varphi'_i = \varphi_i$ if $b(i) = 1$ and $\varphi'_i = \neg\varphi_i$ if $b(i) = 0$, are equivalent.

▶ **Lemma 28.** *If $F^n \subseteq \{0,1\}^n$ is semi-stable, then so is $\mathsf{flip}_b(F^n)$.*

Proof. We assume that F^n is semi-stable. Take $f' = \mathsf{flip}_b(f)$ and $g' = \mathsf{flip}_b(g)$ in $\mathsf{flip}_b(F^n)$, and $s \in \{0,1\}^n$. Then

$$\begin{aligned}(f' \curlywedge s) \curlyvee (g' \curlywedge \bar{s}) &= \big(((f \curlywedge b) \curlyvee (\bar{f} \curlywedge \bar{b})) \curlywedge s\big) \curlyvee \big(((g \curlywedge b) \curlyvee (\bar{g} \curlywedge \bar{b})) \curlywedge \bar{s}\big) \\ &= (((f \curlywedge s) \curlyvee (g \curlywedge \bar{s})) \curlywedge b) \curlyvee \big(((\bar{f} \curlywedge s) \curlyvee (\bar{g} \curlywedge \bar{s})) \curlywedge \bar{b}\big)\end{aligned}$$

Write $\alpha = (f \curlywedge s) \curlyvee (g \curlywedge \bar{s})$ and $\beta = (\bar{f} \curlywedge s) \curlyvee (\bar{g} \curlywedge \bar{s})$. One can easily check that $\beta = \bar{\alpha}$. We then have

$$\begin{aligned}(f' \curlywedge s) \curlyvee (g' \curlywedge \bar{s}) &= (\alpha \curlywedge b) \curlyvee \big(\bar{\alpha} \curlywedge \bar{b}\big) \\ &= \mathsf{flip}_b(\alpha).\end{aligned} \tag{6}$$

This computation being valid for any f and g, we also have

$$\begin{aligned}(g' \curlywedge s) \curlyvee (f' \curlywedge \bar{s}) &= (\gamma \curlywedge b) \curlyvee \big(\bar{\gamma} \curlywedge \bar{b}\big) \\ &= \mathsf{flip}_b(\gamma)\end{aligned} \tag{7}$$

with $\gamma = (g \curlywedge s) \curlyvee (f \curlywedge \bar{s})$. By hypothesis, at least one of α and γ belongs to F^n, so that also at least one of $(f' \curlywedge s) \curlyvee (g' \curlywedge \bar{s})$ and $(g' \curlywedge s) \curlyvee (f' \curlywedge \bar{s})$ belongs to $\mathsf{flip}_b(F^n)$. ◀

▶ **Corollary 29.** *F^n is semi-stable if, and only if, its complement is.*

Proof. Assume F^n is not semi-stable, and pick f and g in F^n and $s \in \{0,1\}^n$ such that none of $\alpha = (f \curlywedge s) \curlyvee (g \curlywedge \bar{s})$ and $\gamma = (g \curlywedge s) \curlyvee (f \curlywedge \bar{s})$ are in F^n. It cannot be the case that $g = f$, as this would imply $\alpha = f \in F^n$. Hence $\alpha \neq \gamma$. We claim that α and γ are our witnesses for showing that the complement of F^n is not semi-stable: both of them belong to the complement of F^n, and $(\alpha \curlywedge s) \curlyvee (\gamma \curlywedge \bar{s})$ can be seen to equal f, hence it is not in the complement of F^n. Similarly for $(\gamma \curlywedge s) \curlyvee (\alpha \curlywedge \bar{s}) = g$. ◀

Using Lemma 28, we can then force a semi-stable set to be upward closed.

▶ **Lemma 30.** *For any semi-stable set F^n, there exists $B \in \{0,1\}^n$ such that $\mathsf{flip}_B(F^n)$ is upward-closed.*

Proof. We start by proving the following lemma:

▶ **Lemma 31.** *If $F^n \subseteq \{0,1\}^n$ is semi-stable, then for any $s \in \{0,1\}^n$ and any non-empty subset H^n of F^n, it holds that*

$$\exists f \in H^n.\ \forall g \in H^n.\ (f \curlywedge s) \curlyvee (g \curlywedge \bar{s}) \in F^n.$$

Proof. For a contradiction, assume that there exist $s \in \{0,1\}^n$ and $H^n \subseteq F^n$ such that, for any $f \in H^n$, there is an element $g \in H^n$ for which $(f \curlywedge s) \curlyvee (g \curlywedge \bar{s}) \notin F^n$. Then there must exist a minimal integer $2 \leq \lambda \leq |H^n|$ and λ elements $\{f_i \mid 1 \leq i \leq \lambda\}$ of H^n such that

$$\forall 1 \leq i \leq \lambda - 1\ (f_i \curlywedge s) \curlyvee (f_{i+1} \curlywedge \bar{s}) \notin F^n \text{ and } (f_\lambda \curlywedge s) \curlyvee (f_1 \curlywedge \bar{s}) \notin F^n.$$

By Corollary 29, the complement of F^n is semi-stable. Hence, considering $(f_{\lambda-1} \curlywedge s) \curlyvee (f_\lambda \curlywedge \bar{s})$ and $(f_\lambda \curlywedge s) \curlyvee (f_1 \curlywedge \bar{s})$, one of the following two vectors is not in F^n:

$$\big([(f_{\lambda-1} \curlywedge s) \curlyvee (f_\lambda \curlywedge \bar{s})] \curlywedge s\big) \curlyvee \big([(f_\lambda \curlywedge s) \curlyvee (f_1 \curlywedge \bar{s})] \curlywedge \bar{s}\big)$$
$$\big([(f_\lambda \curlywedge s) \curlyvee (f_1 \curlywedge \bar{s})] \curlywedge s\big) \curlyvee \big([(f_{\lambda-1} \curlywedge s) \curlyvee (f_\lambda \curlywedge \bar{s})] \curlywedge \bar{s}\big)$$

The second expression equals f_λ, which is in F^n. Hence we get that $(f_{\lambda-1} \curlywedge s) \curlyvee (f_1 \curlywedge \bar{s})$ is not in F^n, contradicting minimality of λ. ◀

The lemma trivially holds for $F^n = \emptyset$ thus, in the following, we assume F^n to be non-empty. For $1 \leq i \leq n$, let $s_i \in \{0,1\}^n$ be the vector such that $s_i(j) = 1$ if, and only if, $j = i$. Applying Lemma 31, we get that for any i, there exists some $f_i \in F^n$ such that for any $f \in F^n$, it holds

$$(f_i \curlywedge s_i) \curlyvee (f \curlywedge \bar{s}_i) \in F^n. \tag{8}$$

We fix such a family $(f_i)_{i \leq n}$ then define $g \in \{0,1\}^n$ as $g = \bigcurlyvee_{1 \leq i \leq n}(f_i \curlywedge s_i)$, i.e. $g(i) = f_i(i)$ for all $1 \leq i \leq n$. Starting from any element of F^n and applying Equation (8) iteratively for each i, we get that $g \in F^n$. Since $g \curlywedge s_i = f_i \curlywedge s_i$, we also have

$$\forall f \in F^n \qquad (g \curlywedge s_i) \curlyvee (f \curlywedge \bar{s}_i) \in F^n$$

By Equation (7), since $\mathsf{flip}_g(g) = \mathbf{1}$, we get

$$\forall f \in F^n \qquad (\mathbf{1} \curlywedge s_i) \curlyvee (\mathsf{flip}_g(f) \curlywedge \bar{s}_i) \in \mathsf{flip}_g(F^n). \tag{9}$$

Now, assume that $\mathsf{flip}_g(F^n)$ is not upward closed: then there exist elements $f \in F^n$ and $h \notin F^n$ such that $\mathsf{flip}_g(f)(i) = 1 \Rightarrow \mathsf{flip}_g(h)(i) = 1$ for all i. Starting from f and iteratively applying Equation (9) for those i for which $\mathsf{flip}_g(h)(i) = 1$ and $\mathsf{flip}_g(f)(i) = 0$, we get that $\mathsf{flip}_g(h) \in \mathsf{flip}_g(F^n)$ and $h \in F^n$. Hence $\mathsf{flip}_g(F^n)$ must be upward closed. ◀

▶ Remark. Notice that being upward-closed is not a sufficient condition for being semi-stable. For instance, the set $F^n = \Uparrow\{(0,0,1,1); (1,1,0,0)\}$ is not semi-stable.

Following Lemma 30, we assume for the rest of the proof that the set F^n of the SL[EG] formula ϕ is upward closed (even if it means negating some of the LTL objectives).

Automata

We build a large set of *deterministic parity word automata* over 2^{AP}. For $s \in \{0,1\}^n$ and $h \in \{0,1\}^n$, we let $D_{s,h}$ be a deterministic parity automaton accepting exactly the words over 2^{AP} along which the following formula holds:

$$\bigvee_{\substack{k \in \{0,1\}^n \\ h\ \preceq_s\ k}} \ \bigwedge_{\substack{j \text{ s.t.} \\ (k \curlywedge s)(j)=1}} \varphi_j. \tag{10}$$

where a conjunction over an empty set (i.e., if $(k \curlywedge s)(j) = 0$ for all j) is true. As an example, take $s \in \{0,1\}^n$ with $|s| = 1$, writing j for the index where $s(j) = 1$, for any $h \in \{0,1\}^n$ we get that $D_{s,h}$ is universal iff there is $k \succeq_s h$ with $k(j) = 0$; otherwise $D_{s,h}$ accepts the set of words over 2^{AP} along which φ_j holds.

Write $\mathcal{D} = \{D_{s,h} \mid s \in \{0,1\}^n,\ h \in \{0,1\}^n\}$ for the set of automata just defined. A *vector of states* of $\mathcal{D}$ is a function associating with each automaton $D \in \mathcal{D}$ one of its states. We write VS for the set of all vectors of states of $\mathcal{D}$. Let d be a vector of states of $\mathcal{D}$ and let q be a state of $\mathcal{G}$. We set $\mathsf{succ}(d, q)$ to be the function associating with each $D \in \mathcal{D}$ the successor of $d(D)$ upon reading the labelling $\mathsf{lab}(q)$ of q; we also extend succ to take an input (d, ρ) and to return the state reachable by ρ from d. As usual, a path $(q_i)_{i\in\mathbb{N}}$ in $\mathcal{G}$ is accepted by an automaton D of $\mathcal{D}$ whenever its labels sequence $(\mathsf{lab}(q_i))_{i\in\mathbb{N}}$ is accepted by D. We use the customary notation $\mathcal{L}(D)$ for the set of words accepted by an automaton D. Finally we denote by $\mathcal{L}(D^d_{s,h})$ the set of words that are accepted by $D_{s,h}$ starting from the state $d(D_{s,h})$.

▶ **Proposition 32.** *The following holds for any $s \in \{0,1\}^n$:*
1. *for any $h_1, h_2 \in \{0,1\}^n$ where $h_1 \preceq_s h_2$, we have $\mathcal{L}(D_{s,h_1}) \supseteq \mathcal{L}(D_{s,h_2})$.*
2. *$D_{s,\mathbf{0}}$ is universal.*
3. *for any $h \in F^n$, $D_{\mathbf{1},h}$ accepts the words satisfying $\bigvee_{f\in F^n} \bigwedge_{j \ s.t.\ f(j)=1} \varphi_j$.*

Proof. The first and third points are immediate. In Formula (10) applied to $h = \mathbf{0}$, take $k = \mathbf{0}$ in the disjunction; then the conjunction is empty thus trivially true and therefore $D_{s,\mathbf{0}}$ accepts any word over 2^{AP}. ◀

Supervising goals going on different paths

Using the automata in $\mathcal{D}$, we define two new families of temporal operators for the proof of Theorem 13. Their semantics differ from the *until* and *next* operators: they are relative to the values of a valuation on the variables and are not asking to assign a strategy to each agent. The first family of operators simply transfers the conditions of the automata of $\mathcal{D}$ onto an operator for a later usage. For any $d \in \mathsf{VS}$ and any two s, h in $\{0,1\}^n$, the *parity* operator $\Gamma^{stick}_{d,s,h}$ obeys the following semantics: given a context χ with $\mathcal{V} \subseteq \mathsf{dom}(\chi)$ and a state q of $\mathcal{G}$,

$$\mathcal{G}, q \models_\chi \Gamma^{stick}_{d,s,h} \quad\Longleftrightarrow\quad \begin{array}{c}\exists\ \rho \text{ infinite in}\\ \mathcal{G} \text{ from } q \text{ with}\end{array} \begin{cases} \forall j \le n,\ s(j) = 1 \Rightarrow \mathsf{out}(\beta_j(\chi), q) = \rho \\ \\ \rho \in \mathcal{L}(D^d_{s,h}) \end{cases}$$

Intuitively, the outcome of the assignments enabled by s must follow a common path that is accepted by $D^d_{s,h}$.

The main difficulty of $\mathsf{SL[EG]}$ (or $\mathsf{SL[BG]}$ more generally) lies in the separation of the different goals along different histories. The second batch of operators must tackle this difficulty but before defining them, we need some formalism (we recall that Q is the set of states of $\mathcal{G}$ and VS is the set of all vectors of states of $\mathcal{D}$):

▶ **Definition 33.** A *partition* of an element $s \in \{0,1\}^n$ is a set $\{s_\kappa \mid 1 \le \kappa < \lambda\}$ of two or more elements of $\{0,1\}^n$ with $s_1 \curlyvee \ldots \curlyvee s_\lambda = s$ and where for any two $\kappa \ne \kappa'$ and any $j \le n$ we have $s_\kappa(j) = 1 \Rightarrow s_{\kappa'}(j) = 0$.

An *extended* partition of s is a set $\tau = \{(s_\kappa, q_\kappa, d_\kappa) \in \{0,1\}^n \times \mathsf{Q} \times \mathsf{VS} \mid 1 \le \kappa \le \lambda\ ,\ \lambda \ge 2\}$ with $(s_\kappa)_{\kappa\le\lambda}$ a partition of s.

Note that we only consider nontrivial partitions. We write $\mathsf{Part}(s)$ for the set of all extended partitions of s. If $|s| =\le 1$, then $\mathsf{Part}(s) = \emptyset$. For any $d \in \mathsf{VS}$, any s in $\{0,1\}^n$ and any set of partitions Υ of s, the condition $\Gamma^{sep}_{d,s,\Upsilon}$ looks for the assignments enabled by s

to all follow a common history ρ for some time then partition themselves according to some partition in Υ. Its semantics are defined upon a context χ with $\mathcal{V} \subseteq \mathsf{dom}(\chi)$ and a state q of $\mathcal{G}$ by the formula below.

$$\mathcal{G}, q \models_\chi \Gamma^{sep}_{d,s,\Upsilon} \iff \begin{array}{c}\exists \tau \in \Upsilon.\ \exists \rho \\ \text{finite history} \\ \text{in } \mathcal{G} \text{ from } q \\ \text{such that}\end{array} \begin{cases} \forall j \leq n, \\ s(j) = 1 \Rightarrow \rho \in \mathsf{Pref}_{<\mathsf{out}(\beta_j(\chi),q)} \\ \\ \forall \kappa \leq |\tau|,\ \forall j \leq n, \\ s_\kappa(j) = 1 \Rightarrow q_\kappa = \Delta(\mathsf{last}(\rho), m_j) \text{ with} \\ \forall A \in \mathsf{Agt},\ m_j(A) = \chi(\beta_j(A), \rho) \\ \\ \forall \kappa \leq |\tau|, \text{ applying } \mathsf{succ} \text{ inductively} \\ \text{from } d \text{ on the path } \rho.q_\kappa \text{ leads to } d_\kappa \end{cases}$$

Finding optimal elements

By an induction on $|s|$ ranging from 1 to n,

1. for every s with $|s| = \alpha$, every $h \in \{0,1\}^n$ and every $d \in \mathsf{VS}$, we define a new temporal operator $\Gamma_{d,s,h}$ based on the Γ^{stick} and Γ^{sep} operators.
   The Γ^{stick}'s operators handle the case where all goals stay on the same path; the Γ^{sep}'s operators handle the case where the goals split in different directions. The operator $\Gamma_{d,s,h}$ will regroup both possibilities and ask that starting with information d, the goals of s do at least as good as h for $\preceq_s$.
2. for every s with $|s| = \alpha$, every $d \in \mathsf{VS}$ and every state q of $\mathcal{G}$, we define an element $b_{q,d,s}$ of $\{0,1\}^n$.
   The $b_{q,d,s}$ element carries the information about the highest $h \in \{0,1\}^n$ possible for $\preceq_s$ so that the operator $\Gamma_{d,s,h}$ is satisfied.
3. if $\alpha \neq n$, for all $s \in \{0,1\}^n$ with $|s| = \alpha + 1$ and all $\tau \in \mathsf{Part}(s)$, we define yet another element $c_{s,\tau}$ of $\{0,1\}^n$.
   The c's elements carry information about previous step of the induction in the form of an element of $\{0,1\}^n$. Past the initial step, $c_{s,\tau}$ is used to determine the b's elements of the form $b_{\star,\star,s}$.

This induction allows us to condense information about the best course possible in the form of elements of $\{0,1\}^n$: the b's and c's elements. Theses elements will then be used to build an optimal behaviour in later sections.

Initial step ($\alpha = 1$)

1. For any $d \in \mathsf{VS}$ and any two s, h of $\{0,1\}^n$ with $|s| = 1$ we set $\Gamma_{d,s,h} = \Gamma^{stick}_{d,s,h}$.
2. For any state q of $\mathcal{G}$, any $d \in \mathsf{VS}$ and any $s \in \{0,1\}^n$ with $|s| = 1$, there is a maximal element $b_{q,d,s} \in \{0,1\}^n$ for the order $\preceq_s$ such that

$$\mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1 \leq i \leq l}\ \Gamma_{d,s,b_{q,d,s}} \tag{11}$$

By Proposition 32, $D^d_{s,\mathbf{0}}$ is universal; therefore, for any complete valuation χ, $\mathcal{G}, q \models_\chi \Gamma_{d,s,\mathbf{0}}$. This trivially implies that any $\mathcal{M}(L,T)$ map Δ is a witness that Formula (11) holds for $b_{q,d,s} = \mathbf{0}$. So there is at least one element of $\{0,1\}^n$ to fill the role of $b_{q,d,s}$ for

Formula (11) and, because $\preceq_s$ is a total quasi order, there must exist a maximal element. On the other hand, unicity is not guaranteed: if $h_1 =_s h_2$ then $\mathcal{L}(D_{s,h_1}) = \mathcal{L}(D_{s,h_2})$ thus $\Gamma_{d,s,h_1} = \Gamma_{d,s,h_2}$ and

$$\mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{d,s,h_1} \iff \mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{d,s,h_2}$$

Characterisation $b_{q,d,s} \in \{0,1\}^n$ is an element such that
a. $\mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{d,s,b_{q,d,s}}$
b. for any $h \in \{0,1\}^n$ with $b_{q,d,s} \prec_s h$, we have $\mathcal{G}, q \not\models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{d,s,h}$

3. Fix some $s \in \{0,1\}^n$ with $|s| = 2$ and an extended partition $\tau = \{(s_\kappa, q_\kappa, d_\kappa) \mid 1 \leq \kappa \leq 2\}$ of s. By definition of τ, for any $\kappa \leq 2$ we have $|s_\kappa| < |s| = 2$ i.e. $|s_\kappa| = 1$ thus $b_{q_\kappa,d_\kappa,s_\kappa}$ have been defined just before. We define $c_{s,\tau}$ by

$$c_{s,\tau} = (s_1 \curlywedge b_{q_1,d_1,s_1}) \curlyvee (s_2 \curlywedge b_{q_2,d_2,s_2})$$

The partition τ models a possible way for the goals to split; $c_{s,\tau}$ then regroups the b elements adequate to τ in a single element of $\{0,1\}^n$. Ergo $c_{s,\tau}$ carries information about the best that can be achieved just after the goals split along τ. The $c_{s,\tau}$ belonging to $\{0,1\}^n$, we can compare it to other elements of $\{0,1\}^n$ carrying other information using the quasi-orders described in Section 5. Using these comparisons, we will then deduce an optimal approach.

Induction step ($1 < \alpha \leq n$)

The induction step is slightly more involved.

1. For any $d \in \mathsf{VS}$ and any two s, h of $\{0,1\}^n$ with $|s| = \alpha$, we define an operator $\Gamma_{d,s,h}$ by

$$\Gamma_{d,s,h} \;=\; \Gamma^{stick}_{d,s,h} \;\vee\; \Gamma^{sep}_{d,s,\Upsilon} \qquad \text{where } \Upsilon = \{\tau \in \mathsf{Part}(s) \mid h \preceq_s c_{s,\tau}\}$$

We recall that $c_{s,\tau}$ was defined at the previous step of the induction. Figure 12 gives an intuition.

2. As before, for any q, any $d \in \mathsf{VS}$ and any $s \in \{0,1\}^n$ with $|s| = \alpha$, there is a maximal element $b_{q,d,s} \in \{0,1\}^n$ for the order $\preceq_s$ such that

$$\mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{d,s,b_{q,d,s}} \tag{12}$$

Similarly to the initial step, we show that such an element $b_{q,d,s}$ exists by proving that Formula (12) holds for $b_{q,d,s} = \mathbf{0}$. F^n is upward closed so $\mathbf{0}$ is a minimal element of $\preceq_s$ (no matter s) and for any $\tau \in \mathsf{Part}(s)$, $\mathbf{0} \preceq_s c_{s,\tau}$. Now, consider any given complete valuation χ. First of two possibilities: after some finite history ρ, χ splits the outcomes of the goals enabled by s into different paths following a partition τ_0, then we get

$$\mathcal{G}, q \models_\chi \Gamma^{sep}_{d,s,\Upsilon} \quad \text{ for } \Upsilon = \{\tau \in \mathsf{Part}(s) \mid \mathbf{0} \preceq_s c_{s,\tau}\} = \mathsf{Part}(s)$$

Second possibility: all the outcomes (enabled by s) follow the same infinite path. $D_{s,\mathbf{0}}$ is universal (Proposition 32) so we get $\mathcal{G}, q \models_\chi \Gamma^{stick}_{d,s,\mathbf{0}}$. This means that, whatever the value of χ, it holds that $\mathcal{G}, q \models_\chi \Gamma_{d,s,\mathbf{0}}$. Hence, as for the initial case, any $\mathcal{M}(L,T)$ map is a witness that Formula (12) holds for $\Gamma_{d,s,\mathbf{0}}$. As for the initial step, unicity is not guaranteed: if $h_1 =_s h_2$ then $\mathcal{L}(D_{s,h_1}) = \mathcal{L}(D_{s,h_2})$ thus $\Gamma^{stick}_{d,s,h_1} = \Gamma^{stick}_{d,s,h_2}$ and $h_1 \preceq_s c_{s,\tau}$ iff $h_2 \preceq_s c_{s,\tau}$, so

$$\mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{d,s,h_1} \iff \mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{d,s,h_2}$$

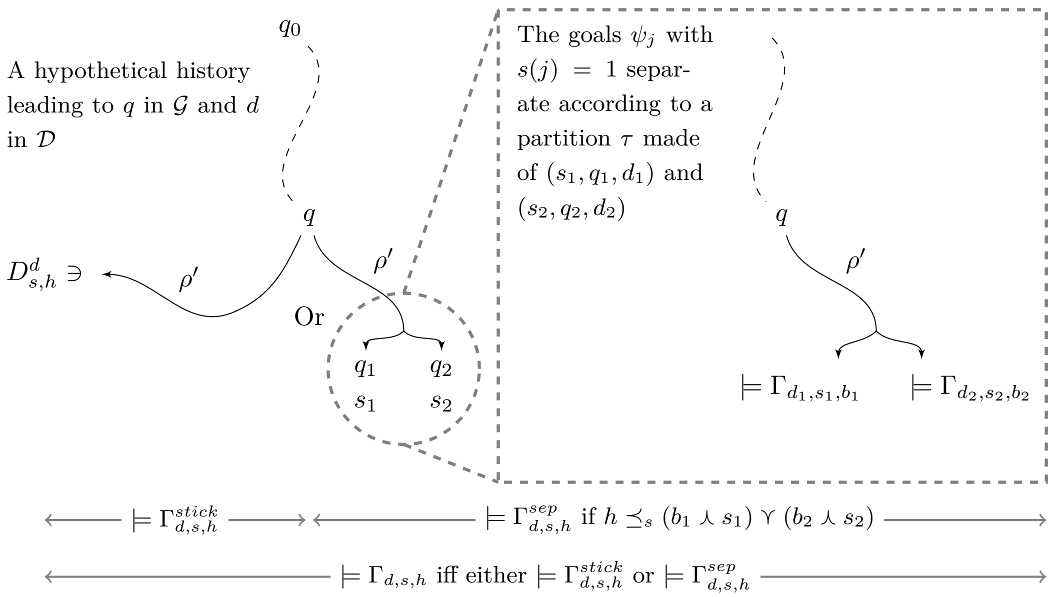


Figure 12 The $\Gamma_{d,s,h}$ operator

Characterisation $b_{q,d,s} \in \{0,1\}^n$ is an element such that

a. $\mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1 \le i \le l}\ \Gamma_{d,s,b_{q,d,s}}$

b. for any $h \in \{0,1\}^n$ with $b_{q,d,s} \prec_s h$, we have $\mathcal{G}, q \not\models^{\mathcal{M}(L,T)} (Q_i x_i)_{1 \le i \le l}\ \Gamma_{d,s,h}$

3. In the case of $\alpha < n$, fix some $s \in \{0,1\}^n$ with $|s| = \alpha + 1$ and an extended partition $\tau = \{(s_\kappa, q_\kappa, d_\kappa) \mid 1 \le \kappa \le \lambda\ ,\ \lambda \ge 2\}$ of s. By definition of τ, for any $\kappa \le \lambda$ we have $|s_\kappa| < |s| = \alpha + 1$, and the element $b_{q_\kappa, d_\kappa, s_\kappa}$ has been defined on previous steps of the induction. We define $c_{s,\tau}$ by

$$c_{s,\tau} = (s_1 \curlywedge b_{q_1,d_1,s_1}) \curlyvee \ldots \curlyvee (s_\lambda \curlywedge b_{q_\lambda,d_\lambda,s_\lambda})$$

▶ Remark. To find each b element effectively, we can build a set of parity games (one for each $k \in \{0,1\}^n$) using techniques similar to the ones developed in Annex B and solve each one. The game with the highest k in which the existential player $P_\exists$ can win is then the b element. More technical details can be found in [9] (Section 7.4.6, though some notations are different).

Intermediary results

We now focus on results derived from the elements defined previously. From the definition of the definitions of the b's elements, we get

▶ **Lemma 34.** *For any state q, any $d \in VS$ and any $s \in \{0,1\}^n$, there is a $\mathcal{M}(L,T)$ map $\varrho_{q,d,s}$ for $(Q_i x_i)_{1 \le i \le l}$ witnessing that*

$$\mathcal{G}, q \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1 \le i \le l}\ \Gamma_{d,s,b_{q,d,s}}$$

We also highlight a peculiar Γ operator whose parameters are set by ϕ. In the big induction, we inductively defined both the Γ's operators, the b's elements and the c's elements. In a way, Γ_{F^n} is what we find at the very top of the induction. Notice that for any two

$f, f' \in F^n$, we have $f =_{\mathbf{1}} f'$ and thus $\mathcal{L}(D_{\mathbf{1},f}) = \mathcal{L}(D_{\mathbf{1},f'})$. By definition of the Γ^{stick} operators, for $d \in \mathsf{VS}$, $\Gamma^{stick}_{d,\mathbf{1},f} = \Gamma^{stick}_{d,\mathbf{1},f'}$. We then set

$$\Gamma^{stick}_{F^n} = \Gamma^{stick}_{d_0,\mathbf{1},f} \qquad \text{and} \qquad \Gamma_{F^n} = \Gamma^{stick}_{F^n} \vee \Gamma^{sep}_{d_0,\mathbf{1},\Upsilon_{F^n}}$$

where f is any element in F^n, d_0 is the initial vector of states and with $\Upsilon_{F^n} = \{\tau \in \mathsf{Part}(\mathbf{1}) \mid c_{\mathbf{1},\tau} \in F^n\}$. Thus the operator Γ_{F^n} is the element of the Γ family of operators where d is the initial vector of state, $s = \mathbf{1}$ and h is an element of F^n.

The same way we highlighted Γ_{F^n} in the family of all Γ's operators, we highlight two $\mathcal{M}(L,T)$ maps $\varrho_{\mathbf{1}}$ and $\overline{\varrho_{\mathbf{1}}}$. The map $\varrho_{\mathbf{1}}$ corresponds to $\varrho_{q_0,d_0,\mathbf{1}}$ while the map $\overline{\varrho_{\mathbf{1}}}$ corresponds to $\overline{\varrho_{q_0,d_0,\mathbf{1}}}$.

▶ **Lemma 35.** *If $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\ \Gamma_{F^n}$, then $\varrho_{\mathbf{1}}$ witness that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\leq i\leq l}\Gamma_{F^n}$.*

The proof is a simple application of Lemma 34.

Assembling optimal $\mathcal{M}(L,T)$ maps

Having done this preliminary work, we may now build a $\mathcal{M}(L,T)$ map θ to define an optimal behaviour for $(Q_i x_i)_{1\leq i\leq l}$ To define the map, we start on the root and progress inductively along the histories. Given a history ρ and a function $w \in (\mathsf{Hist} \to \mathsf{Act})^{\mathcal{V}^{\forall}}$, we can know which goal is still following ρ. Indeed, assume θ has been defined on strict prefixes of ρ, we say that a goal $\psi_j = \beta_j \varphi_j$ is *active* on ρ w.r.t $\theta(w)$ whenever

$$\forall i < |\rho|,\ \Delta(\rho(i), m_i) = \rho(i+1) \begin{cases} \text{with } m_i\colon \mathsf{Agt} \to \mathsf{Act} \text{ is defined by} \\ \forall A \in \mathsf{Agt}.\ m_i(A) = \theta(w)(\beta_j(A))(\rho_{\leq i}) \end{cases}$$

Under these circumstances, we denote by $s_{\rho,\theta(w)} \in \{0,1\}^n$ the unique element such that $s_{\rho,\theta(w)}(j) = 1$ iff β_j is active on ρ w.r.t $\theta(w)$.

The idea behind θ is to combine together the maps defined in Lemmas 34 and 35. We start by defining θ that aims to satisfy $(Q_i x_i)_{i\leq l}\ F^n(\beta_j\varphi_j)_{j\leq n}$.

- If $x_i \in \mathcal{V}^{\forall}$, we must set $\theta(w)(x_i)(\rho) = w(x_i)(\rho)$ whatever the inputs $w \in (\mathsf{Hist} \to \mathsf{Act})^{\mathcal{V}^{\forall}}$ and $\rho \in \mathsf{Hist}$ by definition of maps (of any kind).
- If $x_i \in \mathcal{V}^{\exists}$, we use the maps defined in Lemmas 34 and 35. Consider a history ρ, a function $w \in (\mathsf{Hist} \to \mathsf{Act})^{\mathcal{V}^{\forall}}$ and a variable $x_i \in \mathcal{V}$ such that θ has been defined on strict prefixes of ρ. Then, θ being already defined on strict prefixes of ρ and having unordered prefix dependencies, we can know the active goals on ρ for $\theta(w)$, and we represent them by an element $s_{\rho,\theta(w)}$ of $\{0,1\}^n$. We decompose ρ in two parts ρ_1 and ρ_2 ($\rho = \rho_1.\rho_2$) such that ρ_2 represents the part of ρ that is followed by exactly the goals of $s_{\rho,\theta(w)}$, i.e. ρ_1 is the maximal prefix such that $s_{\rho_1,\theta(w)} \neq s_{\rho,\theta(w)}$.

  - First of two possibilities: $s_{\rho,\theta(w)} = \mathbf{1}$ and $\rho_1 = \epsilon$. Then θ follows the map $\varrho_{\mathbf{1}}$ of Lemma 35 and we set $\theta(w)(x_i)(\epsilon) = \varrho_{\mathbf{1}}(w)(x_i)(\epsilon)$.
  - Second possibility: $s_{\rho,\theta(w)} \neq \mathbf{1}$ and ρ_1 is not empty. The map θ then regroups the important information of ρ_1 in the vector of state $d_{\rho_1} = \mathsf{succ}(d_0, \rho_1)$ of $\mathcal{D}$. The behaviour of θ on ρ then follows the maps of Lemma 34, meaning that we set $\theta(w)(x_i)(\rho) = \varrho_{\mathsf{last}(\rho_1),d_{\rho_1},s_{\rho,u}}(w_{\overrightarrow{\rho_1}})(x_i)(\rho_2)$.

Having defined θ, we proceed similarly to define a map $\overline{\theta}$ for $\neg\phi$ trying to ensure $\overline{F^n}(\beta_j\varphi_j)_{j\leq n}$.

Concluding the proof

▶ **Lemma 36.** *There exists a valuation χ of domain $\mathcal{V}$ such that $\theta(\chi_{|\mathcal{V}^\forall}) = \chi$ and $\overline{\theta}(\chi_{|\mathcal{V}^\exists}) = \chi$. Moreover χ satisfies*

$$\mathcal{G}, q_0 \models_\chi \ \Gamma_{F^n} \quad \Rightarrow \quad \forall w \in (\mathit{Hist}_\mathcal{G} \to \mathit{Act})^{\mathcal{V}^\forall}\ \mathcal{G}, q_0 \models_{\theta(w)} F^n(\beta_j\varphi_j)_{1\le j\le n}$$
$$\mathcal{G}, q_0 \models_\chi \ \neg\Gamma_{F^n} \quad \Rightarrow \quad \forall \overline{w} \in (\mathit{Hist}_\mathcal{G} \to \mathit{Act})^{\mathcal{V}^\exists}\ \mathcal{G}, q_0 \models_{\overline{\theta}(\overline{w})} \overline{F^n}(\beta_j\varphi_j)_{1\le j\le n}$$

Proof. Both θ and $\overline{\theta}$ are $\mathcal{M}(L,T)$ maps, we can therefore apply the technique used in Annex E to get a valuation χ such that $\theta(\chi_{|\mathcal{V}^\forall}) = \chi$ and $\overline{\theta}(\chi_{|\mathcal{V}^\exists}) = \chi$.

It remains to prove the two implications, we start by proving the first one. In the following, assume that

$$\mathcal{G}, q_0 \models_\chi \ \Gamma_{F^n} \tag{13}$$

We start with a preliminary result

▶ **Lemma 37.** *Given a semi-stable set F^n, $s_1, s_2 \in \{0,1\}^n$ such that $s_1 \curlywedge s_2 = \mathbf{0}$ and $f, g \in \{0,1\}^n$ such that $f \preceq_{s_1} g$ and $f \preceq_{s_2} g$. Then $f \preceq_{s_1 \curlyvee s_2} g$.*

Proof. Because $f \preceq_{s_1} g$ and $f \preceq_{s_2} g$, we have

$$\forall i \in \{1,2\}\ \forall h \in \{0,1\}^n \qquad (f \curlywedge s_i) \curlyvee (h \curlywedge \overline{s_i}) \in F^n \Rightarrow (g \curlywedge s_i) \curlyvee (h \curlywedge \overline{s_i}) \in F^n \tag{14}$$

Consider $h' \in \{0,1\}^n$ such that $\alpha = (f \curlywedge (s_1 \curlyvee s_2)) \curlyvee (h' \curlywedge \overline{(s_1 \curlyvee s_2)})$ is in F^n. Define the element $h = \alpha \curlywedge \overline{s_2}$, then $(f \curlywedge s_2) \curlyvee (h \curlywedge \overline{s_2}) = (f \curlywedge (s_1 \curlyvee s_2)) \curlyvee (h' \curlywedge \overline{(s_1 \curlyvee s_2)}) \in F^n$. Using (14) with s_2 and h, we get $\beta = (g \curlywedge s_2) \curlyvee (h \curlywedge \overline{s_2})$. As $s_1 \curlywedge s_2 = \mathbf{0}$, we can write $\beta = (f \curlywedge s_1) \curlyvee (g \curlywedge s_2) \curlyvee (h' \curlywedge \overline{(s_1 \curlyvee s_2)}) \in F^n$.

Now consider $h = \beta \curlywedge \overline{s_1}$, we have $(f \curlywedge s_1) \curlyvee (h \curlywedge \overline{s_1}) = \beta \in F^n$. Using (14) with s_1 and h, we get $(g \curlywedge (s_1 \curlyvee s_2)) \curlyvee (h' \curlywedge \overline{(s_1 \curlyvee s_2)}) \in F^n$. Therefore $\mathbb{F}^n(f, s_1 \curlyvee s_2) \subseteq \mathbb{F}^n(g, s_1 \curlyvee s_2)$ and $f \preceq_{s_1 \curlyvee s_2} g$. ◀

We fix some notations specific to this proof then prove some intermediary results.

Notations For a fixed parameter $w \in (\mathsf{Hist}_\mathcal{G} \to \mathsf{Act})^{\mathcal{V}^\forall}$,

- we call π_j^w the outcome $\mathsf{out}(\beta_j(\theta(w)), q_0)$.
- we set f^w to be the $\{0,1\}^n$ element such that $f^w(j) = 1$ if, and only if, π_j^w satisfies φ_j.
- for any history ρ we call $s_{\rho,w}$ the $\{0,1\}^n$ element such that $s_{\rho,w}(j) = 1$ if, and only if, $\mathsf{out}(\beta_j(\theta(w)), q_0)$ follows ρ.
- finally, we define $\mathbb{R}^w \subseteq \{0,1\}^n \times \mathsf{Hist}_\mathcal{G}$, the relation such that $(s,\rho) \in \mathbb{R}^w$ if, and only if, $s = s_{\rho,w}$ and ρ is minimal (meaning for any prefix ρ' of ρ, $(s,\rho') \notin \mathbb{R}^w$).

▶ **Proposition 38.** *For any $w \in (\mathit{Hist}_\mathcal{G} \to \mathit{Act})^{\mathcal{V}^\forall}$, using the notations presented above, it holds*

$$\forall s \in \{0,1\}^n.\ \forall \rho \in \mathit{Hist}_\mathcal{G}. \quad (s,\rho) \in \mathbb{R}^w \ \Rightarrow\ b_{\mathit{last}(\rho), d_\rho, s} \preceq_s f^w$$

where $d_\rho = \mathit{succ}(d_0, \rho)$ (the vector of states accessible by ρ from the initial vector of states).

Proof. Fix some $w \in (\mathsf{Hist}_\mathcal{G} \to \mathsf{Act})^{\mathcal{V}^\forall}$, we proceed by induction on the size of s from 1 to n.

The initial case ($|s| = 1$) Consider any history ρ such that $(s, \rho) \in \mathbb{R}^w$. As $|s| = 1$ and $(s, \rho) \in \mathbb{R}^w$, there is a unique goal, say $\beta_{j_0}\varphi_{j_0}$, following ρ. By definition of θ, $\pi_{j_0} = \rho.\eta$ where η is the outcome[4] obtained through $\beta_{j_0}\big(\varrho_{\mathsf{last}(\rho),d_\rho,s}(w_{\overrightarrow{\beta}})\big)$ starting in $\mathsf{last}(\rho)$.

Note that because $|s| = 1$, $\Gamma_{d_\rho,s,b_{\mathsf{last}(\rho),d_\rho,s}} = \Gamma^{stick}_{d_\rho,s,b_{\mathsf{last}(\rho),d_\rho,s}}$. The map $\varrho_{\mathsf{last}(\rho),d_\rho,s}$ is a $\mathcal{M}(L,T)$ witness that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\le i\le l}\Gamma_{d_\rho,s,b_{\mathsf{last}(\rho),d_\rho,s}}$, therefore it also witnesses that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\le i\le l}\Gamma^{stick}_{d_\rho,s,b_{\mathsf{last}(\rho),d_\rho,s}}$. By definition of the Γ^{stick} operators, this implies that all its outcomes are accepted by the automaton $D^{d_\rho}_{s,b_{\mathsf{last}(\rho),d_\rho,s}}$; in particular, η is accepted by $D^{d_\rho}_{s,b_{\mathsf{last}(\rho),d_\rho,s}}$.

The automaton $D^{d_\rho}_{s,b_{\mathsf{last}(\rho),d_\rho,s}}$ accepts paths which give better results for the objectives $(\beta_j\varphi_j)_{j|s(j)=1}$ than $b_{\mathsf{last}(\rho),d_\rho,s}$. In our case this means that f^w does better than $b_{\mathsf{last}(\rho),d_\rho,s}$ for s, i.e. $b_{\mathsf{last}(\rho),d_\rho,s} \preceq_s f^w$.

The induction step ($|s| = \alpha$) We assume that the Proposition 38 holds for elements s of size $|s| < \alpha$. Consider for the induction step a history ρ such that $(s, \rho) \in \mathbb{R}^w$.

- Either there exists a common (infinite) path η such that for any j with $s(j) = 1$, $\pi_j = \rho.\eta$, i.e. all goals enabled by s always follow the same path $\rho.\eta$. We then apply the same reasoning as done in the initial case and deduce $b_{\mathsf{last}(\rho),d_\rho,s} \preceq_s f^w$.
- Or, somewhere after ρ, the goals enabled by s split themselves along some extended partition $\tau = (s_\kappa, q_\kappa, d_\kappa)_{\kappa\le\lambda}$. We call η the history from the last state of ρ to the point where the goals split from each other; formally η is obtained by applying $\beta_j\big(\varrho_{\mathsf{last}(\rho),d_\rho,s}(w_{\overrightarrow{\beta}})\big)$ where j is such that $s(j) = 1$.

  We recall the notation for $c_{s,\tau}$:

  $$c_{s,\tau} = (s_1 \curlywedge b_{q_1,d_1,s_1}) \curlyvee \ldots \curlyvee (s_\lambda \curlywedge b_{q_\lambda,d_\lambda,s_\lambda})$$

  The map $\varrho_{\mathsf{last}(\rho),d_\rho,s}$ witnesses that $\mathcal{G}, \mathsf{last}(\rho) \models^{\mathcal{M}(L,T)} \Gamma_{d,s,b_{\mathsf{last}(\rho),d_\rho,s}}$, therefore η may reach only a partition τ such that

  $$b_{\mathsf{last}(\rho),d_\rho,s} \preceq_s c_{s,\tau} \tag{15}$$

  For any $\kappa \le \lambda$ we have $(s_\kappa, \rho.\eta.q_\kappa) \in \mathbb{R}^w$, and using the induction hypothesis we get

  $$s_\kappa \curlywedge b_{q_\kappa,d_\kappa,s_\kappa} \preceq_{s_\kappa} f^w \tag{16}$$

  so, using Lemma 37 repeatedly on the $(s_\kappa)_{\kappa\le\lambda}$ and Inequality 16, we obtain

  $$\begin{aligned} & s_1 \curlywedge b_{q_1,d_1,s_1} \preceq_{s_1} f^w \\ \Rightarrow\ & (s_1 \curlywedge b_{q_1,d_1,s_1}) \curlyvee (s_2 \curlywedge b_{q_2,d_2,s_2}) \preceq_{s_1\curlyvee s_2} f^w \\ & \ldots \\ \Rightarrow\ & (s_1 \curlywedge b_{q_1,d_1,s_1}) \curlyvee \ldots \curlyvee (s_\lambda \curlywedge b_{q_\lambda,d_\lambda,s_\lambda}) \preceq_{s_1\curlyvee\ldots\curlyvee s_\lambda} f^w \\ \Rightarrow\ & c_{s,\tau} \preceq_s f^w \end{aligned}$$

  Combined with Inequality 15, we get $b_{\mathsf{last}(\rho),d_\rho,s} \preceq_s c_{s,\tau} \preceq_s f^w$.

  This concludes the induction and the proof of Proposition 38. ◀

[4] which can be written in the following barbaric way: $\mathsf{out}(\beta_{j_0}(\varrho_{\mathsf{last}(\rho),d_\rho,s}(w_{\overrightarrow{\beta}})), \mathsf{last}(\rho))$

▶ **Proposition 39.** $b_{q_0,d_0,\mathbf{1}} \in F^n$

Proof. Towards a contradiction, assume that $b_{q_0,d_0,\mathbf{1}} \in \overline{F^n}$. Then, by definition of the $b_{q_0,d_0,\mathbf{1}}$ element[5], $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\le i\le l}\ \Gamma_{F^n}$. Applying this to Lemma 35, we have that the map $\overline{\varrho_\mathbf{1}}$ (and therefore $\overline{\theta}$ which act as $\overline{\varrho_\mathbf{1}}$ before goal goes on different paths) witness $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} (Q_i x_i)_{1\le i\le l}\ \Gamma_{F^n}$. This immediately implies that $\mathcal{G}, q_0 \not\models_\chi \Gamma_{F^n}$ which is in contradiction with the Hypothesis 13. ◀

With these preliminary results, we are now ready to prove the first implication of the lemma. Consider a function $w \in (\mathsf{Hist}_\mathcal{G} \to \mathsf{Act})^{\mathcal{V}^\forall}$. By Proposition 38 applied to w, $\mathbf{1}$, ϵ we get that $b_{q_0,d_0,\mathbf{1}} \preceq_\mathbf{1} f^w$. Now by Proposition 39, $b_{q_0,d_0,\mathbf{1}} \in F^n$, therefore the element f^w which is greater than $b_{q_0,d_0,\mathbf{1}}$ for $\preceq_\mathbf{1}$ must also be in F^n, which is equivalent to $\mathcal{G}, q_0 \models_{\theta(w)} F^n(\beta_j\varphi_j)_{1\le j\le n}$.

The second implication of the lemma works similarly. ◀

Lemma 36 allows us to conclude that at least one of ϕ and $\neg\phi$ must hold on $\mathcal{G}$ for $\models^{\mathcal{M}(L,T)}$. Lemma 8 implies that at most one can hold. Combining both we get that exactly one holds. As explained in the paper, a *2*-EXPTIME algorithm can be derived from this work. ◀

J Proof of Proposition 23

▶ **Proposition 23.** *For any non-semi-stable boolean set $F^n \subseteq \{0,1\}^n$, there exists a SL[BG] formula ϕ built on F^n, a game $\mathcal{G}$ and a state q_0 such that*

$$\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi \qquad\qquad \mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \phi$$

Proof. We consider the game $\mathcal{G}$ depicted on Figure 13 with two agents $\square$ and $\bigcirc$. Let F^n be a non-semi-stable set over $\{0,1\}^n$. Then there must exist $f_1, f_2 \in F^n$, and $s \in \{0,1\}^n$, such that $(f_1 \curlywedge s) \curlyvee (f_2 \curlywedge \bar{s}) \notin F^n$ and $(f_2 \curlywedge s) \curlyvee (f_1 \curlywedge \bar{s}) \notin F^n$. We then let

$$\phi = \forall y_t^{\square}\,.\ \forall y_u^{\square}\,.\ \forall x_t^{\bigcirc}\,.\ \exists x_u^{\bigcirc}\,.\ F^n(\beta_1\varphi_1, \ldots, \beta_n\varphi_n)$$

where

$$\beta_i = \begin{cases} \mathsf{assign}(\square \mapsto y_t^{\square};\ \bigcirc \mapsto x_t^{\bigcirc}) & \text{if } s(i) = 1 \\ \mathsf{assign}(\square \mapsto y_u^{\square};\ \bigcirc \mapsto x_u^{\bigcirc}) & \text{if } s(i) = 0 \end{cases}$$

and

$$\varphi_i = \begin{cases} \mathbf{F}\,p_1 \vee \mathbf{F}\,p_2 & \text{if } f_1(i) = f_2(i) = 1 \\ \mathbf{F}\,p_1 & \text{if } f_1(i) = 1 \text{ and } f_2(i) = 0 \\ \mathbf{F}\,p_2 & \text{if } f_1(i) = 0 \text{ and } f_2(i) = 1 \\ \mathtt{false} & \text{if } f_1(i) = f_2(i) = 0 \end{cases}$$

It is not hard to check that the following holds:

▶ **Lemma 40.** *Let ρ be a maximal run of $\mathcal{G}$ from q_0. Let $k \in \{1,2\}$ be such that ρ visits a state labelled with p_k. Then for any $1 \le i \le n$, we have $\rho \models \varphi_i$ if, and only if, $f_k(i) = 1$.*

[5] See the definitions and explanations between Lemma 34 and Lemma 35 page 31

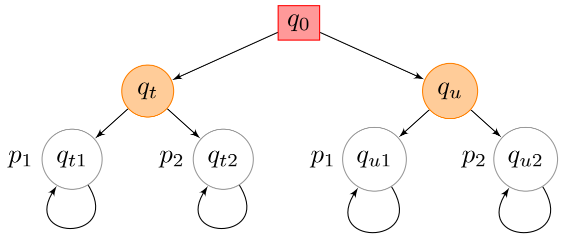


Figure 13 The two-agents turn-based game $\mathcal{G}$

Then:

▶ **Proposition 41.** $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \phi$

Proof. Towards a contradiction, assume that $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \phi$. We let σ_t (resp. σ_u) be the strategy that maps history q_0 to q_t (resp. q_u). We fix strategy τ_t such that $\tau_t(q_0 \cdot q_t) = q_{t1}$. There is a strategy τ_u (with local and timeline dependencies) such that

$$\mathcal{G}, q_0 \models_\chi F^n(\beta_1\varphi_1, \dots, \beta_n\varphi_n)$$

where χ maps $y_t^{\square}$ to σ_t, y° to σ_u, x_t° to τ_t and x_u° to τ_u.

Since x_u° is only jointly applied with $y_u^{\square}$, the only important information about τ_u is its value on history $q_0\sigma_u(q_0) = q_0q_u$. This value is then independent on the value of $\tau_t(q_0q_t) = \tau_t(q_0\sigma_t(q_0))$. In particular, writing χ' for the context obtained from χ by replacing $\chi(y_t^{\square}) = \tau_t$ with τ_t', where $\tau_t'(q_0q_t) = q_{t2}$, we also have

$$\mathcal{G}, q_0 \models_{\chi'} F^n(\beta_1\varphi_1, \dots, \beta_n\varphi_n)$$

Let v and v' be the vectors in $\{0,1\}^n$ representing the values of the goals $(\beta_1\varphi_1, \dots, \beta_n\varphi_n)$ under χ and χ'. Then v and v' are in F^n. However:

- If $\tau_u(q_0q_u) = q_{u1}$, then $v' = (f_1 \curlywedge \bar{s}) \curlyvee (f_2 \curlywedge s)$.
- If $\tau_t(q_0q_t) = q_{t2}$, then $v = (f_1 \curlywedge s) \curlyvee (f_2 \curlywedge \bar{s})$.

In both cases, by hypothesis, this does not belong to F^n, which is a contradiction. ◀

Also,

▶ **Proposition 42.** $\mathcal{G}, q_0 \not\models^{\mathcal{M}(L,T)} \neg\phi$

Proof. Similarly, assume $\mathcal{G}, q_0 \models^{\mathcal{M}(L,T)} \neg\phi$. Fix any three strategies σ_t, σ_u and τ_t respectively intended for the existentially quantified variables $y_t^{\square}$, $y_u^{\square}$ and x_t°. Due to the nature of $\models^{\mathcal{M}(L,T)}$, these three strategies are independent from the strategy τ_u of x_u°. Consider then the following strategy τ_u:

$$\tau_u(q_0.\sigma_u(q_0)) = \tau_t(q_0.\sigma_t(q_0))$$

Let χ be the resulting context and v the vector representing the values of the goals $(\beta_1\varphi_1, \dots, \beta_n\varphi_n)$ under χ. Either $v = f_1$ or $v = f_2$; in both case $v \in F^n$, which is a contradiction. ◀

◀

K Tables for Propositions 9 and 10.

$\forall$	$\exists$		$\forall$	$\exists$	$\forall$	$\exists$	$\forall$	
$x_2(q_3)$ $x_4(q_3)$ $x_5(q_3)$	$x_1(q_3)$ $x_3(q_3)$ $x_6(q_3)$	$z_1(q_3.A)$ $y_1(q_3.A.C)$	$v_5(q_3.D)$	$v_6(q_3.D)$	$w_2(q_3.B)$	$w_3(q_3.B)$	$z_4(q_3.A)$	
$\star,\star,A$ or $\star,\star,B$	$A,\star,\star$	$C,\emptyset$						$\neg\psi_1,\neg\psi_5$
$A,\star,D$	A,A,D	C,p_1	p_5	p_6				ψ_1,ψ_5
			p_6	p_5				$\psi_1,\neg\psi_2,\neg\psi_3,\neg\psi_5,\neg\psi_6$ (OK no matter ψ_4)
B,B,D	B,B,D		p_5	p_6	p_2	p_3		$\neg\psi_1,\psi_2,\psi_3,\neg\psi_4,\psi_5,\psi_6$
			p_5	p_6	p_3	p_2		$\neg\psi_1,\neg\psi_2,\neg\psi_3,\neg\psi_4,\psi_5,\psi_6$
			p_6	p_5	p_2	p_3		$\neg\psi_1,\psi_2,\psi_3,\neg\psi_4,\neg\psi_5,\neg\psi_6$
			p_6	p_5	p_3	p_2	Unordered Prefix	$\neg\psi_1,\neg\psi_2,\neg\psi_3,\neg\psi_4,\neg\psi_5,\neg\psi_6$
B,A,D	A,B,D	C,p_1	p_5	p_6	p_2	p_3	p_4	$\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6$
			p_5	p_6	p_3	p_2		$\psi_1,\neg\psi_2,\neg\psi_3,\psi_4,\psi_5,\psi_6$
			p_6	p_5	p_2	p_3		$\psi_1,\psi_2,\psi_3,\psi_4,\neg\psi_5,\neg\psi_6$
			p_6	p_5	p_3	p_2	Unordered Prefix	$\psi_1,\neg\psi_2,\neg\psi_3,\psi_4,\neg\psi_5,\neg\psi_6$
B,A,D	A,B,D	$C,\emptyset$	p_5	p_6	p_2	p_3	C	$\neg\psi_1,\psi_2,\psi_3,\neg\psi_4,\psi_5,\psi_6$
			p_5	p_6	p_3	p_2		$\neg\psi_1,\neg\psi_2,\neg\psi_3,\neg\psi_4,\psi_5,\psi_6$
			p_6	p_5	p_2	p_3		$\neg\psi_1,\psi_2,\psi_3,\neg\psi_4,\neg\psi_5,\neg\psi_6$
			p_6	p_5	p_3	p_2		$\neg\psi_1,\neg\psi_2,\neg\psi_3,\neg\psi_4,\neg\psi_5,\neg\psi_6$

local *local* *local*

Table 1 Table for the proof of Proposition 9.

$\exists$	$\forall$		$\exists$	$\forall$	$\exists$	$\forall$	$\exists$	
$x_2(q_3)$ $x_4(q_3)$ $x_5(q_3)$	$x_1(q_3)$ $x_3(q_3)$ $x_6(q_3)$	$y_1(q_3.A.C)$ $z_1(q_3.A)$	$v_5(q_3.D)$	$v_6(q_3.D)$	$w_2(q_3.B)$	$w_3(q_3.B)$	$z_4(q_3.A)$	
		$p_4, \star$	p_5					$\neg\psi_1, \psi_2, \neg\psi_3, \psi_5$
B, A, D	$\star, A, \star$ or $\star, D, \star$	$C, \emptyset$	p_5		p_2			$\neg\psi_1, \psi_2, \neg\psi_3, \psi_5$
		C, p_1	p_6					$\psi_1, \psi_2, \neg\psi_3, \neg\psi_5$
B, A, D	$B, B, \star$		p_5		p_3		p_4	$\neg\psi_1, \neg\psi_2, \psi_4, \psi_5$
B, A, D	A, B, A or A, B, B	$p_4, \star$ or $C, \emptyset$	p_5					$\neg\psi_1, \psi_5, \neg\psi_6$
		C, p_1	p_6		p_2		C	$\psi_1, \psi_2, \neg\psi_4, \neg\psi_5, \neg\psi_6$ (OK no matter ψ_3)
B, A, D	A, B, D	$p_4, \star$ or $C, \emptyset$	p_5					
								$\neg\psi_1, \psi_5, \neg\psi_6$
				p_6	p_3		p_4	$\neg\psi_1, \neg\psi_2, \psi_4, \psi_5, \psi_6$ (OK no matter ψ_3)
B, A, D	A, B, D	C, p_1	p_6					
								$\psi_1, \neg\psi_5, \psi_6$
				p_5	p_2		C	$\psi_1, \psi_2, \neg\psi_4, \neg\psi_5, \neg\psi_6$ (OK no matter ψ_3)

local

local

local

side

local

side

local+side

side

local+side

Table 2 Second table for the proof of Prop. 9.

w	$x_2^{\bigcirc}$	0	0	1	1	
	$x_2^{\pentagon}$	0	1	0	1	
$\theta(w)$	$x_{p1}^{\square}$	0	0	0	0	Depend on nothing
	$x_{p2}^{\square}$	1	1	1	1	Depend on nothing
	$x^{\diamond}$	0	1	0	1	Depend on $x_{p1}^{\square}$, $x_{p2}^{\square}$ and $x_2^{\pentagon}$ $(x^{\diamond} = x_2^{\pentagon})$
	$x_1^{\bigcirc}$	0	1	0	1	Depend on $x_{p1}^{\square}$, $x_{p2}^{\square}$ and $x^{\diamond}$ $(x_1^{\bigcirc} = x^{\diamond})$
	$x_1^{\pentagon}$	0	0	1	1	Depend on $x_{p1}^{\square}$, $x_{p2}^{\square}$, $x^{\diamond}$ and $x_2^{\bigcirc}$ $(x_1^{\pentagon} = x_2^{\bigcirc})$
result	Goal 1	ok	no	no	ok	
	Goal 2	no	ok	ok	no	

(a) Building the map θ to prove $\mathcal{G}, q \models^{\mathcal{M}(L,S,T)}$ in Proposition 10.

		if $x_{p2}^{\square} = 0$		if $x_{p1}^{\square} = 1$		if $x_{p1}^{\square} = 0$ and $x_{p2}^{\square} = 1$
		if $(x_1^{\pentagon} = 0)$	if $(x_1^{\pentagon} = 1)$	if $(x_1^{\bigcirc} = 0)$	if $(x_1^{\bigcirc} = 1)$	Case with $x_1^{\pentagon} = 0$ and $x_1^{\bigcirc} = 0$; other cases are similar
w	$x_2^{\bigcirc}$	-	-	0	1	0
	$x_2^{\pentagon}$	1	0	-	-	1
$\theta(w)$	$x_{p1}^{\square}$	-	-	1	1	0
	$x_{p2}^{\square}$	0	0	-	-	1
	$x^{\diamond}$	-	-	-	-	-
	$x_1^{\pentagon}$	0	1	-	-	0
	$x_1^{\bigcirc}$	-	-	0	1	0
result	Goal 1	$q.a.\emptyset$ no	$q.a.\emptyset$ no	$q.b.c.\emptyset$ no	$q.b.c.\emptyset$ no	$q.a.\emptyset$ no
	Goal 2	$q.a.\emptyset$ no	$q.a.\emptyset$ no	$q.b.c.\emptyset$ no	$q.b.c.\emptyset$ no	$q.b.c.\emptyset$ no

(b) Proving $\mathcal{G}, q \not\models^{\mathcal{M}(L,T)}$ in Prop. 10.

Table 3 Tables for Proposition 10.