Discrete approximation of the minimal time function for systems with regular optimal trajectories

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Abstract. In this paper we prove an estimate of the rate of convergence of the approximation scheme for the nonlinear minimum time problem presented in [2]. The estimate holds provided the system have time-optimal controls with bounded variation. This estimate is of order ν with respect to the discretization step in time, if the minimal time function is Hölder continuous of exponent ν. The proof combines the convergence result obtained in [2] by PDE methods, with direct control-theoretic arguments.

1. Introduction.

In this paper we continue the study of an approximation scheme for the classical minimum time problem for nonlinear systems which we began in [2].

We consider the continuous-time controlled dynamical system in $\mathbb{R}^N$

$$\begin{cases} y' = b(y, \alpha) \\ y(0) = x \end{cases} \tag{1.1}$$

and the corresponding discrete-time system, with time step $h > 0$,

$$\begin{cases} x_{j+1} = x_j + h\, b(x_j, a_j) \\ x_0 = x \end{cases} \tag{1.2}$$

where the controls are taken in a given set $A \subseteq \mathbb{R}^M$. For a given compact target set $\mathcal{T}$ we are interested in the minimum times $T(x)$ and $hN_h(x)$ taken respectively by systems (1.1) and (1.2) to reach $\mathcal{T}$, where $N_h(x)$ indicates the minimum number of discrete steps. Note that T is finite only on the set $\mathcal{R}$ of points controllable to the target $\mathcal{T}$ in finite time, that it tends to $+\infty$ near $\partial\mathcal{R}$ and that $\mathcal{R}$ is not known a priori. We recall that the Dynamic Programming method provides time-optimal controls in feedback form for the discrete-time problem, once the discrete Bellman equation for N_h is solved.

In [2] we considered the new unknown functions

$$(1.3)\qquad v(x) := \begin{cases} 1 - e^{-T(x)}, & \text{if } T(x)<+\infty, \\ 1, & \text{if } T(x)=+\infty, \end{cases} \qquad v_h(x) := \begin{cases} 1 - e^{-hN_h(x)}, & \text{if } N_h(x)<+\infty, \\ 1, & \text{if } N_h(x)=+\infty, \end{cases}$$

and, under general controllability assumptions on both (1.1) and (1.2) around $\mathcal{T}$, we proved that v_h converge to v as h tends to 0, uniformly on compact subset of $\mathbb{R}^N$, which implies the convergence of hN_h to T on compact subsets of $\mathcal{R}$. This result was obtained by applying to the Bellman equations for the discrete-time and the continuous-time control problems a technique for passing to the limit in first order nonlinear equations first used by Barles-Perthame [5]. A crucial role in the proof is played by a comparison theorem between semicontinuous viscosity sub and supersolutions proved in [1].

In this paper we give an estimate of the rate of convergence of hN_h to T using direct, control-theoretic methods. Roughly speaking, the main result says that if T is Hölder continuous of exponent ν , for a compact subsets $\mathcal{K}$ of $\mathcal{R}$, there is a constant C such that

$$(1.4)\qquad \left|hN_h(x) - T(x)\right| \le C\, h^{\nu} \qquad \text{for any } x\in \mathcal{K},$$

provided the system satisfies in $\mathcal{K}$ the property of the "bound on the variation of the optimal controls":

(BV) there exists a constant C_0 such that, for all $x\in \mathcal{K}$, there is a time-optimal control with total variation less then C_0, bringing system (1.1) to the target $\mathcal{T}$.

This property is well known for linear systems (see e.g. Hajek [16]), because the optimal control is bang-bang with a bound on the number of switchings. Many papers have been devoted recently to prove similar regularity properties for smooth systems, nonlinear in the state x and affine in the scalar control α , in the case $\mathcal{T}=\{0\}$. Sussman [31] has described a class of such systems which behave as linear systems. In general, however, the optimal control is known not to be bang-bang, and many authors have addressed the problem of proving that optimal trajectories are finite concatenations of bang and singular arcs, with a bound on the number of switchings. From these results one expects that (BV) hold, at least generically, if the state-space dimension N is small: see Sussman [32,33] for $N=2$, Bressan [6] and Schättler [25,26] for $N=3$, Krener-Schättler [19] and Schättler [27] for some results in dimension 4. We recall also the negative results of Kupka [20] and Kawski [18] in higher dimensions. As regards the Hölder continuity of the minimal time function T , we refer to Liverovskii [21], Stefani [30] and the references therein for $\mathcal{T}=\{0\}$, and to Bardi-Soravia [4] and Soravia [28] for more general targets.

In a forthcoming paper [3] we treat the general case where no regularity of the optimal controls is known, by using a completely different method based on the theory of viscosity solutions and introduced by Souganidis [29] and Capuzzo Dolcetta-Ishii [10] for other control problems. This method leads to estimates worse than (1.4), namely of order $\nu/2$ with respect to h . In [10] an estimate of order ν is also proved, under additional assumptions different from ours, and by a different method.

The reader will find in [1,2] a basic bibliography on the minimum time problem and on the theory of viscosity solutions for Hamilton-Jacobi equations. Our approach to numerical solutions of control problems follows Capuzzo Dolcetta [7], Capuzzo Dolcetta-Ishii [9], Souganidis [29], Falcone [13,14], see also the survey paper by Capuzzo Dolcetta-Falcone [8] which traces its origin back to Bellman's time.

The numerical solution of the minimum time problem has been studied by many authors by completely different methods and mostly in the linear case. See e.g. Falb-De Jong [12], Canon-Cullum-Polak [6], Neustadt [26], Eaton [10], Fujisawa-Yasuda [16], Hajek-Krabs [17], Rabinovich [27] and the references therein.

2. Definitions and preliminary results.

We assume that the *set of admissible controls* A is a subset of $\mathbb{R}^M$ and we define the set $\mathcal{A}$ of *admissible control functions* appearing in (1.1) to be

$$\mathcal{A} := \{\alpha(\cdot) : [0,+\infty[\to A \text{ , measurable}\}.$$

We will use also the following subset of control functions

$$\mathcal{P}_h := \{\alpha(\cdot)\in \mathcal{A} \text{ constant on } [ih,(i+1)h[,\ \forall\ i\in \mathbb{N}\},$$

for $h>0$. The general structural assumptions on b and $\mathcal{T}$ we need are the following

$$\text{(A1)}\qquad \begin{cases} b : \mathbb{R}^N \times A \to \mathbb{R}^N \text{ is continuous },\ |b(x,a) - b(y,a)| \le L|x-y| \\ \text{and } |b(y,a)| \le K(1+|y|),\ \forall\ x,y\in \mathbb{R}^N \text{ and } \forall\ a\in A\ ; \\ \mathcal{T} \text{ is compact ;} \end{cases}$$

(A2) there exists γ, $0<\gamma\le 1$, such that $|b(x,a) - b(x,a')| \le L_r|a - a'|^{\gamma}\ \forall a,a'\in A$ and $|x|\le r$.

Let us denote $y(x,t,\alpha) = y(t,\alpha)$ the solution of (1.1) and define $\tilde{y}_h(x,t,\alpha) = \tilde{y}_h(t,\alpha) := x_{[t/h]}$ for any $\alpha(\cdot)\in \mathcal{P}_h$ where $\{x_j\}$ is the solution of (1.2) with $a_j = \alpha(jh)$, and $[\cdot]$ denotes the integer part. Using Gronwall Lemma it is easy to check that if

$$|y(t,\alpha)| \le Y \quad \forall\ t\in[0,s],\ \text{where}\ \ \alpha(\cdot)\in \mathcal{P}_h\ ,$$

then

$$|y(t,\alpha) - \tilde{y}_h(t,\alpha)| \le K(1+Y)\, e^{Lt}\, h\ ,\ \forall\, t\in[0,s]\,. \tag{2.1}$$

We define the set

$$\mathcal{R} := \{x\in \mathbb{R}^N : \text{there exists } \alpha(\cdot)\in \mathcal{A} \text{ and } t\ge 0 \text{ such that } y(x,t,\alpha)\in \mathcal{T}\}.$$

The time necessary to drive system (1.1) to the target $\mathcal{T}$ by applying the control function $\alpha(t)$ is

$$t_x(\alpha) := \inf\{t : y(x,t,\alpha)\in \mathcal{T}\} \le +\infty,$$

where $t_x = +\infty$ if $y(x,\cdot,\alpha)$ never reaches the target. The minimum time function is given by

$$T(x) := \inf_{\alpha\in\mathcal{A}} t_x(\alpha).$$

Notice that $T : \mathcal{R} \to [0,+\infty[$. Obviously $\mathcal{T}\subseteq\mathcal{R}$ and $T(x)=0$ for any $x\in\mathcal{T}$.

A discrete version of the minimum time problem can be obtained replacing (1.1) with the Euler scheme (1.2) with step h, $h > 0$. The state x_j of the discretized problem is then observed only at times $t_j = jh$ and $a_j \in A$ for all $j\in\mathbb{N}$ (just to simplify notations x_j and a_j will sometimes denote also the whole sequences $\{x_j\}$ and $\{a_j\}$). We can define the analogous of $\mathcal{R}$, $t_x(\alpha)$ and $T(x)$ for this new problem,

$$\mathcal{R}_h := \{x\in\mathbb{R}^N : \text{there exists } \{a_j\} \text{ and } j\in\mathbb{N} \text{ such that } x_j\in\mathcal{T}\}$$

$$n_h(a_j, x) := \min\{j\in\mathbb{N} : x_j\in\mathcal{T}\} \le +\infty ,$$

where $n_h = +\infty$ if x_j never reaches the target,

$$N_h(x) := \inf_{\{a_j\}} n_h(a_j, x) .$$

Besides the previous assumptions, we need the small time local controllability of both the continuous-time and discrete-time systems, in a slightly stronger form than in [2]. Set

$$d(x) := \operatorname{dist}(x, \partial\mathcal{T}), \qquad X_\delta := \{x : \operatorname{dist}(x, \partial X) < \delta\}.$$

We will assume for some $\delta > 0$, $0 < \beta, \zeta, \eta \le 1$,

(A3) $$T(x) \le C_1\, d^{\beta}(x) \quad \text{for all } x\in\mathcal{T}_\delta ,$$

(A4) $$h\, N_h(x) \le C_2\, d^{\zeta}(x) + C_3\, h^{\eta} \quad \text{for all } x\in\mathcal{T}_\delta .$$

In [2] we proved that a sufficient condition for both (A3) and (A4) to hold with $\beta = \zeta = \eta = C_3 = 1$ is the following

$$
\text{(A5)}\quad \begin{cases} \mathcal{T} = \{x : g_i(x) \le 0 \ \forall\ i=1,..,M\} \ \text{ where } \ g_i \in C^2(\mathbb{R}^N) \ \text{ and} \\ \qquad |\nabla g_i(x)| > 0 \text{ for all } x \text{ such that } g_i(x)=0 \ ; \\ \forall\ x \in \partial\mathcal{T}\ \exists\ a \in A \ \text{ such that } \ g_i(x)=0 \ \text{ implies } b(x,a)\cdot\nabla g_i(x) < 0. \end{cases}
$$

Hypothesis (A5) means that $\partial\mathcal{T}$ is piecewise C^2 and at each point of the boundary the controller can choose a vector field pointing inward $\mathcal{T}$. A simple special case of (A5) is

$$
\begin{cases} \mathcal{T} \text{ is the closure of an open set with } C^2 \text{ boundary ;} \\ \inf\limits_{a \in A} b(x, a)\cdot n(x) < 0 \text{ for all } x \in \partial\mathcal{T}, \\ \text{where } n(x) \text{ represents the outward normal to } \mathcal{T} \text{ at } x. \end{cases}
$$

(A3) is essentially equivalent to the Hölder continuity with exponent β of the minimum time function T. Necessary and sufficient conditions for it have been studied e.g. by Petrov [23], Liverovskii [21], Stefani [30] for the case $\mathcal{T}=\{0\}$ (see also the references therein), and by the authors [2], Bardi-Soravia [4] and Soravia [28] for more general targets and $\beta = 1$ or $1/2$.

Due to the discretization of the dynamics, a crucial role will be played by control functions in $\mathcal{P}_h$, so it is interesting to approximate controls in $\mathcal{A}$ with controls belonging to $\mathcal{P}_h$. For a function $\alpha : I \to \mathbb{R}^M$ we denote the total variation by $V(\alpha, I) := \sum\limits_{i=1,..,M} V(\alpha_i, I)$,

where $V(\alpha_i, I)$ is the usual total variation of the i-th component of α in the bounded interval I.

Lemma 2.1

If $\alpha \in \mathcal{A}$ *has bounded variation in* $[0,R]$, *then for any* $h > 0$ *there exists an* $\alpha_h \in \mathcal{P}_h$ *such that*

$$
\int_0^R \left|\alpha(s) - \alpha_h(s)\right| ds \le h\ V(\alpha,[0,R]) \ . \tag{2.2}
$$

Proof.

Set $I_k := [kh, (k+1)h[$ for $k=0,1,...,n$, n such that $(n+1)h \le R < (n+2)h$ and $I_{n+1} := [(n+1)h, R]$. We can define α_h in I_k as any value taken up by α in the same interval, or for instance

$$
\alpha_h(s) := \lim_{t \to kh^+} \alpha(t) \quad \text{for } s \in I_k.
$$

Then

$$
\int_{I_k} \left|\alpha(s) - \alpha_h(s)\right| ds \le h\ V(\alpha, I_k) \ ,
$$

and (2.2) follows easily. #

In the previous paper [2] we have proved the following result (the definition of v and v_h is (1.3)):

Theorem 2.2

Assume (A1), (A3), (A4). *Then* v_h *converge to* v *uniformly on compact subsets of* $\mathbb{R}^N$.

Corollary 2.3

Under the assumptions of Theorem 2.2 for any given compact subset $\mathcal{K}$ *of* $\mathcal{R}$ *there exist* $\bar{h}$ *and* $\tilde{T}$, *such that*

$$\mathcal{K} \subseteq \mathcal{R}_h \ \textit{and}\ hN_h(x) \leq \tilde{T},\ \forall\, h \leq \bar{h}, \forall\, x \in \mathcal{K}.$$

Proof.

Since v_h converges uniformly to v on $\mathcal{K}$ and $v \leq 1 - \varepsilon$ on $\mathcal{K}$ for some positive ε, we have

$$v_h \leq 1 - \frac{\varepsilon}{2}, \quad \text{on } \mathcal{K},$$

for $h \leq \bar{h}$, which gives the conclusion. #

3. Estimates.

Lemma 3.1

Assume (A1), (A3), (A4) *and let* $\mathcal{K}$ *be a compact subset of* $\mathcal{R}$. *Then there exist two positive constants* $\bar{h}$ *and* C *such that*

$$T(x) - hN_h(x) \leq Ch^\beta \quad \textit{for any}\ x \in \mathcal{K} \quad \forall\, h \leq \bar{h}.$$

Proof.

For x fixed in $\mathcal{K}$ we choose $\{a_j\}$ such that $n_h(a_j, x) = N_h(x)$ and define $\alpha(s) = a_{[s/h]}$. Let us choose $\tilde{T}$ as in Corollary 2.3. Then (A1) and Gronwall's Lemma imply (see Lemma 1 in [1]) the existence of a constant Y, depending only on $\mathcal{K}$, such that

$$\left|y(x,t,\alpha)\right| \leq Y \quad \forall\, x \in \mathcal{K},\ t \leq \tilde{T}.$$

Then (2.1) gives

$$\left|y\big(hN_h(x), \alpha\big) - \tilde{y}_h\big(hN_h(x), \alpha\big)\right| \leq K(1+Y)\, e^{L\tilde{T}} h =: C_4 h,$$

and we have, for $h \leq \frac{\delta}{C_4}$,

$$y\big(hN_h(x), \alpha\big) \in \mathcal{T}_\delta.$$

Thus by (A3)

$$T(x) \leq hN_h(x) + C_1\, C_4^\beta\, h^\beta,$$

which ends the proof. #

Theorem 3.2

Assume (A1), (A2), (A4) *and let* $x \in \mathcal{R}$. *Suppose there is a sequence* α_n *in* $\mathcal{A}$ *with*

$\lim_{n \to +\infty} t_x(\alpha_n) = T(x)$ *such that*

$$V(\alpha_n,[0,T(x)+1]) \le C_o \quad \textit{for all } n.$$

Then there exist two positive constants $\bar{h}$ *and* C, *depending only on* $|x|$, *an upper bound on* T(x), *and the constants appearing in the hypotheses, such that*

$$hN_h(x) - T(x) \le Ch^{\nu} \quad \textit{for all } h \le \bar{h} ,$$

where $\nu = \min\{\gamma\zeta, \eta\}$.

Proof.

Let $\tilde{T}$ be an upper bound on $T(x)$. Since (A1) implies $|y(x,t,\alpha)-x| \le (1+|x|)(e^{Kt}-1)$, we can choose a constant Y, depending only on $|x|$, K and $\tilde{T}$ such that $|y(t,\alpha)| < Y$, for all $t \le T(x)+1$. Fix $h > 0$ and let $\alpha_n \in \mathcal{A}$ be such that

$$t_x(\alpha_n) - h \le T(x). \tag{3.1}$$

By Lemma 2.1 there exist $\bar{\alpha}_h = (\alpha_n)_h \in \mathcal{P}_h$ such that

$$\int_0^{\tilde{T}+1} \left|\alpha_n(s) - \bar{\alpha}_h(s)\right| ds \le hC_o , \quad \forall\, n .$$

We have the following estimate

$$\left|y(t,\alpha_n) - y(t,\bar{\alpha}_h)\right| \le \int_0^t \left|b(y(s,\alpha_n(s)),\alpha_n(s)) - b(y(s,\bar{\alpha}_h(s)),\bar{\alpha}_h(s))\right| ds \le$$

$$\le L_Y\, C_0^{\gamma} h^{\gamma}\, \tilde{T}^{1-1/\gamma} + \int_0^t L \left|y(s,\alpha_n(s)) - y(s,\bar{\alpha}_h(s))\right| ds \quad ,$$

which implies

$$\left|y(t,\alpha_n) - y(t,\bar{\alpha}_h)\right| \le L_Y\, C_o^{\gamma}\, h^{\gamma} \tilde{T}^{1-1/\gamma} e^{Lt}, \quad \forall\, t \le \tilde{T} + 1.$$

Combining this with (2.1) we get

$$\left|y(t,\alpha_n) - \tilde{y}_h(t,\bar{\alpha}_h)\right| \le C_5 h^{\gamma} , \forall\, t \le \tilde{T} + 1, \tag{3.2}$$

for a positive constant C_5 depending only on $|x|$, $\tilde{T}$, C_o, and the constants in (A1)(A2). Next (3.1), $h \le 1$ and (3.2) imply $\tilde{y}_h\big(t_x(\alpha_n),\bar{\alpha}_h\big) \in \mathcal{T}_{C_5 h^{\gamma}}$. Thus we take $h \le \left(\frac{\delta}{C_5}\right)^{1/\gamma}$ and obtain from (A4)

$$hN_h(x) \le T(x) + h + C_2\, C_\zeta^{\zeta}\, h^{\gamma\zeta} + C_3 h^{\eta},$$

which concludes the proof. #

The following result gives conditions under which the convergence of the discrete minimum time function to T(x) is of order 1 with respect to h .

Corollary 3.3
Assume (A1), (A2) *with* $\gamma = 1$ *and* (A5). *Suppose* $\mathcal{K}$ *is a compact subset of* $\mathcal{R}$ *where* (BV) *holds. Then there exist two positive constants* $\bar{h}$ *and* C *such that*

$$| T(x) - hN_h(x) | \le Ch \quad \textit{for any } x \in \mathcal{K} \qquad \forall\, h \le \bar{h}.$$

Proof.
By Lemma 5.1 in [2], (A5) implies (A3) with $\beta = 1$ and the continuity of T on the points of $\partial\mathcal{T}$. Then v is continuous in $\mathbb{R}^N$ by the argument at the end of the proof of Thm. 3.3 of [2]. Since $\mathcal{R}=\{ x: v(x) <1\}$, $\mathcal{R}$ is open. Then ,

$$T(x) \le \tilde{T} < +\infty, \quad \forall\, x \in \mathcal{K}.$$

By Lemma 4.1 in [2], (A5) implies also (A4) with $\zeta = \eta = C_3 = 1$. Then the conclusion follows from (BV), Lemma 3.1 and Theorem 3.2. #

Corollary 3.4
Under the hypothesis of Corollary 3.3, there exist two positive constants $\bar{h}$ *and* C *such that*

$$\left| v(x) - v_h(x) \right| \le Ch \quad \forall\, x \in \mathcal{K} \textit{ and } h \le \bar{h}.$$

Proof.
We prove the estimate for $v - v_h$, the other being completely analogous. Without any loss of generality we consider the case $T(x) \ge hN_h(x)$. We have

$$v(x) - v_h(x) = - e^{-\vartheta} \left(hN_h(x) - T(x) \right) \le e^{-\tilde{T}}\, Ch$$

where in the equality we used the mean value theorem, for some $\vartheta \in [hN_h(x), T(x)]$. #

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