Zonotopes Techniques for Reachability Analysis

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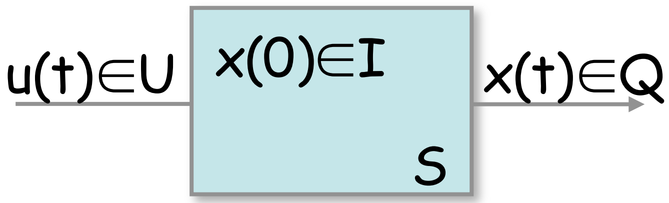
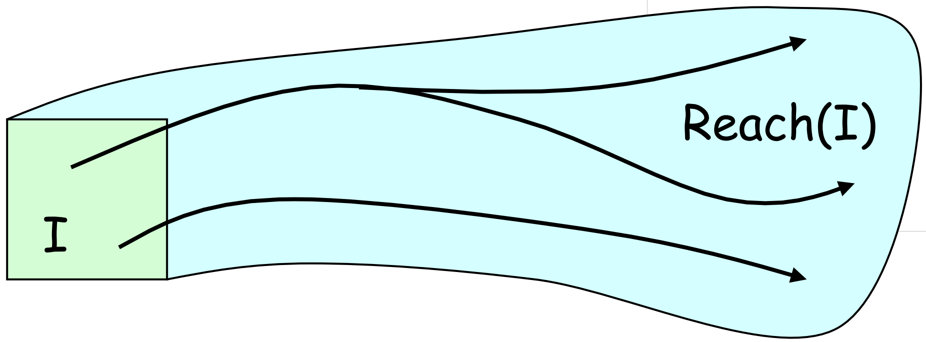
Reachability Analysis



- Computation of the states that are reachable by a system S:
  - from a set of initial states I
  - subject to a set of admissible inputs (*disturbances*)



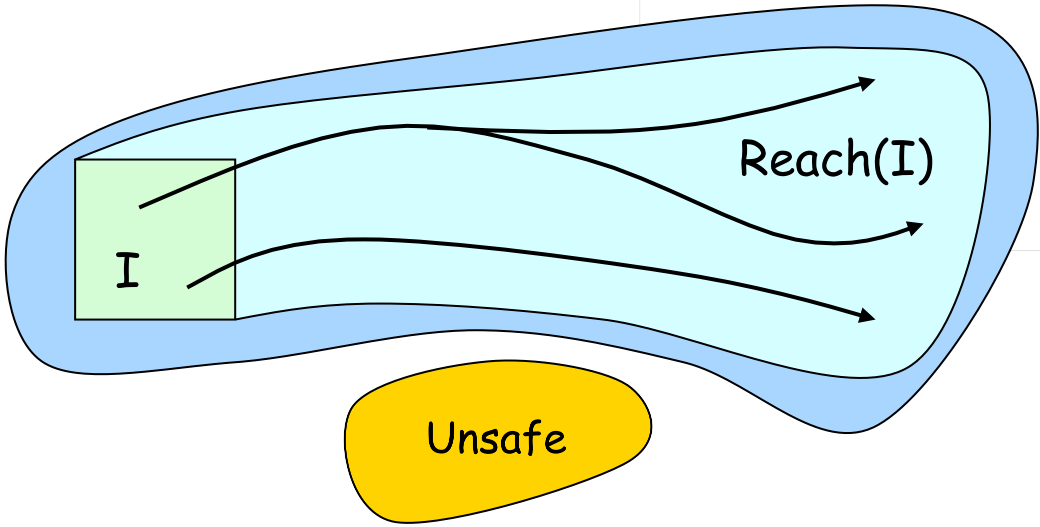
- Can be thought as *exhaustive simulation* of a system

Algorithmic Verification

- Algorithmic proof of the safety of a system:

  No trajectory of the system can reach
  a set of unsafe states.

- Can be solved by computing:
  - the *exact* reachable set (*LHA, some linear systems*)
  - *an over-approximation* of the reachable set



Outline

1. Reachability computations for continuous systems.
   - Flow pipe approximation
   - Computations for linear systems
2. Scalable computations using zonotopes.
3. Extensions to nonlinear/hybrid systems.

Continuous Dynamics

- Nondeterministic continuous system S is represented by a flow Φ :

  $\Phi(X,t)$ denotes the set of states reachable from X at time t.

- Note that we must have the semi-group property

$$\Phi(X,t+t') = \Phi(\Phi(X,t),t')$$

- Example: $\dot{x} = Ax \Leftrightarrow \Phi(X,t) = e^{At}X$

- The reachable set of S on a time interval [t,t'] is formally defined by the flow pipe :

$$Reach_{[t,t']}(I) = \bigcup_{s \in [t,t']} \Phi(I,s)$$

Flow Pipe Computation

- Choose a time step r (*arbitrarily small*) and remark that

$$Reach_{[0,t]}(I) = \bigcup_{k=0}^{k=N-1} \Phi(Reach_{[0,r]}, kr), \text{ with } t = Nr.$$

- Algorithm for reachability computation :

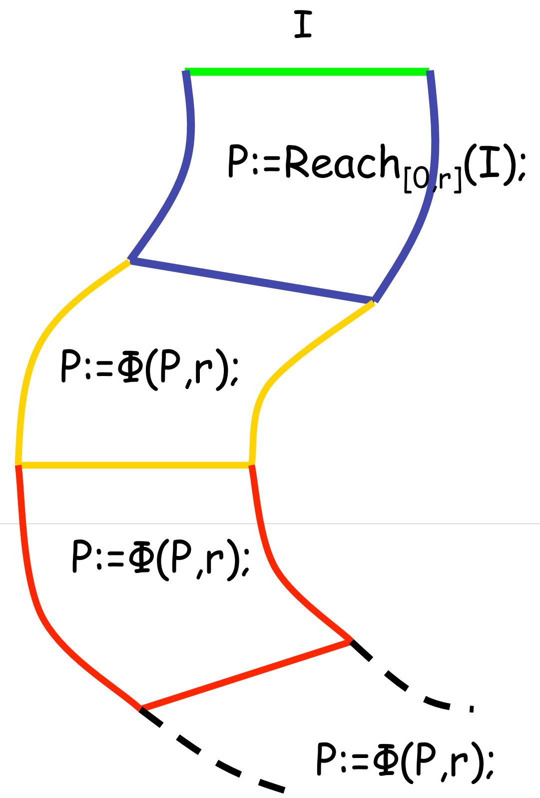
$P := Reach_{[0,r]}(I);\ R := P;$

for k from 0 to N - 1

  $P := \Phi(P, r);$

  $R := R \cup P;$

end



Implementation

- Choice of representation for the set P ($P \in C$):

  C can be the set of *polytopes, ellipsoids, level sets…*

- Let us assume that the initial set $I \in C$, then define two functions

$$\overline{R}_r : S \to S; \quad \forall P \in S, \mathrm{Reach}_{[0,r]}(P) \subseteq \overline{R}_r(P)$$
$$\overline{\Phi}_r : S \to S; \quad \forall P \in S, \Phi(P,r) \subseteq \overline{\Phi}_r(P)$$

- Implement the previous algorithm with the functions :

  - Over-approximation of the reachable set $\mathrm{Reach}_{[0,r]}(I)$

  - Under some assumptions, we can prove convergence as $r \to 0$.

Computations for Linear Systems

- Linear systems of the form:

$$\dot{x}(t) = Ax(t) + Bu(t), u(t) \in U$$

where U is assumed to be bounded convex set of the class C.

- Then, the flow of the system is

$$\Phi(X,r) = \left\{ e^{rA}x + \int_0^r e^{(r-s)A}Bu(s)ds \;\middle|\; x \in X \text{ and } \forall s, u(s) \in U \right\}$$

which can be over-approximated by

$$\Phi(X,r) \subseteq \left\{ e^{rA}x + \int_0^r e^{(r-s)A}Buds \;\middle|\; x \in X \text{ and } u \in U \right\} \oplus \beta(\varepsilon_r)$$

where $\beta(\varepsilon_r)$ is a ball of radius $\varepsilon_r = O(r^2)$.

Computations for Linear Systems

- If the class of sets C is closed under:
  - Linear transformations
  - Minkowski sum

- Then, the approximate flow can be chosen as

$$\bar{\Phi}_r(X) = e^{rA}X \oplus V_r \text{ where}$$

$$V_r = \left\{ \int_0^r e^{(r-s)A}Buds \;\middle|\; u \in U \right\} \oplus \beta(\varepsilon_r)$$

- Something similar can be done for $\bar{R}_r(X)$

$$\bar{R}_r(X) \approx \text{Convex Hull}(X, \bar{\Phi}_r(X)) \oplus \beta(\delta_r)$$

- Convergence as $r \to 0$.

- Example of such a class: polytopes (*d/dt, Checkmate*)

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Polytopes and Large Scale Systems

- Minkowski sum of N polytopes with at most K vertices in R^d :

  computational complexity in $O(N^{d-1} K^{2d-1})$

- Reachability computations of a d-dimensional system involve the Minkowski sum of N polytopes in R^d.

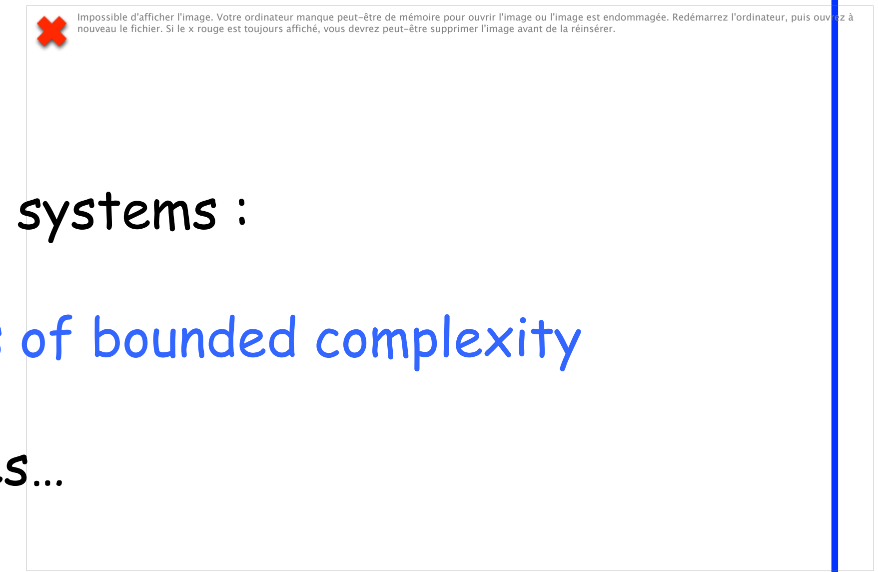
- (Expected) complexity of the reachability algorithm :

  exponential in the dimension of the system

- Polytope based reachability computations are limited to :
  - relatively small systems ($d \leq 10$)
  - relatively small time horizon N.

Reachability of Large Scale Systems

- Large scale systems (dimension ≥ 100) arise in :
  - Biology,
  - Circuits,
  - Networked systems…



- Idea for reachability of large scale systems :

  use alternative classes of sets of bounded complexity

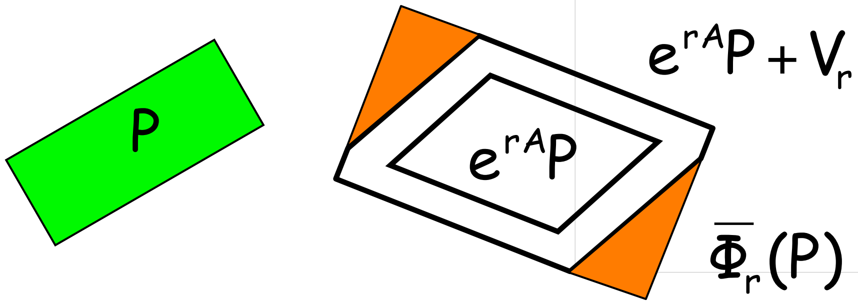
- Ellipsoids, Oriented hyperrectangles…

Reachability using Hyperrectangles

- Oriented hyperrectangles: polytopes of bounded complexity (d^2)

- But not closed under:
  - Linear transformations
  - Minkowski sum

- ORH based reachability computations:



- Additional inaccuracies which propagates (*wrapping effect*)

- No more convergence as $r \to 0$.

Summary

Polytope based reachability computations:

1. Accurate approximation of the reachable set
   (closed linear transformations and Minkowski sum)
2. Intractable for large scale systems
   (exponential complexity in dimension)

Impossible d'afficher l'image. Votre ordinateur manque peut-être de mémoire pour ouvrir l'image ou l'image est endommagée. Redémarrez l'ordinateur, puis ouvrez à nouveau le fichier. Si le x rouge est toujours affiché, vous devrez peut-être supprimer l'image avant de la réinsérer.

The missing link: Zonotopes

Oriented Hyperrectangles based reachability computations:

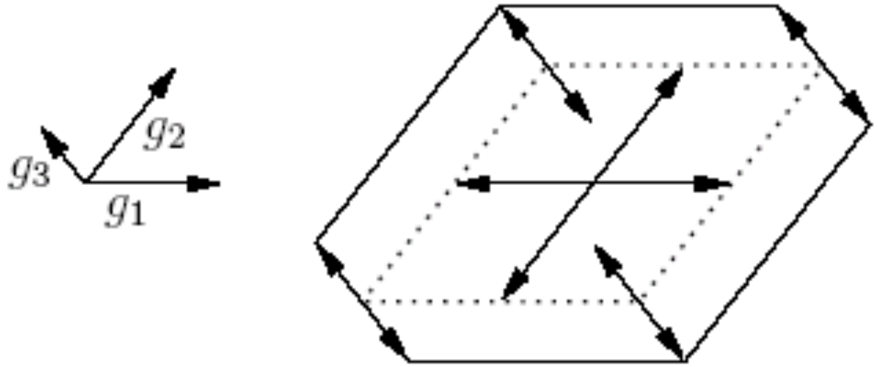
1. Can be used for large scale systems
   (polynomial complexity in dimension)
2. Inaccurate approximation of the reachable set
   (wrapping effect)

What is a Zonotope?

- Zonotope: Minkowski sum of a finite number of segments.

$$Z = \left\{ x \in R^n,\ x = c + \sum_{i=1}^{i=p} x_i g_i,\ -1 \leq x_i \leq 1 \right\}.$$

- c is the center of the zonotope, $\{g_1, \dots, g_p\}$ are the generators. The ratio p/n is the order of the zonotope.



Two dimensional zonotope with 3 generators

Some Properties of Zonotopes

- A generic d-dimensional zonotope of order p has

  more than $(2p)^{d-1}/\sqrt{d}$ vertices.

- The set of zonotopes is closed under linear transformation

$$Z = (c, < g_1, \ldots, g_p >), LZ = (Lc, < Lg_1, \ldots, Lg_p >).$$

- The set of zonotopes is closed under the Minkowski sum

$$Z_1 = (c_1, < g_1, \ldots, g_p >), Z_2 = (c_2, < h_1, \ldots, h_q >),$$
$$Z_1 \oplus Z_2 = (c_1 + c_2, < g_1, \ldots, g_p, h_1, \ldots, h_q >).$$

- Suitable for accurate and efficient reachability computations.

Reachability using Zonotopes

- Implementation of the reachability algorithm consists of:
  - Matrix products
  - List concatenations

- Computational complexity of the zonotope based algorithm is:

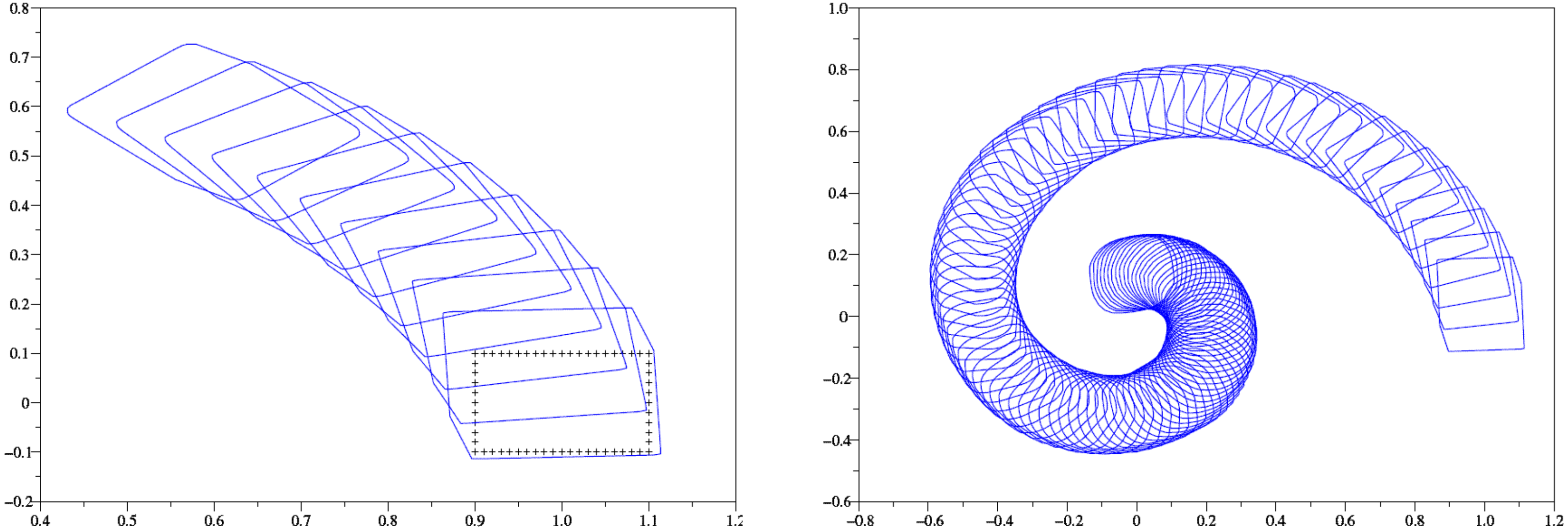
  $O(d^3N^2)$ compared to more than $O(N^{d-1})$ for polytopes.

- Polynomial complexity in the dimension

- Convergence of the approximation as $r \to 0$.

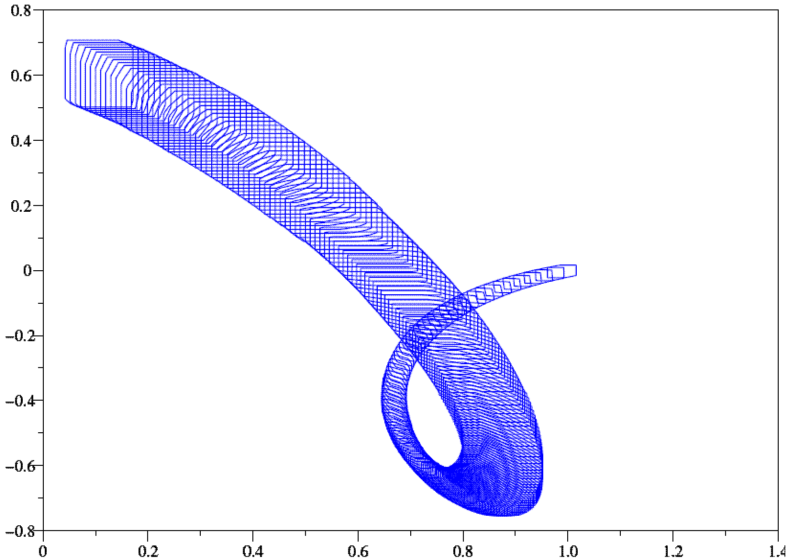
- Suitable for large scale systems (in practice up to dimension 100)

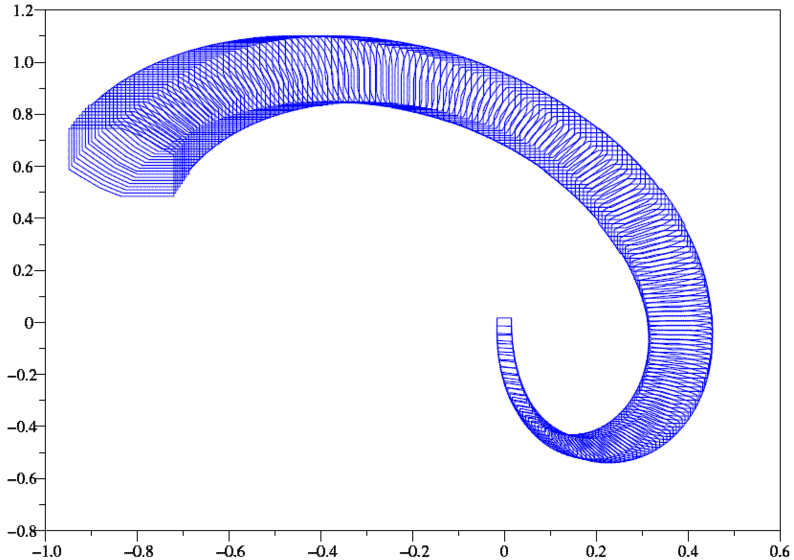
Two Dimensional Example



Reachable set on the interval [0,2].

Five Dimensional Example





Projections of the reachable set on the interval [0,1].

Reachability using Zonotopes

- Complexity $O(d^3N^2)$ can be annoying for large time horizons N.
- Solution to this problem:
  - use a new implementation scheme of the recurrence relation

$$P_{k+1} = e^{rA}P_k \oplus V_r$$

  - complexity becomes $O(d^3N)$
- Subject of Colas Le Guernic's talk on Wednesday afternoon.

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Hybrid Systems

We consider the class of hybrid systems that consists of:

1. A finite set Q of modes.
2. In each mode q, the continuous dynamics is given by a linear system:

$$\dot{x}(t) = A_q x(t) + B_q u(t),\ u(t) \in U_q$$

3. Switching conditions (Guards) are given by linear inequalities:

$$q \to q' \Leftrightarrow x(t) \in S_{q,q'} = \{ m_{q,q'} \le h_{q,q'} \cdot x \le M_{q,q'} \}$$

Reachability of Hybrid Systems

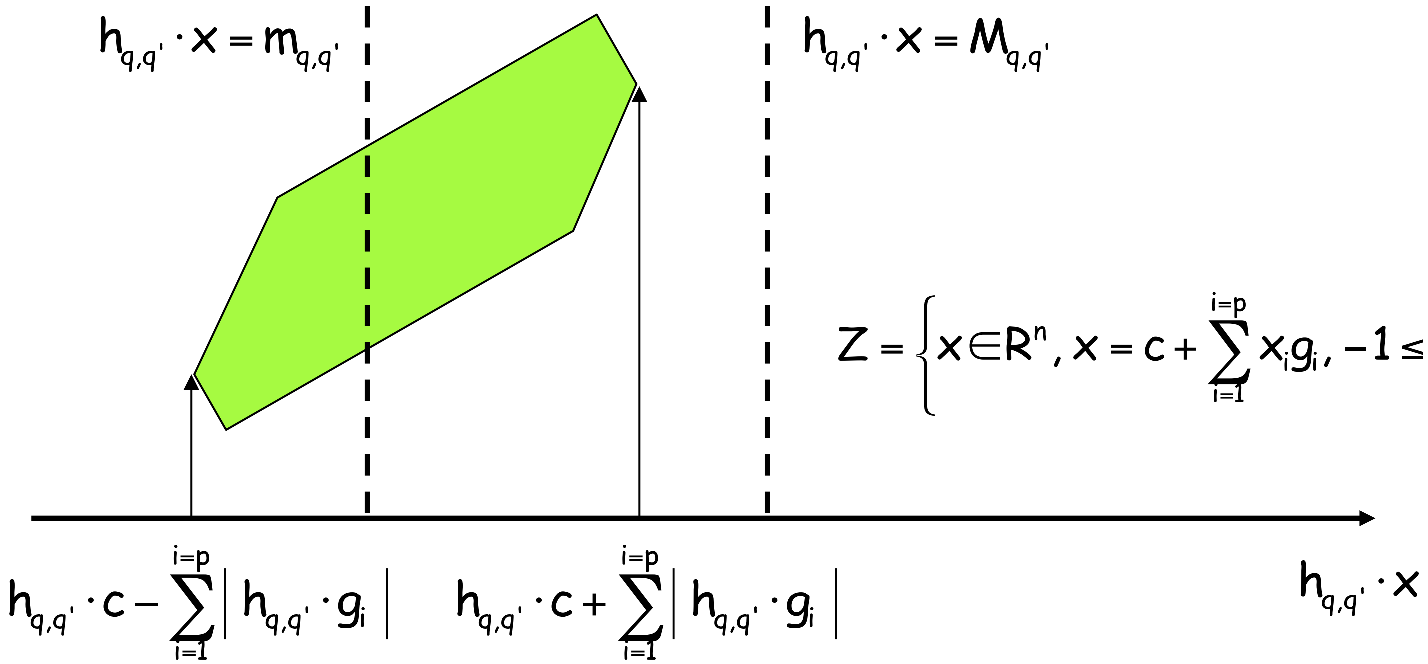
Following the classical scheme for reachability of hybrid systems:

- In each mode, the reachability analysis of the continuous dynamics is handled by our algorithm.

- Processing of discrete transitions requires:
  1. Detection of the intersection of a zonotope with a guard.
  2. Computation of this intersection
     - The intersection of zonotope with a band is not a zonotope.
     - Over-approximation algorithms.

Event Detection

- Detection of the intersection of a zonotope with a guard.

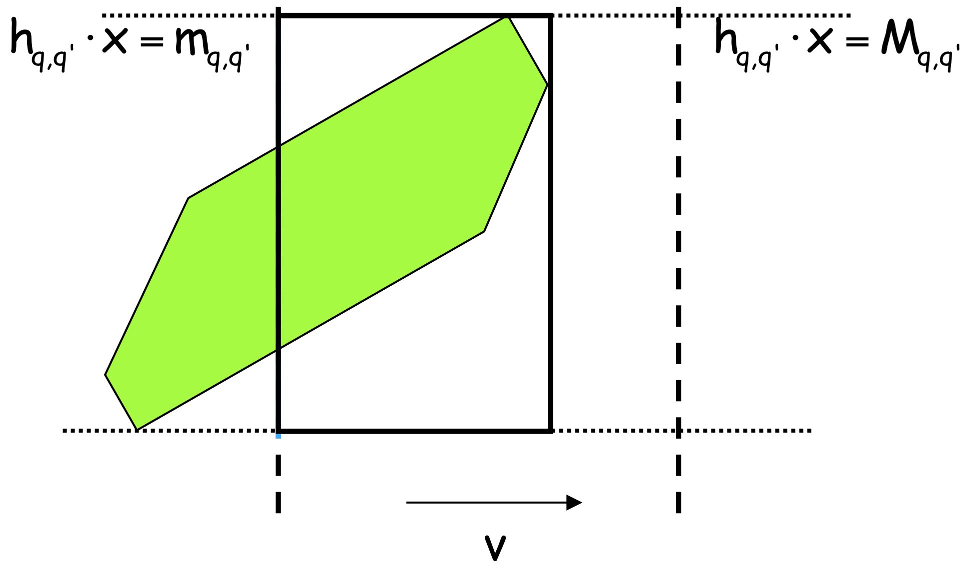


$$Z = \left\{ x \in R^n,\ x = c + \sum_{i=1}^{i=p} x_i g_i,\ -1 \leq x_i \leq 1 \right\}.$$

$$\text{Intersection} \Leftrightarrow \left[h_{q,q'} \cdot c - \sum_{i=1}^{i=p} \left| h_{q,q'} \cdot g_i \right|, h_{q,q'} \cdot c + \sum_{i=1}^{i=p} \left| h_{q,q'} \cdot g_i \right| \right] \cap \left[m_{q,q'}, M_{q,q'} \right] \neq \emptyset$$

Computing the Intersection

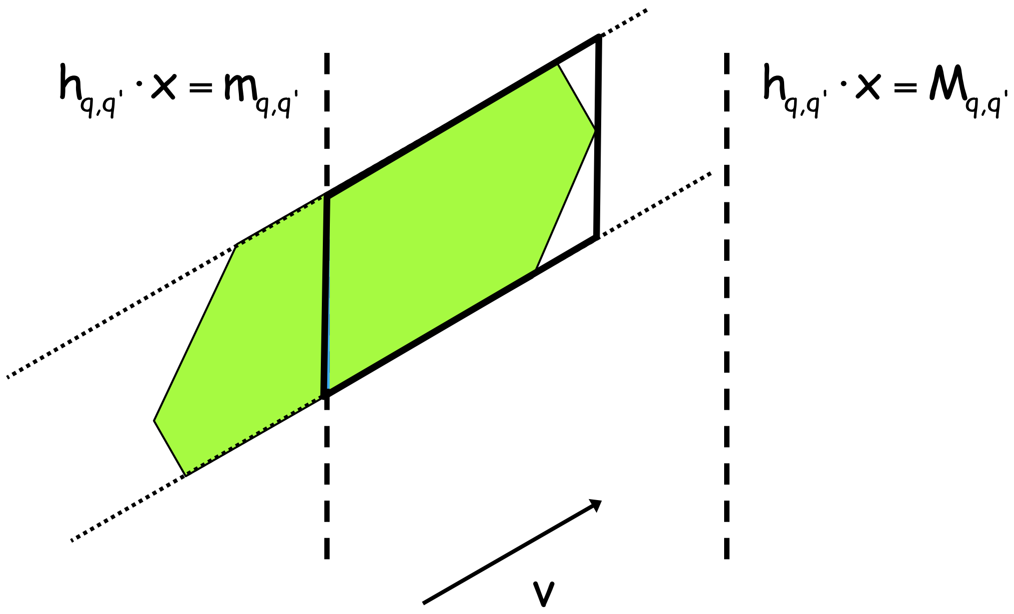
- Over-approximation by projection and bloating:



- The over-approximation I is
  - a zonotope: $I = P_v Z \oplus [\alpha, \beta] v.$
  - included in the guard

Computing the Intersection

- You can project in an other direction:



- Find the direction which results in the *best* over-approximation.

Direction of Projection

- Computation of the best direction is feasible but difficult

- Heuristics:
  - direction as weighted sum of generators (C. Le Guernic):

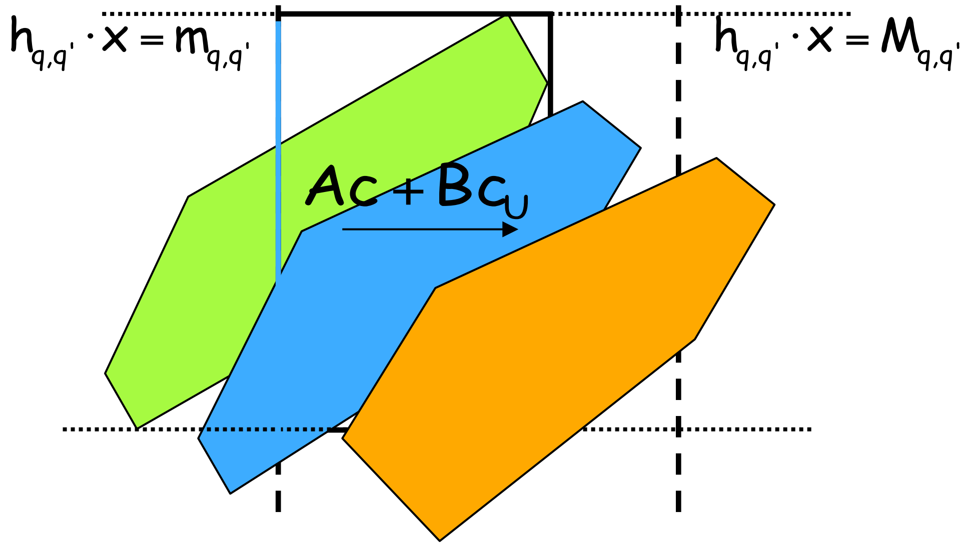
$$v = \sum_{i=1}^{i=p} \frac{\left| g_i \cdot h_{q,q'} \right|}{\left\| g_i \right\|} g_i .$$

  - use the dynamics of the system:

$$v = Ac + Bc_U .$$

Direction of Projection

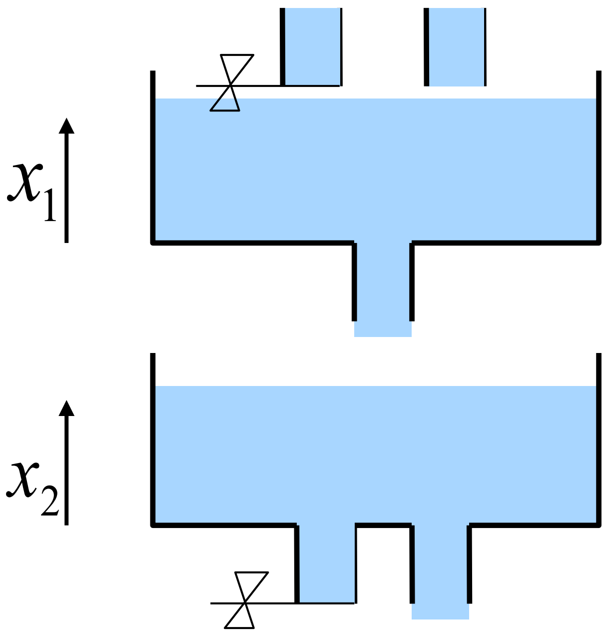
- *Dynamic* heuristic:

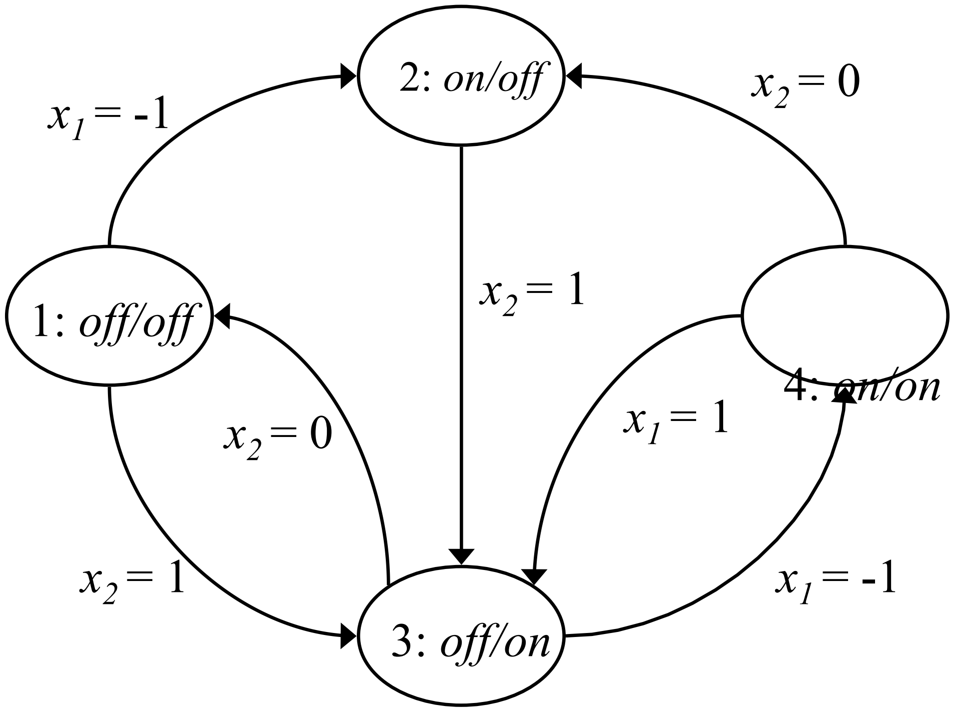


- A large part of the over-approximation at step k is actually reachable at steps k+1, k+2…

Example

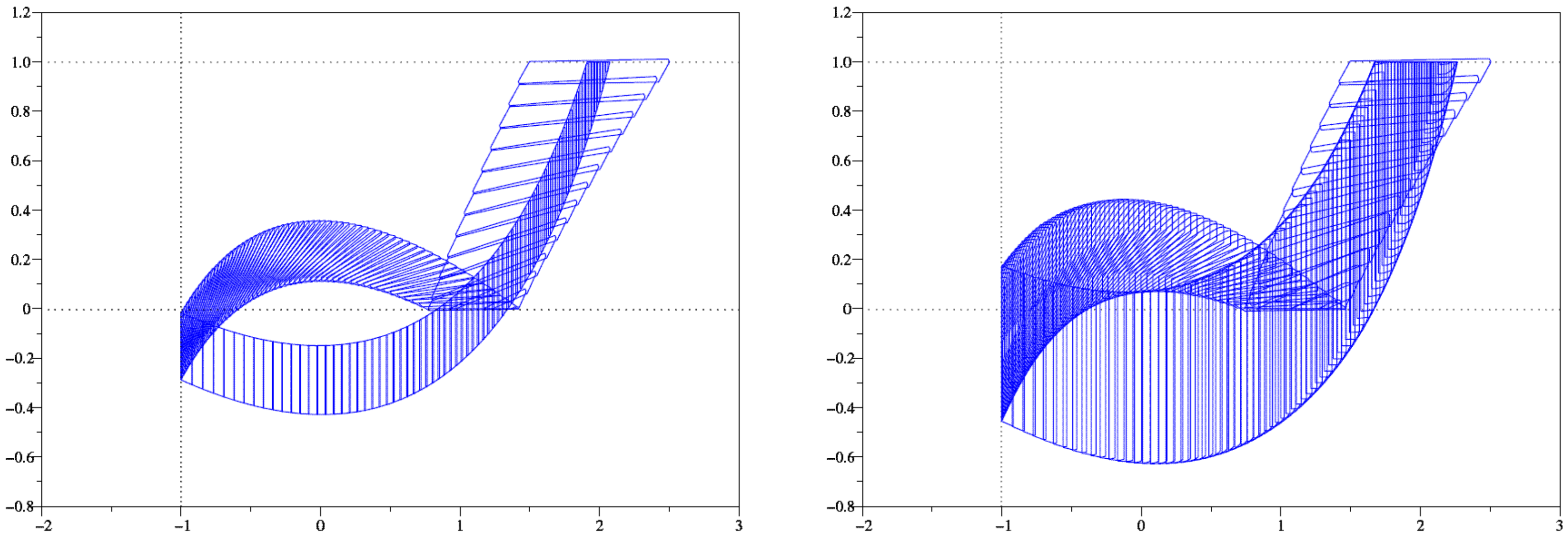
Two tank system:





Want to check robustness of periodic behavior.

Example



Reachable set of the two tank system for $\mu = 0.01$ and $\mu = 0.1$

Hybrid reachability needs to be tested for large scale examples.

Reachability of Nonlinear Systems

Two approaches for reachability analysis of nonlinear systems:

- Hybridization approach [Asarin, Dang, Girard]:
  - state space is partitioned
  - in each region, linear conservative approximation of the nonlinear vector field
  - accurate approximation
- Trajectory piecewise linearization [Han, Krogh]:
  - at each time step, vector field linearized around the center of the zonotope
  - efficient computations

Conclusions

- Class of zonotopes for reachability computations:
  - nice balance between efficiency and accuracy
  - was proved efficient for high-dimensional linear systems
  - ongoing research on nonlinear/hybrid dynamics
- Future work:
  - software development
  - reachability framework based on support functions, unifying zonotopes and ellipsoids approaches
- Thank you to Colas Le Guernic and Oded Maler