Euler's Method applied to the control of switched systems

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joint work with: A. Le Cöent, F. de Vuyst, L. Chamoin, J. Alexandre dit Sandretto, A. Chapoutot

Outline

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Switched systems

A continuous switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t))$$

- state $x(t) \in \mathbb{R}^n$
- control rule $\sigma(\cdot) : \mathbb{R}^+ \longrightarrow U$
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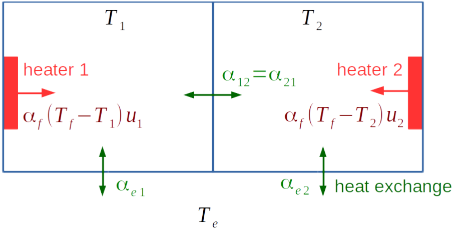
The control σ is a piecewise constant function
with equal steps of length τ, and height value in U

Example: Two-room apartment



$$\begin{pmatrix}\dot{T}_1\\ \dot{T}_2\end{pmatrix} = \begin{pmatrix}-\alpha_{21}-\alpha_{e1}-\alpha_f u_1 & \alpha_{21}\\ \alpha_{12} & -\alpha_{12}-\alpha_{e2}-\alpha_f u_2\end{pmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix}+\begin{pmatrix}\alpha_{e1}T_e+\alpha_f T_f u_1\\ \alpha_{e2}T_e+\alpha_f T_f u_2\end{pmatrix}.$$

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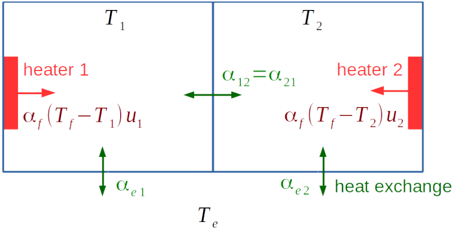

$$\begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \end{pmatrix} = \begin{pmatrix} -\alpha_{21} - \alpha_{e1} - \alpha_f u_1 & \alpha_{21} \\ \alpha_{12} & -\alpha_{12} - \alpha_{e2} - \alpha_f u_2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} \alpha_{e1} T_e + \alpha_f T_f u_1 \\ \alpha_{e2} T_e + \alpha_f T_f u_2 \end{pmatrix}.$$

- Modes: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; stepsize τ

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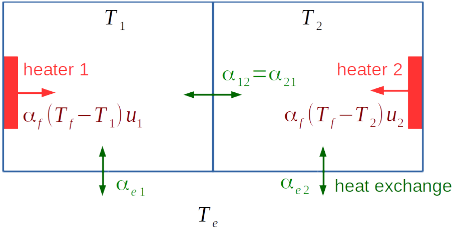

$$\dot{T}_1(t) = f_1(T_1(t), T_2(t), u_1)$$
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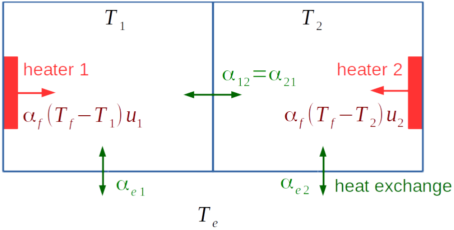

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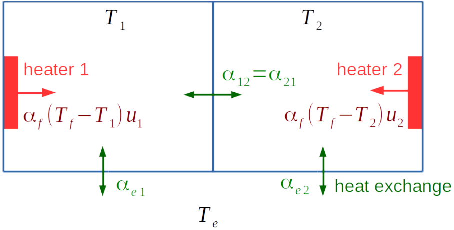

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- state-dependent control: select at each τ a mode/pattern according to current state value x, in order to satisfy a desired property (eg: stability)

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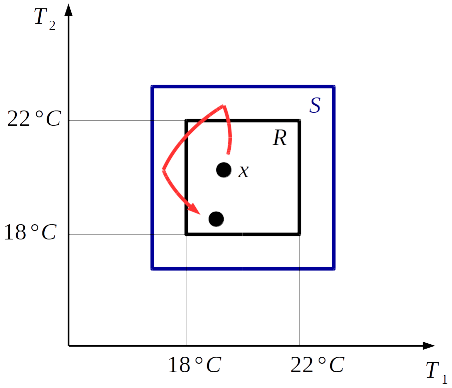
(R, S)-stability:
$x(t)$ returns to R
while never leaving S



Outline

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- they can account for
  - rounding errors
  - inaccuracies in measurements of inputs
  - uncertainty on parameters, disturbance, errors from the model

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Interval-based integration[2]

For $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we consider the ODE

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0$$

solution denoted by $x(t; x_0)$ (or simply $x(t)$)

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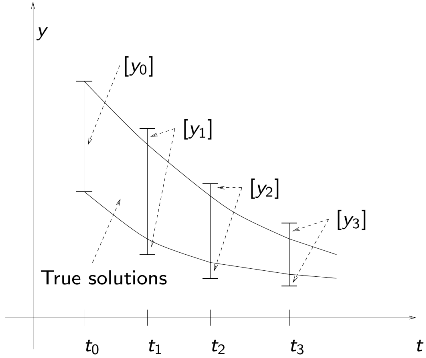

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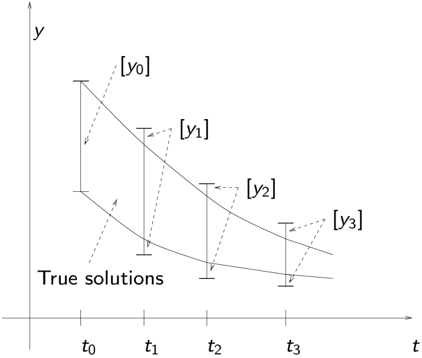

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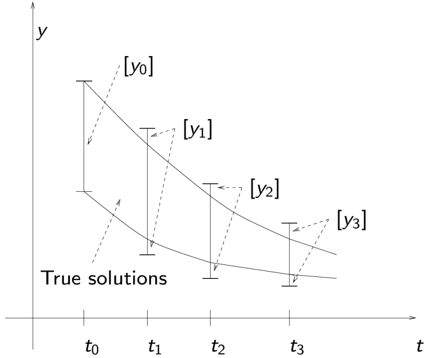

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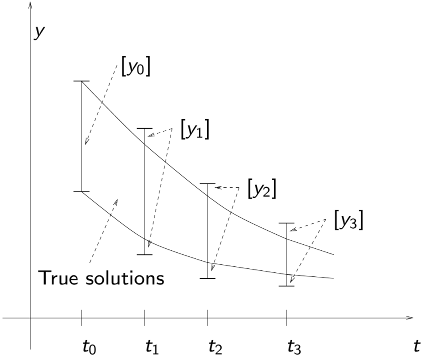

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Given I_j an interval for $t = t_j$, compute a (super)set of solutions I_{j+1} at $t_{j+1} = t_j + \tau$ via a two-step method:

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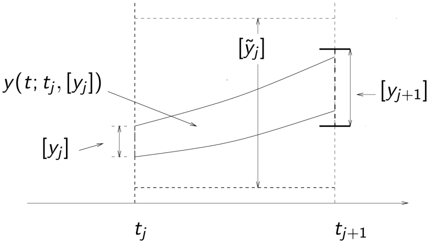
2. Algorithm II: compute a tighter enclosure I_{j+1}:

$$x(t; I_j) \subseteq I_{j+1} \subset F_j \text{ at } t = t_{j+1}$$



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The construction of F relies on fixed-point acceleration heuristics ("widening") using adjustment of stepsize τ.

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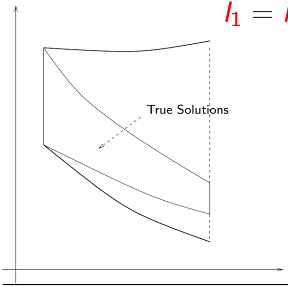

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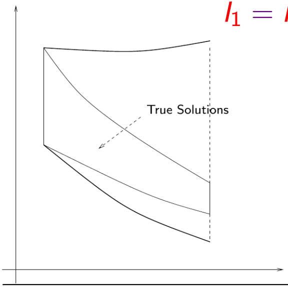


NB: with this algo, $|I_1| > |I_0|$

– even if the true solutions contract!

$\rightarrow$ further refinement needed

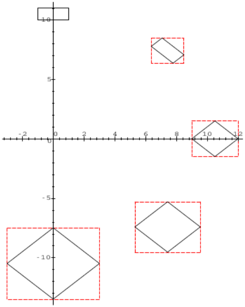
[5] idem

Wrapping effect[6]

A simple rotation:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x; \quad x_0 \in I_0$$

The solution is $x(t) = \begin{pmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{pmatrix} x_0$, where $x_0 \in I_0$



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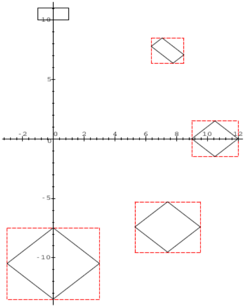

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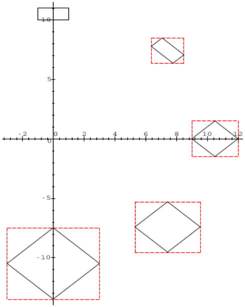

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At $t = 2\pi$, the blow up factor is by a factor $e^{2\pi} \approx 535$, as the stepsize tends to zero.

[6] idem

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Advantages over standard numerical methods:

1. ensure a unique solution exists
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Disadvantages

1. computation is time consuming
2. harder to implement than standard numerical methods
3. error bounds may be too large

[7] idem

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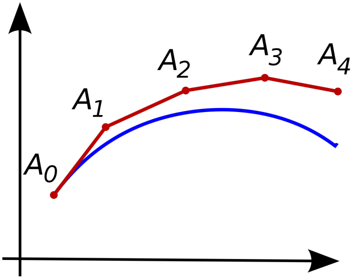

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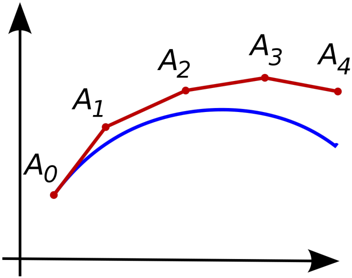

Piecewise linear fn.:
at each step, constant derivative of $\tilde{x}(t)$ $(= f(\tilde{x}(t_i))$ deriv. at starting pt)

Classical error bound (using Lipschitz constant L)

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  If f is Lipschitz cont. ($\|f(y) - f(x)\| \leq L\|y - x\|$), then:

  $$\text{error}(t) \leq \frac{\tau M}{2L}(e^{L(t-t_0)} - 1)$$

  where L is the Lipschitz constant of f (and M an upper bound on f'').

Classical error bound (using Lipschitz constant L)

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Idea: exploit another constant λ that will allow for a sharper estimation of Euler’s error

One-sided Lipschitz (OSL) constant λ

- $\lambda \in \mathbb{R}$ is a constant s.t., for all $x, y \in S$:

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- λ can be computed using constraint optimization algorithms

Hypotheses

$(H0)$ (Lipschitz): for all $j \in U$, there exists a constant $L_j > 0$ such that:

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The constants C_j for all $j \in U$ are defined as follows:

$$C_j = \sup_{x \in S} \; L_j \|f_j(x)\|.$$

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Notations

Let $x_j(t)$ the solution at time t of the system under mode j with (implicit) initial point x^0

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assuming $\text{error}_j(0) \equiv \| x^0 - \tilde{x}^0 \| \leq \delta^0$ for some $\delta^0 \in \mathbb{R}_+$.

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Given a ball $B^0 \equiv B(\tilde{x}^0, \delta^0) \subset S$, safely control B^0 during one step, select $j \in U$:

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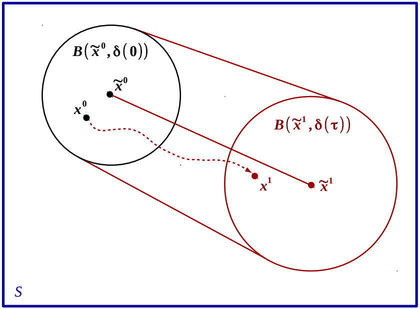

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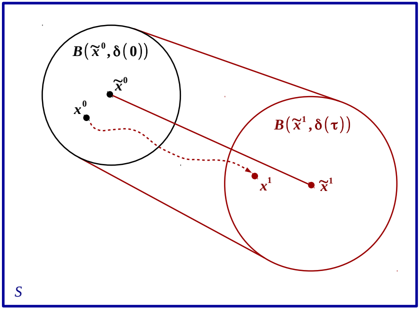

it suffices to find $j \in U$: $\quad B^1 \equiv B(\tilde{x}^1, \delta_j(\tau)) \subseteq S \quad$ with $\tilde{x}^1 \equiv \tilde{x}^0 + \tau \cdot f_j(\tilde{x}^0)$

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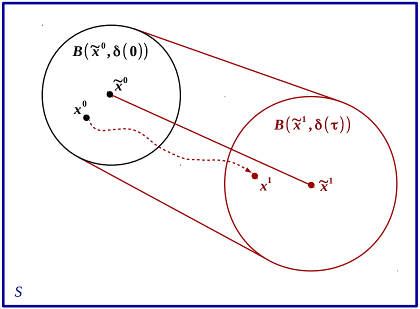

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provided δ_j verified to be convex on $[0, \tau]$

Remarks on the form of $\delta_j(\cdot)$

ex: DC-DC converter

modes given by $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}$ with $\sigma(t) \in U = \{1, 2\}$,

$$A_1 = \begin{pmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{pmatrix} \quad B_1 = \begin{pmatrix} \frac{v_s}{x_l} \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -\frac{1}{x_l}(r_l + \frac{r_0.r_c}{r_0+r_c}) & -\frac{1}{x_l}\frac{r_0}{r_0+r_c} \\ \frac{1}{x_c}\frac{r_0}{r_0+r_c} & -\frac{1}{x_c}\frac{r_0}{r_0+r_c} \end{pmatrix} \quad B_2 = \begin{pmatrix} \frac{v_s}{x_l} \\ 0 \end{pmatrix}$$

with $x_c = 70$, $x_l = 3$, $r_c = 0.005$, $r_l = 0.05$, $r_0 = 1$, $v_s = 1$.

λ_1	-0.0142
λ_2	0.142
C_1	6.7126×10^{-5}
C_2	2.6229×10^{-2}

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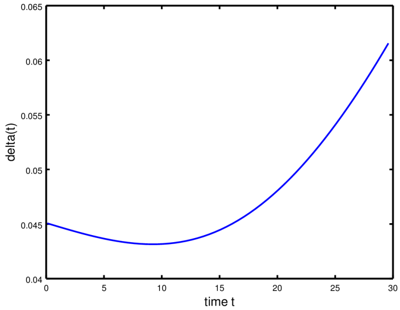
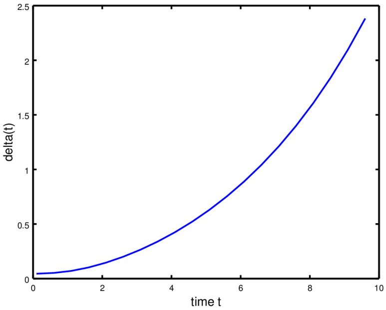


$\lambda_1 = -0.0142 < 0$



$\lambda_2 = 0.142 > 0$

For mode 1 ($\lambda_1 < 0$): optimal stepsize τ corresponding to minimum of δ_1

For mode 2 ($\lambda_2 > 0$): δ_2 always $\nearrow$

$\rightarrow$ suggests subsampling of τ for achieving better precision

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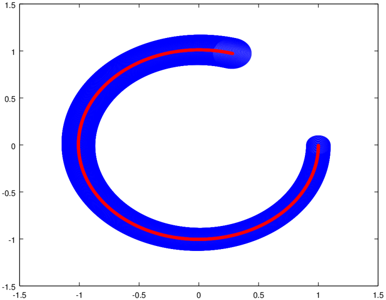
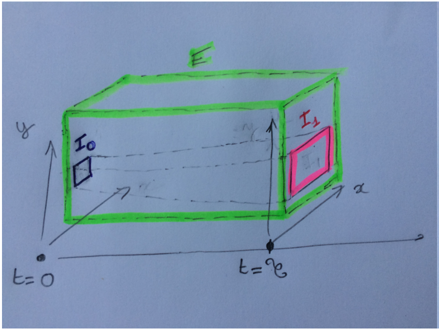
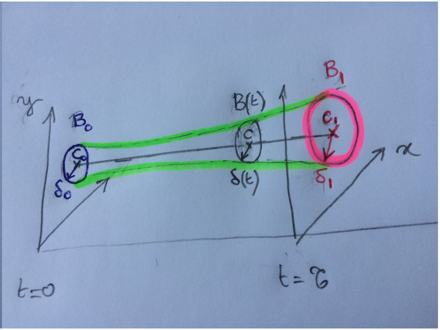


Recap': Interval-based vs. Euler-based method





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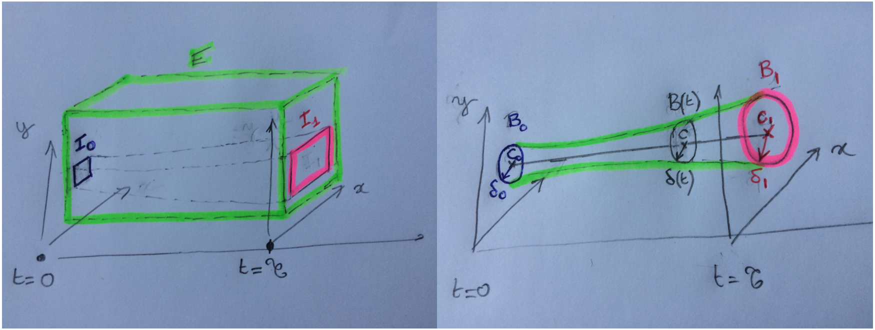

- <u>input/output</u>: intervals I_0, I_1 vs ball $B_0 \equiv B(C_0, \delta_0)$, $B_1 \equiv B(C_1, \delta_1)$

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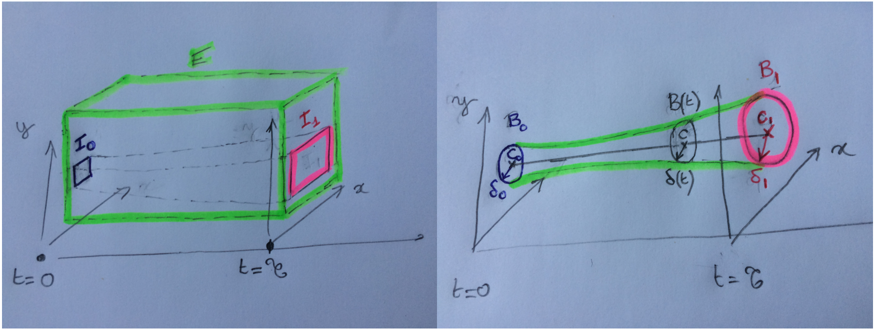

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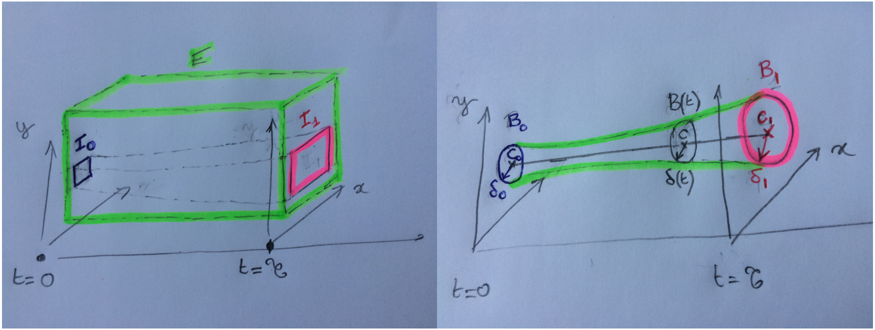

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  vs. B_1 evaluated directly from C_0 and δ_0

Euler-based integration (vs. interval integration)

- Advantages:
  1. Computationally very cheap (standard arithmetic, no need for computation of f derivatives, δ_j pre-computed)
  2. allows a priori for longer stepsize τ (often)
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- Limits:
  - less precise than interval-based integration method

    (1st order Taylor method vs. higher order Taylor method)

Outline

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Given a ball $B^0 \equiv B(\tilde{x}^0, \delta^0) \subseteq S$, select a mode j:

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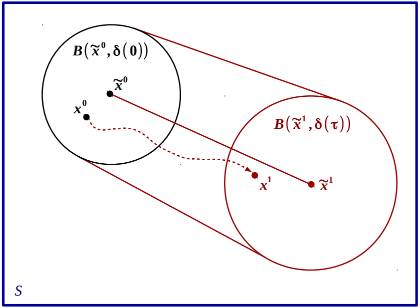

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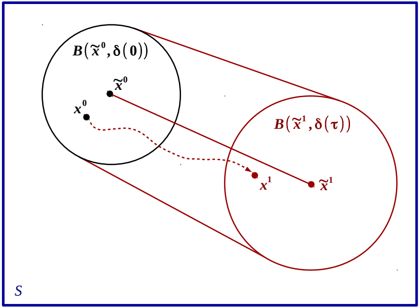

It suffices to find j:

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assuming $\delta_j(\cdot)$ convex

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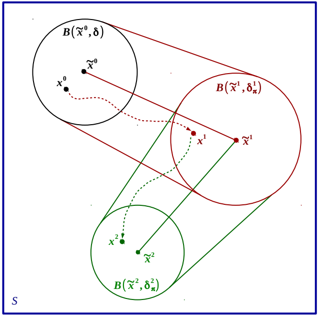

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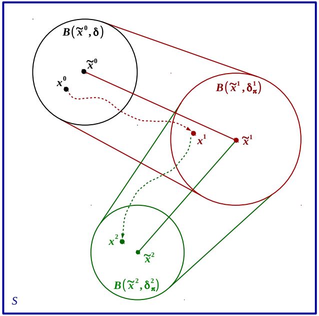

It suffices to find a pattern $\pi \equiv j_1 \cdots j_k$:

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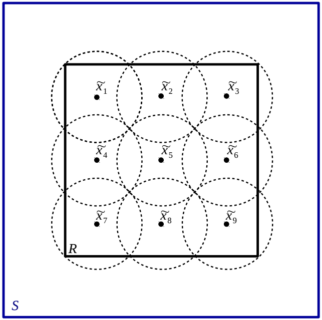
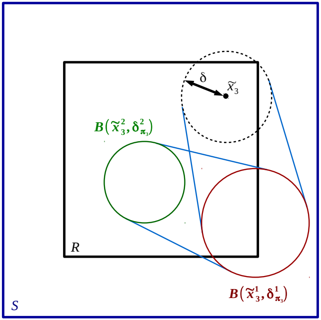
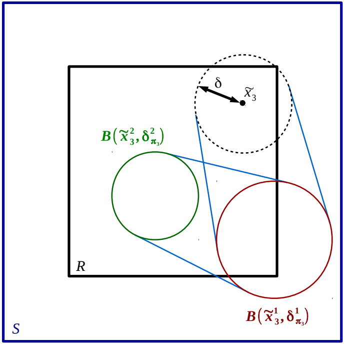
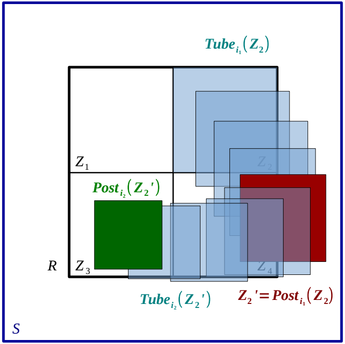




Euler-based control vs. interval-based control





Example: Building ventilation

[Meyer, Nazarpour, Girard, Witrant, 2014]

Dynamics of a four-room apartment:

$$\frac{dT_i}{dt} = \sum_{j \in \mathcal{U}^*} a_{ij}(T_j - T_i) + \delta_{s_i} b_i (T_{s_i}^4 - T_i^4) + c_i \max\left(0, \frac{V_i - V_i^*}{\bar{V}_i - V_i^*}\right)(T_u - T_i).$$

with $\mathcal{U}^* = \{1, 2, 3, 4, u, o, c\}$

16 switching modes

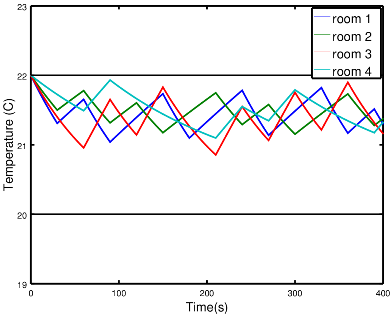
(control inputs: V_1, $V_4 \in \{$ 0V, 3.5V$\}$, and V_2, $V_3 \in \{$ 0V, 3V$\}$)

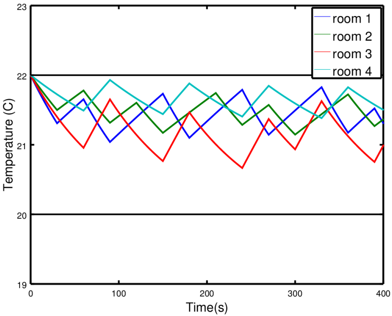
Building ventilation

	Euler	DynIBEX
R S	$[20,22]^4$ $[19,23]^4$	
τ	30	
Time subsampling	No	
Complete control	Yes	Yes
$\max_{j=1,\ldots,16} \lambda_j$ $\max_{j=1,\ldots,16} C_j$	-6.30×10^{-3} 4.18×10^{-6}	
Number of balls/tiles Pattern length	4096 1	252 1
CPU time	63 seconds	249 seconds

Control based on Euler (left) and interval (right).

Building ventilation





Control based on Euler (left) and interval (right).

Outline

Robust OSL condition w.r.t. disturbance $w \in W$

Consider: $\dot{x}(t) = f(x(t), w(t))$ with $w(t) \in W$ for all $t \in [0, \tau]$.

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(H_W): $\forall x, x' \in T$, $\forall w, w' \in W$

$$\langle f(x, w) - f(x', w'), x - x' \rangle \leq \lambda \|x - x'\|^2 + \gamma \|x - x'\| \|w - w'\|.$$

E. error function δ_W in presence of disturbance $w \in W$

Consider the ODE: $\dot{x}(t) = f(x(t), w(t))$ with $w(t) \in W$ for all $t \in [0, \tau]$.

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The fn δ_W (s.t: for all $t \in [0, \tau]$, $w(t) \in W$: $\|x(t) - \tilde{x}(t)\| \leq \delta_W(t)$)
can now be defined by:

E. error function δ_W in presence of disturbance $w \in W$

- if $\lambda < 0$,

$$\delta_W(t) = \left(\frac{C^2}{-\lambda^4} \left(-\lambda^2 t^2 - 2\lambda t + 2e^{\lambda t} - 2 \right) \right.$$
$$+ \frac{1}{\lambda^2} \left(\frac{C\gamma|W|}{-\lambda} \left(-\lambda t + e^{\lambda t} - 1 \right) \right.$$
$$\left. \left. + \lambda \left(\frac{(\gamma)^2(|W|/2)^2}{-\lambda}(e^{\lambda t} - 1) + \lambda(\delta^0)^2 e^{\lambda t} \right) \right) \right)^{1/2} \quad (1)$$

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- if $\lambda = 0$,

$$\delta_W(t) = \left(C^2 \left(-t^2 - 2t + 2e^t - 2 \right) + \left(C\gamma|W| \left(-t + e^t - 1 \right) + ((\gamma)^2(|W|/2)^2(e^t - 1) + (\delta^0)^2 e^t) \right) \right)^{1/2} \tag{3}$$

Compositional $(R_1 \times R_2, S_1 \times S_2)$-stability

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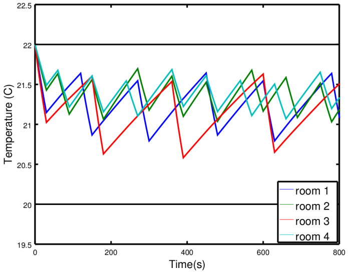
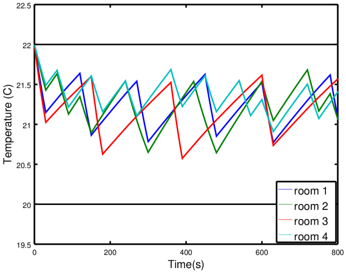
Then: $\sigma = \sigma_1 | \sigma_2$ is an $(R_1 \times R_2, S_1 \times S_2)$-stable control of $x(t) = (x_1(t), x_2(t))$.

Ex. of centralized vs. distributed Euler-based control





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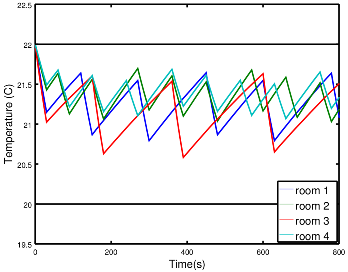

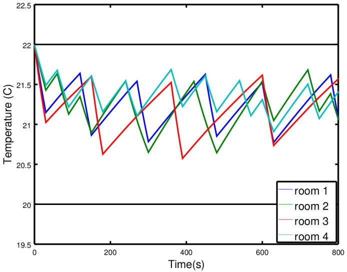

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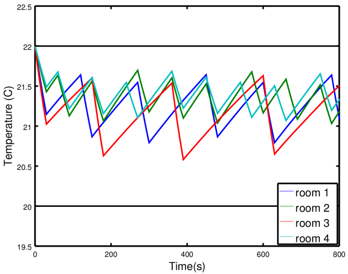

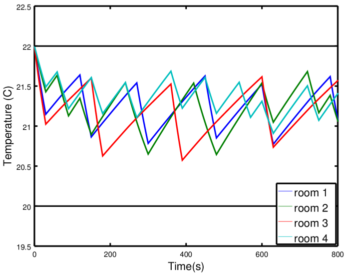

- (left) E.-based centralized control: subsampling $h = \frac{\tau}{20}$,
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- (right) E.-based distributed control: subsampling $h = \frac{\tau}{10} \mid h = \frac{\tau}{1}$,
  $2^2 \mid 2^2$ modes, $16 \mid 16$ balls $\rightarrow$ < 1 s. of CPU time.

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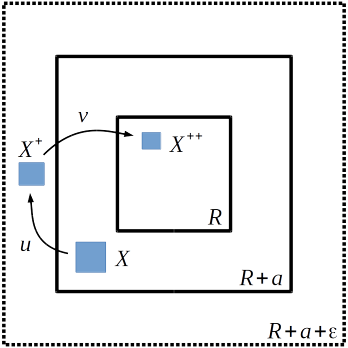
THANKS!

Distributed vs. Centralized Control

Centralized control synthesis

$$\dot{x}(t) = f_u(x(t))$$

Example of a validated pattern of length 2 mapping the "ball" X into R with $S = R + a + \varepsilon$ as safety box:



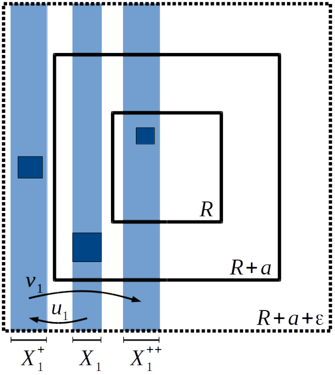
- $X \subset R$
- $X^+ = f_u(X) \subset S$
- $X^{++} = f_v(X^+) \subset R$

- Pattern $u \cdot v$ depends on X

Distrib. Control Synth. (of x_1 using S_2 as approx. of x_2)

$$\dot{x}_1(t) = f^1_{u_1}(x_1(t), x_2(t))$$
$$\dot{x}_2(t) = f^2_{u_2}(x_1(t), x_2(t))$$

Target zone: $R = R_1 \times R_2$



- $X_1 \subset R_1$
- $X_1^+ = f^1_{u_1}(X_1, S_2) \subset S_1$
- $X_1^{++} = f^1_{v_1}(X_1^+, S_2) \subset R_1$

- Pattern $u_1 \cdot v_1$ depends only on X_1

Distrib. Control Synth. (of x_2 using S_1 as approx. of x_1)

$$\dot{x}_1(t) = f^1_{u_1}(x_1(t), x_2(t))$$
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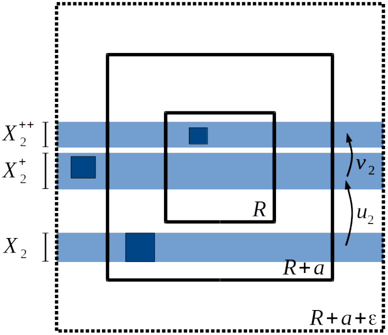
Target zone: $R = R_1 \times R_2$



- $X_2 \subset R_2$
- $X_2^+ = f^2_{u_2}(S_1, X_2) \in S_2$
- $X_2^{++} = f^2_{v_2}(S_1, X_2^+) \in R_2$
- Pattern $u_2 \cdot v_2$ depends only on X_2