An Introduction to Numerical Methods for Differential Games 2/2

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Outline

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Numerical approximation

We will describe a method to construct approximation schemes for the Isaacs equation which keep the main informations of the game/control problem.

This approach leads to the numerical approximation of a first order PDE derived by a discretization of the original control problem and by a discrete DP principle.

Naturally, one can also choose to construct directly an approximation scheme for the Isaacs equation based on the discretization of the PDE, e.g. using a Finite Difference (FD) scheme.

Features of the DP scheme

- The schemes have a natural interpretation which comes from the Discrete Dynamic Programming Principle
- Approximate feed-back controls can be obtained without extra computations on the nodes.
- Once the value function is computed we easily obtain approximate optimal trajectories.
- Natural extensions to high-order methods.

Outline

Time discretization

By applying the change of variable (Kružkov)

$$v(x) = 1 - e^{-T(x)}$$

we rewrite the equation in the new variable

$$v(x) + \sup_{a \in A}[-f(x,a) \cdot \nabla v] = 1 \tag{HJ}$$

$$v(x) = 0 \text{ on } \mathcal{T}$$
$$v(x) = 1 \text{ on } \partial\mathcal{R}$$

As we have seen, we can drop the second boundary condition.

Time discretization

Time step $h = \Delta t > 0$
Discrete times $t_j = jh$, $j \in \mathbb{N}$

Discrete dynamical system

$$\begin{cases} x_{j+1} = x_j + hf(x_j, a_j) \\ x_0 = x \end{cases}$$

We define the reachable set for the discrete dynamical system

$$\mathcal{R}_h \equiv \{x \in \mathbb{R}^N : \exists \{a_j\} \text{ and } j \in \mathbb{N} \text{ such that } x_j \in \mathcal{T}\}$$

Discrete Minimum Time Function

Let us define

$$n_h(\{a_j\}, x) = \begin{cases} +\infty & x \notin \mathcal{R}_h \\ \min\{\in \mathbb{N} : x_j \in \mathcal{T}\} & \forall\, x \in \mathcal{R}_h \end{cases}$$

$$N_h(x) = \min_{\{a_j\}} n_h(\{a_j\}, x)$$

The discrete analogue of the minimum time function is $N_h(x)h$.

The discrete time Bellman equation

As in the continuous problem, we change the variable

$$v_h(x) = 1 - e^{-h\,N_h(x)},$$

again we have $0 \leq v_h \leq 1$.
By the Discrete Dynamic Programming Principle we get

$$v_h(x) = S(v_h)(x) \qquad \text{on } \mathcal{R}_h \setminus \mathcal{T}. \tag{HJ_h}$$

where

$$S(v_h)(x) \equiv \min_{a \in A} \left[e^{-h} v_h(x + hf(x,a)) \right] + 1 - e^{-h}$$

which is complemented by the boundary condition

$$v_h(x) = 0 \qquad \text{on } \mathcal{T} \tag{BC}$$

Characterization of v_h

Note that $x \in \mathbb{R}^N \setminus \mathcal{R}_h$ implies that $x + hf(x, a) \in \mathbb{R}^N \setminus \mathcal{R}_h$

So we can extend v_h to $\mathbb{R}^N$ setting

$$v_h(x) = 1 \qquad \text{on } \mathbb{R}^N \setminus \mathcal{R}_h \ .$$

THEOREM

v_h is the unique bounded solution of $(HJ_h) - (BC)$.

Local Controllability

Let $\mathcal{T}$ be defined as

$$\mathcal{T} \equiv \{x : g_i(x) \leq 0 \quad \forall\, i = 1, \ldots, M\}$$

where $g_i \in C^2(\mathbb{R}^N)$ and $|\nabla g_i(x)| > 0$ for any x such that $g_i(x) = 0$.

LOCAL CONTROLLABILITY

$\forall\, x \in \mathcal{T}$ such that $g_i(x) = 0$ (i.e. belonging to ∂T) $\exists\, a \in A$ for which

$$f(x, a) \cdot \nabla g_i(x) < 0.$$

Convergence

THEOREM (convergence, Bardi-F (1990)
Let the assumptions of the Lemma be satisfied and let $\mathcal{T}$ be compact with nonempty interior.
Then, for $h \to 0^+$

$$v_h \to v \text{ locally uniformly in } \mathbb{R}^N$$

$$h\, N_h \to T \text{ locally uniformly in } \mathcal{R}$$

.

Error estimate

Let us assume Q is a compact subset of $\mathcal{R}$ where the following condition holds:

$$\exists\ C_0 > 0 :\ \forall x \in Q \text{ there is a time optimal control with total variation less than } C_0 \text{ bringing the system to } \mathcal{T}. \qquad \text{(BV)}$$

THEOREM (Bardi-F. (1990))
Let the assumptions of theorem hold true and let Q be a compact subset of $\mathcal{R}$ where (BV) holds.
Then $\exists\, \overline{h}, C > 0$:

$$|T(x) - h\, N_h(x)| \le Ch\ , \qquad \forall\, x \in Q, \quad \forall\, h \le \overline{h}$$

First order scheme

COROLLARY

Under the same hypotheses there exists two positive constants $\overline{h}$ and C:

$$|v(x) - v_h(x)| \leq Ch \qquad \forall\, x \in Q, h \leq \overline{h}$$

This means that the rate of convergence of the approximation scheme is 1. Note that also high-order methods can be obtained more accurate schemes for the dynamics (e.g. Runge-Kutta).

Space discretization

We need a grid to obtain a fully discrete scheme.
We can use a lattice or a triangulation of a rectangle Q in $\mathbb{R}^2$, $Q \supset \mathcal{T}$.

NOTATIONS

x_i : nodes of the grid

d : the number of nodes

$k :=$ max diameter of the cells (or triangles)

Sets of Indices:

$$I_{\mathcal{T}} := \{i \in \mathbb{N} : x_i \in \mathcal{T}\} \text{(target nodes)}$$

$$I_{out} := \{i \in \mathbb{N} : x_i + hf(x_i, a) \notin Q, \forall\, a\} \text{(internal nodes)}$$

$$I_{in} := \{i \in \mathbb{N} : x_i + hf(x_i, a) \in Q\} \text{(boundary nodes)}$$

Fully discrete scheme

We want to solve the the problem

$$\begin{aligned} v(x_i) &= \min_{a \in A}[e^{-h} v(x_i + hf(x_i, a)] + 1 - e^{-h}, \quad \forall x_i \in I_{in}, \\ v(x_i) &= 0 \qquad \forall x_i \in I_{\mathcal{T}} \\ v(x_i) &= 1 \qquad \forall x_i \in I_{out} \end{aligned}$$

To get a finite dimensional problem we introduce the space of piecewise linear functions

$$W^k := \{w : Q \to [0,1] : w \in C(Q) \text{ and } \nabla w = constant \text{ in } S_j\}$$

Fixed point problem

This means that we choose a piecewise linear (P_1) reconstruction for $v(x_i + hf(x_i, a))$.
In fact, for any $i \in I_{\text{in}}$, $x_i + hf(x_i, a) \in Q$, there exists a vector of coefficients,$\lambda_{ij}(a)$:

$$0 \leq \lambda_{ij}(a) \leq 1, \quad \sum_{j=1}^{d} \lambda_{ij}(a) = 1$$

and

$$x_i + hf(x_i, a) = \sum_{j=1}^{L} \lambda_{ij}(a) x_j \quad \text{(convex combination)}$$

so we are writing $x_i + hf(x_i, a)$ in local coordinates.

Fixed point problem

Let us define the matrix $\Lambda := \{\lambda_{ij}\}$. We define the operator $S : \mathbb{R}^d \to \mathbb{R}^d$

$$S_i(U) := \begin{cases} \min\limits_{a \in A}[e^{-h}\Lambda_i(a)U] + 1 - e^{-h}, & \forall\, i \in I_{in} \\ 0 & \forall\, i \in I_{\mathcal{T}} \\ 1 & \forall\, i \in I_{\text{out}} \end{cases}$$

Since $\beta := e^{-h} \in (0,1)$, the operator $S = \{S_i\}$ takes its values in $[0,1]^d$

$$S : [0,1]^d \to [0,1]^d$$

and has a unique fixed point.

S properties

THEOREM

$S : [0,1]^d \to [0,1]^d$ and

$$\|S(U) - S(V)\|_\infty \leq \beta \|U - V\|_\infty$$

Sketch of the proof

S is monotone, i.e.

$$U \leq V \Rightarrow S(U) \leq S(V)$$

Then, for any $U \in [0,1]^d$

$$1 - \beta = S_i(0) \leq S_i(U) \leq S_i(\mathbf{1}) = 1, \qquad \forall\, i \in I_{in}$$

where $\mathbf{1} \equiv (1, 1, \ldots, 1)$. This implies, $S : [0,1]^d \to [0,1]^d$

S is a contraction

For any $i \in I_{in}$, we have

$$S_i(U) - S_i(V) \leq \beta \Lambda_i(\hat{a})(U - V)$$

and since $\|\Lambda_i(a)\| \leq 1$, $\forall\, a \in A$, this implies

$$\|S_i(U) - S(V)\|_\infty \leq \beta \|U - V\|_\infty.$$

Monotone convergence

The scheme

$$U^{n+1} \equiv S(U^n)$$

converges for every initial condition U^0.

However, choosing $U^0 \in [0,1]^d$

$$U_i^0 = \begin{cases} 0 & \forall\, i \in I_{\mathcal{T}} \\ 1 & \mathit{elsewhere} \end{cases}$$

we have

$$U^0 \in U^+ \equiv \{U \in [0,1]^L : U \geq S(U)\}$$

Monotone convergence

The sequence U^n starting from that U^0 (which is a discrete supersolution) s monotone decreasing .
This stems from the monotonicity of the discrete operator S, so

$$U^n \searrow U^*$$

by the fixed point argument.
Note that monotonicity allows to accelerate convergence.

How the informations flow

The information flows from the target to the other nodes of the grid.
In fact, on the nodes in $Q \setminus \mathcal{T}$, $U_i^0 = 1$.

But starting from the first step of the algorithm the value of the internal nodes immediately decreases since, by the local controllability assumption, the Euler scheme drives them to the target where the value is set to 0.

Outline

Back to the Pursuit-Evasion game

Using the same change of variable $v(x) = 1 - e^{-T(x)}$ we can set the Isaacs equation in $\mathbb{R}^N$

$$\begin{cases} v(x) + \min_{b \in B} \max_{a \in A}[-f(x,a,b) \cdot \nabla v(x)] = 1 & \text{in } \mathbb{R}^N \setminus \mathcal{T} \\ v(x) = 0 & \text{for } x \in \partial\mathcal{T} \end{cases} \tag{I}$$

The discretization in time and space leads to the

Fully discrete scheme for games

$$\begin{aligned} w(x_i) &= \max_b \min_a [\beta w(x_i + hf(x_i,a,b)] + 1 - \beta \quad \text{for } i \in I_{in} \\ w(x_i) &= 1 \qquad \text{for } i \in I_{out_2} \\ w(x_i) &= 0 \qquad \text{for } i \in I_{\mathcal{T}} \cup I_{out_1} \end{aligned}$$

Fully discrete scheme for games

where

$$\beta = e^{-h}$$
$$I_{in} = \{i : x_i \in Q \setminus \mathcal{T}\}$$
$$I_{\mathcal{T}} = \{i : x_i \in \mathcal{T} \cap Q\}$$
$$I_{out_1} = \{i : x_i \notin Q_2\}$$
$$I_{out_2} = \{i : x_i \notin Q_2 \setminus Q\}$$
$$Q = Q_1 \cap Q_2$$

Properties of the scheme for games

THEOREM

The operator $S : [0,1]^L \to [0,1]^L$, moreover:

$$U \leq V \Rightarrow S(U) \leq S(V)$$

S is a contraction map.

Let U^* be the unique fixed point, we define

$$\begin{aligned} w(x_i) = & \quad U_i^* \quad \forall\, i \\ w(x) = & \sum_j \lambda_{ij}(a,b) w(x_j) \end{aligned}$$

Convergence for games

Naturally w depends on the discretization steps, h and k.

THEOREM

Let $\mathcal{T}$ be the closure of an open set with Lipschitz boundary, "diam $Q \to +\infty$" and v continuous. Then

$$w^{h,k} \to v \quad \text{on compact sets of } \mathbb{R}^N$$

for $h \to 0^+$ and $\frac{k}{h} \to 0^+$.

Convergence: discontinuous value

Let w_n^ε be the sequence generated by the numerical scheme with target $\mathcal{T}_\varepsilon = \{x : d(x, \mathcal{T}) \leq \varepsilon\}$

THEOREM

For all x there exists the limit

$$\overline{w}(x) = \lim_{\substack{\varepsilon \to 0^+ \\ n \to +\infty \\ n \geq n(\varepsilon)}} w_n^\varepsilon(x)$$

and it coincides with the lower value V of the game with target $\mathcal{T}$, i.e.

$$\overline{w} = V$$

The convergence is uniform on every compact set where V is continuous.

Error estimates (Soravia (1998))

Assume for simplicity $L_f \leq 1$ and v Lipschitz continuous.

Then,

$$\|w^{h,k} - v\|_\infty \leq Ch^{1/2}\left(1 + \left(\frac{k}{h}\right)^2\right)$$

Synthesis of Feedback Controls

The algorithm computes also an approximate optimal control at each point of the grid . However by w we can also compute an approximate optimal feedback at each point of Q, i.e. we can define the feedback map $F : Q \to A$

$$x \to F(x) = a_x^k \quad \text{feedback}$$

where a_x^k is the argmin of

$$I^k(x, a) \equiv e^{-h} w(x + hf(x, a)) + 1 - e^{-h}$$

Note that $I^k(x, \cdot)$ has a minimum over A (compact), but the minimum point may be not unique.

Feedback selection

We want to construct a selection, e.g. take a strictly convex ϕ and define

$$A_x^k = \{a^* \in A : I_x^k(x, a^*) = \min_A I^k(x, a)\}$$

The selection is

$$\arg \min_{A_x^k} \phi(a)$$

Discrete optimal trajectories

To compute the discrete optimal trajectories, we define the piecewise constant control (in time)

$$a^k(s) = a^k_{y_{n,h}} \quad s \in [nh, (n+1)h[$$

where $y_{n,h}$ = state of the Euler scheme, step h.

Error estimates of the approximation of feedbacks and optimal trajectories are available for control problems (F. 2001).

Feedback controls for games

The algorithm computes an approximate optimal control couple (a^*, b^*) at each point of the grid. By w we can also compute an approximate optimal feedback at every point of Q.

$$(a^*, b^*) \equiv \text{argminmax}\{e^{-h} w(x + hf(x, a, b))\} + 1 - e^{-h}$$

In case of multiple solutions we can select a unique couple, e.g. minimizing two convex functionals.

We can also introduce an inertia criterium to stabilize the trajectories, i.e. if a at step $n+1$ the set of optimal couples contains (a^*_n, b^*_n) we keep it.

Eikonal equation

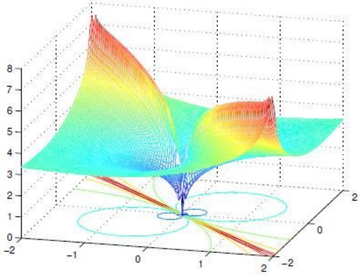
Distance from the origin. $\Omega = \{(0,0)\}$, $\quad c(x,y) \equiv 1$
Exact solution: $T(x,y) = \sqrt{(x^2+y^2)}$

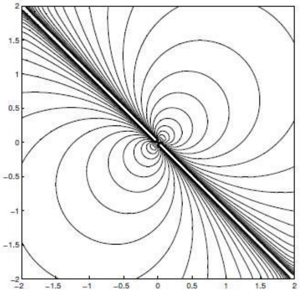
method	Δx	L^∞ error	L^1 error	CPU time (sec)
SL (46 its.)	0.08	**0.0329**	**0.3757**	**8.4**
FM-FD	0.08	0.0875	0.7807	0.5
FM-SL	0.08	**0.0329**	**0.3757**	**0.7**

FM-FD: Fast Marching Method based on define differences
FM-SL: Fast Marching Method based on the DP scheme (semi-lagrangian)

Minimum time problem with variable velocity

Eikonal equation with velocity $c(x, y) = |x + y|$.





The Tag-Chase Game

Dynamics

$$f_P(y, a, b) = v_P a \qquad f_E(y, a, b) = v_E b$$

$$v_P = 2 \qquad v_E = 1$$

Admissible control sets

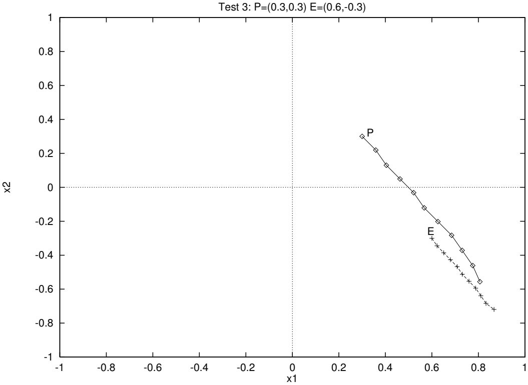
$$A = B = B(0, 1)$$

Relative coordinates

$\tilde{x} = (x_E - x_P)\cos\theta - (y_E - y_P)\sin\theta$

$\tilde{y} = (x_E - x_P)\sin\theta - (y_E - y_P)\cos\theta$

Optimal trajectories



The Tag-Chase game with directional constraints

Dynamics

$$f_P(y,a,b) = v_P a \qquad f_E(y,a,b) = v_E b$$

$$v_P = 2 \qquad v_E = 1$$

Admissible control sets

$$A = B(0,1)$$

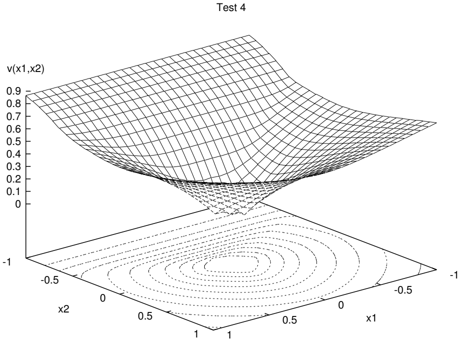
$$B = B(0,1) \setminus S$$

where

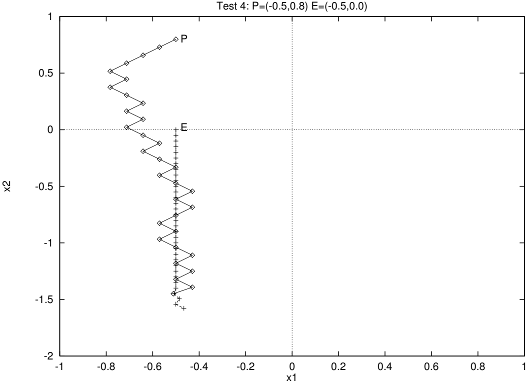
$$S = (\rho\cos\theta, \rho\sin\theta), \theta \in (\theta_1, \theta_2),\ |\rho| \le 1$$

So the pursuer has a forbidden cone of directions.

Value Function



Optimal Trajectories

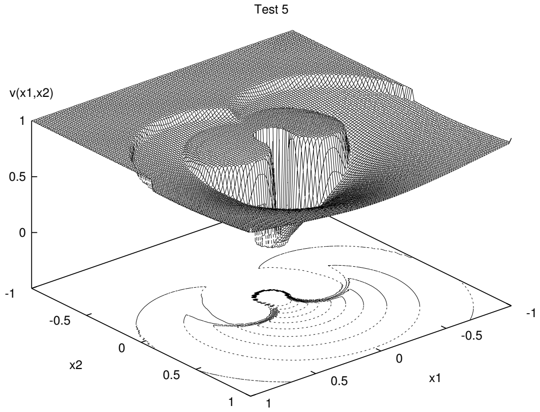


The Homicidal chauffeur

Dynamics

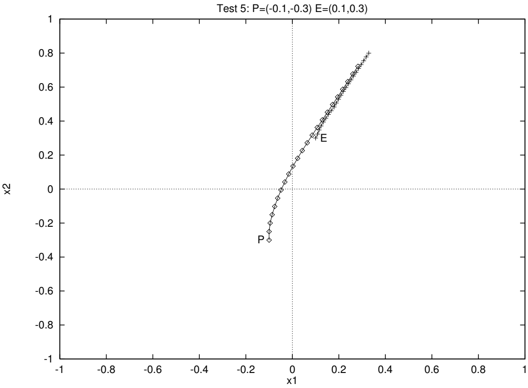
$$\begin{cases} \dot{x}_P & = v_P \sin\theta \\ \dot{y}_P & = v_P \cos\theta \\ \dot{x}_E & = v_E \sin b \\ \dot{y}_E & = v_E \cos b \\ \dot{\theta} & = \frac{R}{v_P} a \end{cases}$$

Value Function

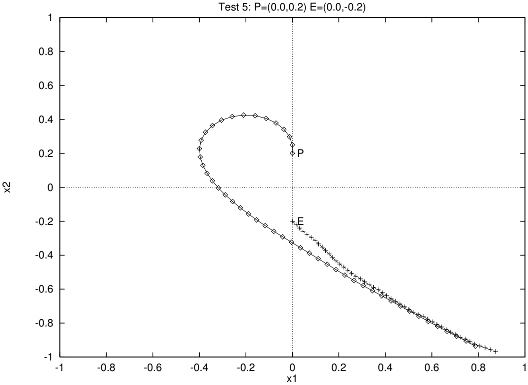


The value function is discontinuous on the barriers

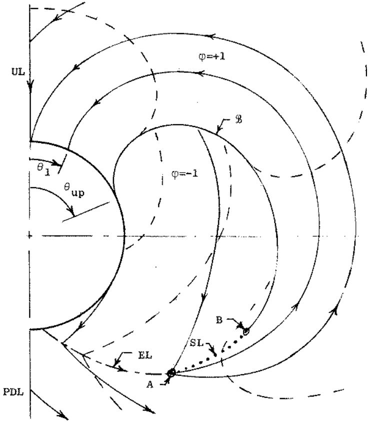
Optimal Trajectories



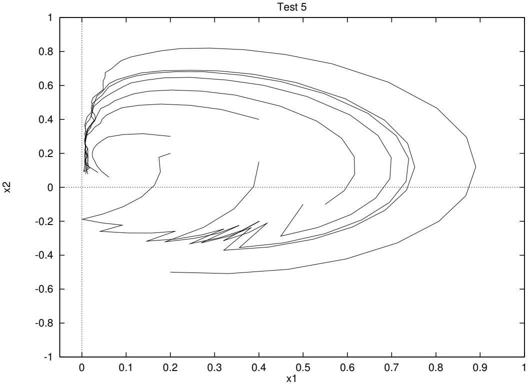
Optimal Trajectories



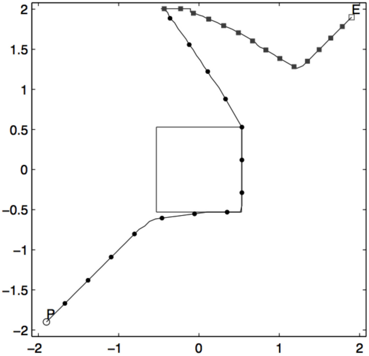
Optimal Trajectories (Merz Thesis)



Optimal Trajectories (computed)



The Tag-chase game in a courtyard, $V_p > V_e$



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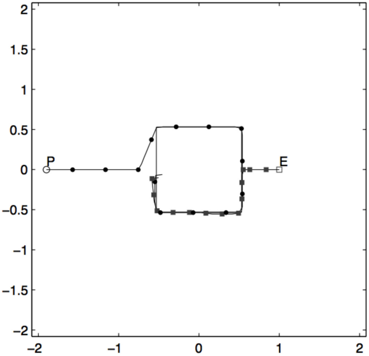

Efficient methods

In order to overcome the curse of dimensionality and reduce the memory allocation requirements one can modify the standard approximation scheme based on a fixed point iteration of a discrete Hamilton-Jacobi equation derived by DP.

Recent extensions include:
Fast Marching/Fast Sweeping Methods
Parallel Methods
Domain Decomposition techniques

Basic References

APPROXIMATION OF DETERMINISTIC CONTROL PROBLEMS AND DIFFERENTIAL GAMES

The following references will contain many information and several links to the literature

M. Falcone, R. Ferretti, Semi-Lagrangian Approximation Schemes for Linear and Hamilton-Jacobi Equations, SIAM, 2014

M. Bardi, I. Capuzzo Dolcetta, Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkhäuser, 1997

M. Falcone, *Numerical Methods for Differential Games via PDEs*, International Game Theory Review, vol. 8, 2 (2006), 231-272.