TD 11: Petri Nets

Exercise 1 (Dickson’s Lemma). A quasi-order \((A, \leq)\) is a set \(A\) endowed with a reflexive and transitive ordering relation \(\leq\). A well quasi order (wqo) is a quasi order \((A, \leq)\) s.t., for any infinite sequence \(a_0a_1\cdots\) in \(A^\omega\), there exist indices \(i < j\) with \(a_i \leq a_j\).

1. Let \((A, \leq)\) be a wqo and \(B \subseteq A\). Show that \((B, \leq)\) is a wqo.

2. Show that \((\mathbb{N} \cup \{\omega\}, \leq)\) is a wqo.

3. Let \((A, \leq)\) be a wqo. Show that any infinite sequence \(a_0a_1\cdots\) in \(A^\omega\) embeds an infinite increasing subsequence \(a_{i_0} \leq a_{i_1} \leq a_{i_2} \leq \cdots\) with \(i_0 < i_1 < i_2 < \cdots\).

4. Let \((A, \leq_A)\) and \((B, \leq_B)\) be two wqo’s. Show that the cartesian product \((A \times B, \leq_A \times \leq_B)\), where the product ordering is defined by \((a, b) \leq_A (a', b')\) iff \(a \leq_A a'\) and \(b \leq_B b'\), is a wqo.

Exercise 2 (Coverability Graph). The coverability problem for Petri nets is the following decision problem:

Instance: A Petri net \(\mathcal{N} = \langle P, T, F, W, m_0 \rangle\) and a marking \(m_1\) in \(\mathbb{N}^P\).

Question: Does there exist \(m_2\) in \(\text{reach}_{\mathcal{N}}(m_0)\) such that \(m_1 \leq m_2\)?

For 1-safe Petri nets, coverability coincides with reachability, and is thus PSPACE-complete.

One way to decide the general coverability problem is to use Karp and Miller’s coverability graph (see the lecture notes). Indeed, we have the equivalence between the two statements:

\[ i. \text{ there exists } m_2 \text{ in } \text{reach}_{\mathcal{N}}(m_0) \text{ such that } m_1 \leq m_2, \text{ and} \]

\[ ii. \text{ there exists } m_3 \text{ in } \text{CoverabilityGraph}_{\mathcal{N}}(m_0) \text{ such that } m_1 \leq m_3. \]

1. In order to prove that \([\text{i}]\) implies \([\text{ii}]\), we will prove a stronger statement: for a marking \(m\) in \((\mathbb{N} \cup \{\omega\})^P\), write \(\Omega(m) = \{p \in P \mid m(p) = \omega\}\) for the set of \(\omega\)-places of \(m\).

Show that, if \(m_0 \xrightarrow{u} \mathcal{N} m_2\) in the Petri net \(\mathcal{N}\) for some \(u\) in \(T^*\), then there exists \(m_3\) in \((\mathbb{N} \cup \{\omega\})^P\) such that \(m_2(p) = m_3(p)\) for all \(p\) in \(P \setminus \Omega(m_3)\) and \(m_0 \xrightarrow{u} G m_3\) in the coverability graph.

2. Let us prove that \([\text{ii}]\) implies \([\text{i}]\). The idea is that we can find reachable markings that agree with \(m_3\) on its finite places, and that can be made arbitrarily high on its \(\omega\)-places. For this, we need to identify the graph nodes where new \(\omega\) values were introduced, which we call \(\omega\)-nodes.
(a) The threshold $\Theta(u)$ of a transition sequence $u$ in $T^*$ is the minimal marking $m$ in $\mathbb{N}^P$ s.t. $u$ is enabled from $m$. Show how to compute $\Theta(u)$. Show that $\Theta(u \cdot v) \leq \Theta(u) + \Theta(v)$ for all $u, v$ in $T^*$.

(b) Recall that an $\omega$ value is introduced in the coverability graph thanks to Algorithm 1.

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1 repeat
  2 saved ← $m'$;
  3 foreach $m'' \in V$ s.t. $\exists v \in T^*, m'' \xrightarrow{G} m$ do
    4 if $m'' < m'$ then
      5 $m' \leftarrow m' + ((m' - m'') \cdot \omega)$
    6 end
  7 end
  8 until saved = $m'$;
  9 return $m'$
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**Algorithm 1: AddOmegas($m, m', V$)**

We consider a call to $\text{AddOmegas}(m, m', V)$ on line 8 of the COVERABILITY-GRAph algorithm from the course notes, where $m \xrightarrow{t} m'$ for $t$ the transition chosen at line 6 of the COVERABILITY-GRAph algorithm.

Let $\{v_1, \ldots, v_\ell\}$ be the set of “vt” sequences, where $v$ is found on line 3 of $\text{AddOmegas}(m, m', V)$. These sequences $vt$ resulted in adding at least one $\omega$ value to $m'$ on line 5. Let $w = v_1 \cdots v_\ell$. Show that, for any $k$ in $\mathbb{N}$, the marking $\nu_k$ defined by

$$
\nu_k(p) = \begin{cases} 
  m'(p) & \text{if } p \in P \setminus \Omega(m) \\
  \Theta(w^k)(p) & \text{if } p \in \Omega(m)
\end{cases}
$$

allows to fire $w^k$. How does the marking $\nu'_k$ with $w^k \xrightarrow{\nu'_k} \nu'_k$ compare to $\nu_k$?

(c) Prove that, if $m_0 \xrightarrow{u} G m_3$ for some $u$ in $T^*$ in the coverability graph and $m'$ in $\mathbb{N}^\Omega(m_3)$ is a partial marking on the places of $\Omega(m_3)$, then there are

- a decomposition $u = u_1 u_2 \cdots u_{n+1}$ with each $u_i$ in $T^*$ (where the markings $\mu_i$ reached by $m_0 \xrightarrow{u_1 \cdots u_i} G \mu_i$ for $i \leq n$ have new $\omega$ values),
- sequences $w_1, \ldots, w_n$ in $T^+$,
- numbers $k_1, \ldots, k_n$ in $\mathbb{N}$,

such that $m_0 \xrightarrow{u_1 k_1 u_2 \cdots u_n k_n u_{n+1}} G m_2$ with $m_2(p) = m_3(p)$ for all $p$ in $P \setminus \Omega(m_3)$ and $m_2(p) \geq m'(p)$ for all $p$ in $\Omega(m_3)$.
Exercise 3 (Decidability of Model-checking Action-based LTL).

1. Let \( \mathcal{N} \) be Petri net, \( G \) its coverability graph, and \( m \) some marking in \( \mathbb{N}^P \). An infinite computation is a sequence \( m_0m_1\cdots \) in \( (\mathbb{N}^P)^\omega \) where for all \( i \in \mathbb{N} \), \( m_i \rightarrow_{\mathcal{N}} m_{i+1} \) is a transition step. The effect \( \Delta(u) \) of a transition sequence \( u \) in \( T^* \) is defined by \( \Delta(\varepsilon) = 0^P \) and \( \Delta(ut) = \Delta(u) - W(P,t) + W(t,P) \).

Show that there exists an infinite computation s.t. \( m \leq m_i \) for infinitely many indices \( i \) iff there exists an accessible loop \( m' \xrightarrow{A_{G}} m' \) in \( G \) s.t. \( m \leq m' \) and \( \Delta(v) \geq 0^P \).

2. Show that action-based LTL model-checking is decidable for labeled Petri nets.

Exercise 4 (Rackoff’s Algorithm). A rather severe issue with the coverability graph construction is that it can generate a graph of Ackermannian size compared to that of the original Petri net. We show here a much more decent ExpSpace upper bound, which is matched by an ExpSpace hardness proof by Lipton.

Let us fix a Petri net \( \mathcal{N} = \langle P, T, F, W, m_0 \rangle \). We consider generalized markings in \( \mathbb{Z}^P \). A generalized computation is a sequence \( \mu_1 \cdots \mu_n \) in \( (\mathbb{Z}^P)^* \) such that, for all \( 1 \leq i < n \), there is a transition \( t \) in \( T \) with \( \mu_{i+1}(p) = \mu_i(p) - W(p,t) + W(t,p) \) for all \( p \in P \) (i.e. we do not enforce enabling conditions). For a subset \( I \) of \( P \), a generalized sequence is \( I \)-admissible if furthermore \( \mu_i(p) \geq W(p,t) \) for all \( p \in I \) at each step \( 1 \leq i < n \). For a value \( B \) in \( \mathbb{N} \), it is \( I \)-\( B \)-bounded if furthermore \( \mu_i(p) < B \) for all \( p \in I \) at each step \( 1 \leq i \leq n \). A generalized sequence is an \( I \)-covering for \( m_1 \) if \( \mu_1 = m_0 \) and \( \mu_n(p) \geq m_1(p) \) for all \( p \) in \( I \).

Thus a computation is a \( P \)-admissible generalized computation, and a \( P \)-admissible \( P \)-covering for \( m_1 \) answers the coverability problem.

For a Petri net \( \mathcal{N} = \langle P, T, F, W, m_0 \rangle \) and a marking \( m_1 \) in \( \mathbb{N}^P \), let \( \ell(\mathcal{N}, m_1) \) be the length of the shortest \( P \)-admissible \( P \)-covering for \( m_1 \) in \( \mathcal{N} \) if one exists, and otherwise \( \ell(\mathcal{N}, m_1) = 0 \). For \( L, k \) in \( \mathbb{N} \), define

\[
M_L(k) = \sup\{ \ell(\mathcal{N}, m_1) \mid |P| = k, \max_{p \in P,t \in T} W(p,t) + \max_{p \in P} m_1(p) \leq L \}
\]

the maximal \( \ell(\mathcal{N}, m_1) \) over all Petri nets \( \mathcal{N} \) of dimension \( k \) and all markings \( m_1 \) to cover, under some restrictions on incoming weights \( W(p,t) \) in \( \mathcal{N} \) and values in \( m_1 \).

1. Show that \( M_L(0) \leq 1 \).

2. We want to show that

\[
M_L(k) \leq (L \cdot M_L(k-1))^{k} + M_L(k-1)
\]

for all \( k \geq 1 \). To this end, we prove that, for every marking \( m_1 \) in \( \mathbb{N}^P \) for a Petri net \( \mathcal{N} \) with \( |P| = k \),

\[
\ell(\mathcal{N}, m_1) \leq (L \cdot M_L(k-1))^{k} + M_L(k-1) \ .
\]

(*)
Let
\[ B = M_L(k - 1) \cdot \max_{p \in P, t \in T} W(p, t) + \max_{p \in P} m_1(p) . \]
and suppose that there exists a \( P \)-admissible \( P \)-covering \( w = \mu_1 \cdots \mu_n \) for \( m_1 \) in \( \mathcal{N} \).

(a) Show that, if \( w \) is \( P \)-\( B \)-bounded, then \((\ast)\) holds.

(b) Assume the contrary: we can split \( w \) as \( w_1w_2 \) such that \( w_1 \) is \( P \)-\( B \)-bounded and \( w_2 \) starts with a marking \( \mu_j \) with a place \( p \) such that \( \mu_j(p) \geq B \). Show that \((\ast)\) also holds.

3. Show that \( M_L(|P|) \leq L^{(3 \cdot |P|)!} \) for \( L \geq 2 \).

4. Given a Petri net \( \mathcal{N} = (P, T, W, m_0) \) and a marking \( m_1 \), set \( L = 2 + \max_{p \in P} m_1(p) + \max_{p \in P, t \in T} W(p, t) \). Assuming that the size \( n \) of the instance \((\mathcal{N}, m_1)\) of the coverability problem is more than
\[ \max(\log L, |P|, \max_{p \in P, t \in T} \log W(t, p)) , \]
deduce that we can guess a \( P \)-admissible \( P \)-covering for \( m_1 \) of length at most \( 2^{2c \cdot n \log n} \) for some constant \( c \). Conclude that the coverability problem can be solved in \textit{ExpSpace}. 