TD 9: BDDs

Exercise 1 (Some BDDs). Draw the BDDs for the following functions, using the order of your choice on the variables \{x_1, x_2, x_3\}:

1. The majority function \(m(x_1, x_2, x_3)\): its value is 1 iff the majority of the input bits are 1’s,
2. The hidden weighted bit function \(h(x_1, x_2, x_3)\): its value is that of variable \(x_s\), where \(s = \sum_{i=1}^3 x_i\) and \(x_0\) is defined as 0.

Exercise 2 (Symmetric Functions). A symmetric function of \(n\) variables has the same value for all permutations of the same \(n\) tuple of arguments.

Show that a BDD for a symmetric function has at most \(\frac{n(n+1)}{2} + 1\) nodes (when omitting the 0-node).

Exercise 3 (Counting Solutions). Write a linear time algorithm for counting the number of solutions of a boolean function \(f\) represented by a BDD, i.e., of the number of valuations \(\nu\) s.t. \(\nu \models f\).

Exercise 4 (An Upper Bound on the Size of BDDs). The size \(B(f)\) of a BDD for a function \(f\) is defined as the number of its nodes. Consider an arbitrary boolean function \(f\) on the ordered set \(x_1 \cdots x_n\), and consider a variable \(x_k\).

1. Show that we can bound the number of nodes labeled by \(\{x_1, \ldots, x_k\}\) by \(2^k - 1\).
2. How many different subfunctions on the ordered set of variables \(x_k+1 \cdots x_n\) exist? Deduce another bound for the number of nodes labeled by \(\{x_{k+1}, \ldots, x_n\}\).
3. What global bound do you obtain for \(k = n - \log_2(n - \log_2 n)\)?

Exercise 5 (Finding the Optimal Order). There are in general \(n!\) different orders for the variables \(\{x_1, \ldots, x_n\}\), and building the BDD for each of these is computationally expensive. One can nevertheless design an exponential time algorithm for finding the optimal order. Indeed, an optimal ordering on a subset \(X\) of variables does not depend on the order in which \(X' = \{x_1, \ldots, x_n\}\) has been accessed.

1. Fix a boolean function \(f\) over variables \(\{x_1, \ldots, x_n\}\). We assume that \(f\) is provided as a BDD \(B\) for the ordering \(x_1, x_2, \ldots, x_n\).

Given a subset \(X\) of \(\{x_1, \ldots, x_n\}\) and a variable \(x\) in \(X\), how many nodes labeled by \(x\) does any BDD \(B'\) for \(f\) have if it first treats \(X' = \{x_1, \ldots, x_n\}\) \(X\), then \(x\), and last \(X\) \(\{x\}\)? How can you compute this number on the provided BDD \(B\) for \(f\)?
2. Reduce the optimal order problem to the search of a path of minimal weight in a weighted graph with subsets of \( \{x_1, \ldots, x_n\} \) as vertices.