# TD 6

**Exercise 1** (Ehrenfeucht-Fraïssé games)**.** Let $w_0 = (\mathbb{T}_0, <, h_0)$ and $w_1 = (\mathbb{T}_1, <, h_1)$ be two temporal structures. Let $i_0 \in \mathbb{T}_0$ and $i_1 \in \mathbb{T}_1$. Let $k \in \mathbb{N}$. We say that $(w_0, i_0)$ and $(w_1, i_1)$ are *k-equivalent*, denoted $(w_0, i_0) \equiv_k (w_1, i_1)$, if they satisfy the same formulas in $\mathrm{TL}(\mathrm{AP}, \mathsf{SU}, \mathsf{SS})$ of temporal depth at most $k$.

This can also be described through *Ehrenfeucht-Fraïssé games* (EF-games). EF-games for $\mathrm{TL}(\mathrm{AP}, \mathsf{SU}, \mathsf{SS})$ are defined as follows. The game has two players : *Spoiler* and *Duplicator*, and the *game board* consists of two temporal structures $w_0 = (\mathbb{T}_0, <, h_0)$ and $w_1 = (\mathbb{T}_1, <, h_1)$. There are two tokens, one on each structure: $i_0 \in \mathbb{T}_0$ and $i_1 \in \mathbb{T}_1$. A *configuration* is a tuple $(w_0, i_0, w_1, i_1)$, or simply $(i_0, i_1)$ if the game board is understood. Intuitively, Duplicator will try to prove that $(w_0, i_0)$ and $(w_1, i_1)$ are equivalent, while Spoiler will try to distinguish them. Let $k \in \mathbb{N}$. The $k$-round EF-game from a configuration proceeds with (at most) $k$ moves. There are two available moves for $\mathrm{TL}(\mathrm{AP}, \mathsf{SU}, \mathsf{SS})$: $\mathsf{SU}$-moves, or $\mathsf{SS}$-moves. Spoiler choses which move is played in each round.

**$\mathsf{SU}$-move.**

- Spoiler chooses one of the two structures, i.e., $\varepsilon \in \{0, 1\}$, and a position $k_\varepsilon \in \mathbb{T}_\varepsilon$ such that $i_\varepsilon < k_\varepsilon$.
- Duplicator chooses $k_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ such that $i_{1-\varepsilon} < k_{1-\varepsilon}$. Spoiler wins if there is no such $k_{1-\varepsilon}$.

Either Spoiler chooses $(k_0, k_1)$ as the next configuration, or the move continues as follows.

- Spoiler choses $j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon}$ with $i_{1-\varepsilon} < j_\varepsilon < k_\varepsilon$.
- Duplicator chooses $j_\varepsilon \in \mathbb{T}_\varepsilon$ with $i_\varepsilon < j_\varepsilon < k_\varepsilon$. Spoiler wins if there is no such $j_\varepsilon$.

The next configuration is $(j_0, j_1)$.

**$\mathsf{SS}$-move.** The definition is symetric, substituting $>$ for $<$.

Spoiler wins if either duplicator cannot answer during a move, or a configuration such that $(w_0, i_0) \not\equiv_0 (w_1, i_1)$ is reached. Otherwise, Duplicator wins.

We say that Duplicator has a *winning strategy* in the $k$-round EF-game starting from $(w_0, i_0, w_1, i_1)$ if she can win all plays starting from this configuration. This is denoted $(w_0, i_0) \sim_k (w_1, i_1)$.

We say that Spoiler has a winning strategy in the $k$-round EF-game starting from $(w_0, i_0, w_1, i_1)$ if he can win all plays starting from this configuration. The game is determined: from each initial configuration, either Spoiler or Duplicator has a winning strategy.

1. Let $w_0 = \{p\}\{p\}\{p\}\{p\}\{q\}\{q\}\{r\}$, and $w_1 = \{p\}\{p\}\{p\}\{q\}\{p\}\{q\}\{r\}$. Does Duplicator have a winning strategy in the $k$-round EF-game starting from $(w_0, 0)$, $(w_1, 0)$ for $k = 0$ ? $k = 1$ ? $k = 2$ ?

2. Show that $(w_0, i_0) \equiv_k (w_1, i_1)$ iff $(w_0, i_0) \sim_k (w_1, i_1)$.

3. Show that $p \,\mathsf{SU}\, (q \,\mathsf{SU}\, r)$ cannot be expressed with a formula of temporal depth at most 1.

4. Show that on finite linear time flows, the property of being of even length cannot be expressed in $\mathrm{TL}(\mathrm{AP}, \mathsf{SU}, \mathsf{SS})$.

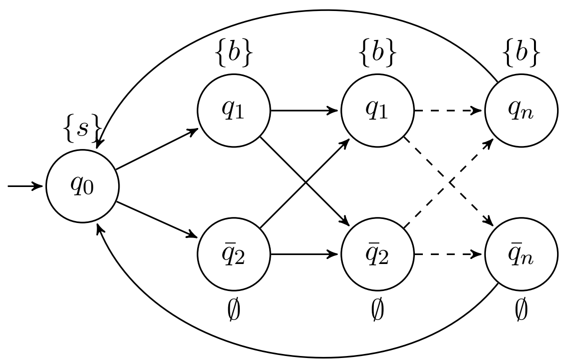
5. Propose a definition for $\mathsf{U}$-moves and $\mathsf{S}$-moves, and show that $\mathsf{SU}$ is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{U}, \mathsf{S})$ over the time flow $(\mathbb{N}, <)$.

6. Show that $\mathsf{SU}$ is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{U}, \mathsf{S})$ over the time flow $(\mathbb{R}, <)$.

7. Propose a definition for $\mathsf{SF}$-moves and $\mathsf{SP}$-moves, and show that $\mathsf{SU}$ is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{SP}, \mathsf{SF})$ over the time flow $(\mathbb{N}, <)$.

**Exercise 2** (Hardness of $\mathrm{LTL}(\mathsf{X}, \mathsf{F})$)**.** Adapt the proof given during the lecture to show that $\mathsf{MC}^{\exists}(\mathsf{X}, \mathsf{F})$ is PSPACE-hard.

As a preliminary question, consider the following Kripke structure $M$ over $\mathrm{AP} = \{s, b\}$:




Any infinite word $\sigma$ generated by $M$ is in $\big(\{s\}(\{b\}+\emptyset)^n\big)^\omega$, where each segment between two $s$'s can be seen as describing a value from 0 to $2^n - 1$ encoded in binary. Provide an $\mathrm{LTL}(\mathsf{X}, \mathsf{F})$ formula $\varphi$ that selects runs $\rho$ where the successive values form the sequence $0, 1, \ldots, 2^n - 1, 0, 1, \ldots$, i.e. count modulo $2^n$.

**Exercise 3** (Linear Orders with Gaps)**.** In this exercise we assume $(\mathbb{T}, <)$ to be a linear time flow.

1. Let us define a new unary "gap" modality $\mathsf{gap}$:

$$\begin{aligned} w,i \models \mathsf{gap}\varphi \text{ iff } \forall k.k > i \to (&\exists \ell.k < \ell \wedge \forall j.i < j < \ell \to w,j \models \varphi)\\ &\vee (\exists j.i < j < k \wedge w,j \models \neg\varphi)\\ \wedge\ &\exists k_1.k_1 > i \wedge \forall j.i < j \leq k_1 \to w,j \models \varphi\\ \wedge\ &\exists k_2.k_2 > i \wedge w,k_2 \models \neg\varphi\ . \end{aligned}$$

The intuition behind $\mathsf{gap}$ is that $\varphi$ should hold for some time until a gap occurs in the time flow, after which $\neg\varphi$ holds at points arbitrarily close to the gap.

   (a) Express $\mathsf{gap}\varphi$ using the standard $\mathsf{SU}$ modality.

   (b) Show that, if $(\mathbb{T},<)$ is Dedekind-complete, then $\mathsf{gap}p$ for $p \in \mathrm{AP}$ cannot be satisfied.

2. Consider the temporal flow $(\{0\} \times \mathbb{Z}_{<0} \times \mathbb{Z} \cup \{1\} \times \mathbb{Z} \times \mathbb{Z}, <)$ where $<$ is the lexicographic ordering and $\mathrm{AP} = \{p\}$. Let $n$ be an even integer in $\mathbb{Z}$, and define

$$\begin{aligned} h_0(p) &= \{(0,i,j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1,i,j) \in \mathbb{T} \mid i \text{ is odd}\}\\ h_1(p) &= \{(0,i,j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1,i,j) \in \mathbb{T} \mid i > n \text{ is odd}\}\ . \end{aligned}$$

   (a) Show that $w_0,(x,i,j) \models \mathsf{gap}p$ for any $x \in \{0,1\}$, odd $i$, and $j$.

   (b) Show that no $\mathrm{TL}(\{p\},\mathsf{SS},\mathsf{SU})$ formula can distinguish between $(w_0,(0,-1,0))$ and $(w_1,(0,-1,0))$.

   (c) Here is the definition of the Stavi "until" modality:

$$\begin{aligned} w,i \models \varphi \,\overline{\mathsf{U}}\, \psi \text{ iff } \exists \ell.i < \ell &\\ \wedge \forall k.i < k < \ell \to\ &[\exists j_1.k < j_1 \wedge \forall j.i < j < j_1 \to w,j \models \varphi]\\ &\vee [(\forall j_2.k < j_2 < \ell \to w,j_2 \models \psi)\\ &\quad \wedge (\exists j_3.i < j_3 < k \wedge w,j_3 \models \neg\varphi)]\\ \wedge \exists k_1.i < k_1 < \ell \wedge\ & w,k_1 \models \neg\varphi\\ \wedge \exists k_2.i < k_2 < \ell \wedge\ & \forall j.i < j < k_2 \to w,j \models \varphi \end{aligned}$$

   This modality is quite similar to $\mathsf{gap}\varphi$, but further requires $\psi$ to hold for some time after the gap (the "$j_2$" condition above).

   Show that $w_1,(0,-1,0) \models p \,\overline{\mathsf{U}}\, \neg\,\mathsf{gap}\,p$ but $w_0,(0,-1,0) \not\models p \,\overline{\mathsf{U}}\, \neg\,\mathsf{gap}\,p$.