**Exercise 1** (Ehrenfeucht-Fraïssé Games). Let \( w_0 = (\mathbb{T}_0, <, h_0) \) and \( w_1 = (\mathbb{T}_1, <, h_1) \) be two temporal structures. Let \( i_0 \in \mathbb{T}_0 \) and \( i_1 \in \mathbb{T}_1 \). Let \( k \in \mathbb{N} \). We say that \( (w_0, i_0) \) and \( (w_1, i_1) \) are \( k \)-equivalent, denoted \( (w_0, i_0) \equiv_k (w_1, i_1) \), if they satisfy the same formulas in \( \text{TL}(\text{AP}, \text{SU}, \text{SS}) \) of temporal depth at most \( k \).

This can also be described through **Ehrenfeucht-Fraïssé games** (EF-games). EF-games for \( \text{TL}(\text{AP}, \text{SU}, \text{SS}) \) are defined as follows. The game has two players: **Spoiler** and **Duplicator**, and the **game board** consists of two temporal structures \( w_0 = (\mathbb{T}_0, <, h_0) \) and \( w_1 = (\mathbb{T}_1, <, h_1) \). There are two tokens, one on each structure: \( i_0 \in \mathbb{T}_0 \) and \( i_1 \in \mathbb{T}_1 \). A **configuration** is a tuple \( (w_0, i_0, w_1, i_1) \), or simply \( (i_0, i_1) \) if the game board is understood. Intuitively, Duplicator will try to prove that \( (w_0, i_0) \) and \( (w_1, i_1) \) are equivalent, while Spoiler will try to distinguish them. Let \( k \in \mathbb{N} \). The \( k \)-round EF-game from a configuration proceeds with (at most) \( k \) moves. There are two available moves for \( \text{TL}(\text{AP}, \text{SU}, \text{SS}) \): **SU-moves**, or **SS-moves**. Spoiler choses which move is played in each round.

**SU-move.**

- Spoiler chooses one of the two structures, i.e., \( \varepsilon \in \{0, 1\} \), and a position \( k_\varepsilon \in \mathbb{T}_\varepsilon \) such that \( i_\varepsilon < k_\varepsilon \).
- Duplicator chooses \( k_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon} \) such that \( i_{1-\varepsilon} < k_{1-\varepsilon} \). Spoiler wins if there is no such \( k_{1-\varepsilon} \).

Either Spoiler chooses \( (k_0, k_1) \) as the next configuration, or the move continues as follows.

- Spoiler choses \( j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon} \) with \( i_{1-\varepsilon} < j_\varepsilon < k_\varepsilon \).
- Duplicator chooses \( j_\varepsilon \in \mathbb{T}_\varepsilon \) with \( i_\varepsilon < j_\varepsilon < k_\varepsilon \). Spoiler wins if there is no such \( j_\varepsilon \).

The next configuration is \( (j_0, j_1) \).

**SS-move.** The definition is symmetric, substituting \( > \) for \( < \).

Spoiler wins if either duplicator cannot answer during a move, or a configuration such that \( (w_0, i_0) \not\equiv (w_1, i_1) \) is reached. Otherwise, Duplicator wins.

We say that Duplicator has a **winning strategy** in the \( k \)-round EF-game starting from \( (w_0, i_0, w_1, i_1) \) if she can win all plays starting from this configuration. This is denoted \( (w_0, i_0) \sim_k (w_1, i_1) \).

We say that Spoiler has a winning strategy in the \( k \)-round EF-game starting from \( (w_0, i_0, w_1, i_1) \) if he can win all plays starting from this configuration. The game is determined: from each initial configuration, either Spoiler or Duplicator has a winning strategy.

1. Let \( w_0 = \{p\}{p}\{p\}{p}\{q\}{q}\{r\} \), and \( w_1 = \{p\}{p}\{p\}{p}\{q\}{q}\{q\}\{r\} \). Does Duplicator have a winning strategy in the \( k \)-round EF-game starting from \( (w_0, 0), (w_1, 0) \) for \( k = 0 \)? \( k = 1 \)? \( k = 2 \)?
2. Show that \((w_0, i_0) \equiv_k (w_1, i_1)\) iff \((w_0, i_0) \sim_k (w_1, i_1)\).

3. Show that \(p \text{ SU } (q \text{ SU } r)\) cannot be expressed with a formula of temporal depth at most 1.

4. Show that on finite linear time flows, the property of being of even length cannot be expressed in \(\text{TL}(\text{AP, SU, SS})\).

5. Propose a definition for \(U\)-moves and \(S\)-moves, and show that \(\text{SU}\) is not expressible in \(\text{TL}(\text{AP, U, S})\) over the time flow \((\mathbb{N}, <)\).

6. Show that \(\text{SU}\) is not expressible in \(\text{TL}(\text{AP, U, S})\) over the time flow \((\mathbb{R}, <)\).

7. Propose a definition for \(SF\)-moves and \(SP\)-moves, and show that \(\text{SU}\) is not expressible in \(\text{TL}(\text{AP, SP, SF})\) over the time flow \((\mathbb{N}, <)\).

**Exercise 2** (Hardness of \(\text{LTL}(X, F)\)). Adapt the proof given during the lecture to show that \(\text{MC}^3(X, F)\) is \(\text{PSpace}\)-hard.

As a preliminary question, consider the following Kripke structure \(M\) over \(\text{AP} = \{s, b\}\):

![Kripke Diagram](image)

Any infinite word \(\sigma\) generated by \(M\) is in \((\{s\}(\{b\} + \emptyset)^n)^\omega\), where each segment between two \(s\)'s can be seen as describing a value from 0 to \(2^n - 1\) encoded in binary. Provide an \(\text{LTL}(X, F)\) formula \(\varphi\) that selects runs \(\rho\) where the successive values form the sequence 0, 1, \ldots, \(2^n - 1\), 0, 1, \ldots, i.e. count modulo \(2^n\).

**Exercise 3** (Linear Orders with Gaps). In this exercise we assume \((T, <)\) to be a linear time flow.
1. Let us define a new unary “gap” modality \( \text{gap} \):

\[
w, i \models \text{gap} \varphi \text{ iff } \forall k. k > i \rightarrow (\exists \ell. \ell < k \land \forall j. i < j < \ell \rightarrow w, j \models \varphi) \\
\lor (\exists j. i < j < k \land w, j \models \neg \varphi) \\
\land \exists k_1. k_1 > i \land \forall j. i < j < k_1 \rightarrow w, j \models \varphi \\
\land \exists k_2. k_2 > i \land w, k_2 \models \neg \varphi.
\]

The intuition behind \( \text{gap} \) is that \( \varphi \) should hold for some time until a gap occurs in the time flow, after which \( \neg \varphi \) holds at points arbitrarily close to the gap.

(a) Express \( \text{gap} \varphi \) using the standard SU modality.

(b) Show that, if \((\mathbb{T}, <)\) is Dedekind-complete, then \( \text{gap} p \) for \( p \in \text{AP} \) cannot be satisfied.

2. Consider the temporal flow \( \{(0) \times \mathbb{Z} \land 0 \leq \mathbb{Z} \cup (1) \times \mathbb{Z} \times \mathbb{Z}, <\) where \( < \) is the lexicographic ordering and \( \text{AP} = \{p\} \). Let \( n \) be an even integer in \( \mathbb{Z} \), and define

\[
h_0(p) = \{(0, i, j) \in \mathbb{T} | i \text{ is odd}\} \cup \{(1, i, j) \in \mathbb{T} | i \text{ is odd}\}
\]

\[
h_1(p) = \{(0, i, j) \in \mathbb{T} | i \text{ is odd}\} \cup \{(1, i, j) \in \mathbb{T} | i > n \text{ is odd}\}.
\]

(a) Show that \( w_0, (x, i, j) \models \text{gap} p \) for any \( x \in \{0, 1\} \), odd \( i \), and \( j \).

(b) Show that no \( \text{TL}(\{p\}, \text{SS}, \text{SU}) \) formula can distinguish between \((w_0, (0, -1, 0))\) and \((w_1, (0, -1, 0))\).

(c) Here is the definition of the Stavi “until” modality:

\[
w, i \models \varphi \mathcal{U} \psi \text{ iff } \exists \ell. i < \ell \\
\land \forall k. i < k < \ell \rightarrow [\exists j_1. k < j_1 \land \forall j. i < j < j_1 \rightarrow w, j \models \varphi] \\
\lor [(\exists j_2. k < j_2 < \ell \rightarrow w, j_2 \models \psi) \\
\land (\exists j. i < j < k \land w, j \models \neg \varphi)] \\
\land \exists k_1. i < k_1 < \ell \land w, k_1 \models \neg \varphi \\
\land \exists k_2. i < k_2 < \ell \land \forall j. i < j < k_2 \rightarrow w, j \models \varphi
\]

This modality is quite similar to \( \text{gap} \varphi \), but further requires \( \psi \) to hold for some time after the gap (the “\( j_2 \)” condition above).

Show that \( w_1, (0, -1, 0) \models p \mathcal{U} \neg \text{gap} p \) but \( w_0, (0, -1, 0) \not\models p \mathcal{U} \neg \text{gap} p \).