

## TD 4: Büchi Automata

**Exercise 1** (Generalized Acceptance Condition). A *generalized* Büchi automaton  $\mathcal{A} = (Q, \Sigma, I, T, (F_i)_{0 \leq i < n})$  has a finite set of accepting sets  $F_i$ . An infinite run  $\sigma$  of  $\mathcal{A}$  satisfies this generalized acceptance condition if each set  $F_i$  is visited infinitely often.

Show that for any generalized Büchi automaton, one can construct an equivalent Büchi automaton.

**Exercise 2** (Rational Languages). A *rational language*  $L$  of infinite words over  $\Sigma$  is a finite union

$$L = \bigcup X \cdot Y^\omega$$

where  $X$  is in  $\text{Rat}(\Sigma^*)$  and  $Y$  in  $\text{Rat}(\Sigma^+)$ . We denote the set of *rational* languages of infinite words by  $\text{Rat}(\Sigma^\omega)$ .

Show that  $\text{Rec}(\Sigma^\omega) = \text{Rat}(\Sigma^\omega)$ .

**Exercise 3** (Deterministic Büchi Automata). A Büchi automaton is *deterministic* if  $|I| \leq 1$ , and for each state  $q$  in  $Q$  and symbol  $a$  in  $\Sigma$ ,  $|\{(q, a, q') \in T \mid q' \in Q\}| \leq 1$ .

1. Give a nondeterministic Büchi automaton for the language  $L \subseteq \{a, b\}^\omega$  described by the expression  $(a + b)^* a^\omega$ , and a deterministic Büchi automaton for  $\bar{L}$ .
2. Show that there does not exist any deterministic Büchi automaton for  $L$ .
3. Let  $\mathcal{A} = (Q, \Sigma, T, q_0, F)$  be a finite deterministic automaton that recognizes the language of finite words  $L \subseteq \Sigma^*$ . We can also interpret  $\mathcal{A}$  as a deterministic Büchi automaton with a language  $L' \subseteq \Sigma^\omega$ ; our goal here is to relate the languages of finite and infinite words defined by  $\mathcal{A}$ .

Let the *limit* of a language  $L \subseteq \Sigma^*$  be

$$\vec{L} = \{w \in \Sigma^\omega \mid w \text{ has infinitely many prefixes in } L\}.$$

Characterize the language  $L'$  of infinite words of  $\mathcal{A}$  in terms of its language of finite words  $L$  and of the limit operation.

**Exercise 4** (Closure by Complementation). The purpose of this exercise is to prove that  $\text{Rec}(\Sigma^\omega)$  is closed under complement. We consider for this a Büchi automaton  $\mathcal{A} = (Q, \Sigma, T, I, F)$ , and want to prove that its complement language  $\overline{L(\mathcal{A})}$  is in  $\text{Rec}(\Sigma^\omega)$ .

We write  $q \xrightarrow{u} q'$  for  $q, q'$  in  $Q$  and  $u = a_1 \cdots a_n$  in  $\Sigma^*$  if there exists a sequence of states  $q_0, \dots, q_n$  such that  $q_0 = q$ ,  $q_n = q'$  and for all  $0 \leq i < n$ ,  $(q_i, a_{i+1}, q_{i+1})$  is in  $T$ .

We write in the same way  $q \xrightarrow{u}_F q'$  if furthermore at least one of the states  $q_0, \dots, q_n$  belongs to  $F$ .

We define the *congruence*  $\sim_{\mathcal{A}}$  over  $\Sigma^*$  by

$$u \sim_{\mathcal{A}} v \text{ iff } \forall q, q' \in Q, (q \xrightarrow{u} q' \Leftrightarrow q \xrightarrow{v} q') \text{ and } (q \xrightarrow{u}_F q' \Leftrightarrow q \xrightarrow{v}_F q').$$

1. Show that  $\sim_{\mathcal{A}}$  has finitely many congruence classes  $[u]$ , for  $u$  in  $\Sigma^*$ .
2. Show that each  $[u]$  for  $u$  in  $\Sigma^*$  is in  $\text{Rec}(\Sigma^*)$ , i.e. is a regular language of finite words.
3. Consider the language  $K(L)$  for  $L \subseteq \Sigma^\omega$

$$K(L) = \bigcup_{\substack{u, v \in \Sigma^* \\ [u][v]^\omega \cap L \neq \emptyset}} [u][v]^\omega$$

Show that  $K(L)$  is in  $\text{Rec}(\Sigma^\omega)$  for any  $L \subseteq \Sigma^\omega$ .

4. Show that  $K(L(\mathcal{A})) \subseteq L(\mathcal{A})$  and  $K(\overline{L(\mathcal{A})}) \subseteq \overline{L(\mathcal{A})}$ .
5. Prove that for any infinite word  $\sigma$  in  $\Sigma^\omega$  there exist  $u$  and  $v$  in  $\Sigma^*$  such that  $\sigma$  belongs to  $[u][v]^\omega$ . The following theorem might come in handy when applied to couples of positions  $(i, j)$  inside  $\sigma$ :

**Theorem 1** (Ramsey, infinite version). *Let  $E = \{(i, j) \in \mathbb{N}^2 \mid i < j\}$ , and  $c : E \rightarrow \{1, \dots, k\}$  a  $k$ -coloring of  $E$ . There exists an infinite set  $A \subseteq \mathbb{N}$  and a color  $i \in \{1, \dots, k\}$  such that for all  $(n, m) \in A^2$  with  $n < m$ ,  $c(n, m) = i$ .*

6. Conclude.

**Exercise 5** (Muller Automata). A nondeterministic Muller automaton is a tuple  $\mathcal{A} = (Q, \Sigma, I, T, \mathcal{F})$ , where  $Q, \Sigma, I, T$  are as for Büchi automata and  $\mathcal{F} \subseteq 2^Q$  is the acceptance condition. For a run  $\sigma$  of  $\mathcal{A}$ , denote by  $\text{Inf}(\sigma)$  the set of states which are visited infinitely often. A run  $\sigma$  is accepting if  $\text{Inf}(\sigma) \in \mathcal{F}$ .

1. Give a *deterministic* Muller automaton for the language  $(a + b)^* a^\omega$ .
2. Show that for any Muller automaton  $\mathcal{A}$ ,  $L(\mathcal{A})$  is  $\omega$ -regular.
3. Show that any  $\omega$ -regular language is accepted by some (nondeterministic) Muller automaton.

Remark: in fact, any  $\omega$ -regular language can be recognized by some *deterministic* Muller automaton.