TD 3: CTL, CTL*

**Exercise 1** (Equivalences). Are the following formulæ equivalent?

1. $A X A G \varphi$ and $A X G \varphi$
2. $EXE G \varphi$ and $EX G \varphi$
3. $A(\varphi \land \psi)$ and $A \varphi \land A \psi$
4. $E(\varphi \land \psi)$ and $E \varphi \land E \psi$
5. $\neg A(\varphi \Rightarrow \psi)$ and $E(\varphi \land \neg \psi)$

**Exercise 2** (Semantics of CTL*). Compute $\llbracket \varphi \rrbracket$, where:

$$
\begin{align*}
M &= \{q\} \\
1 &\xrightarrow{} 2 \\
3 &\xrightarrow{} 4 \\
4 &\xrightarrow{} 5 \\
\{p\} &\xrightarrow{} \{p, q\} \\
\{p, q\} &\xrightarrow{} \{q\}
\end{align*}
$$

$\varphi = A[(Xq) \lor FA((EFGp)U(AGq))]$

**Exercise 3** (CTL Model-Checking). Let $M = (S, T, I, AP, \ell)$ be a finite Kripke structure, and $\varphi$ a CTL formula.

1. Let $M_\varphi$ be the restriction of $M$ to states satisfying $\varphi$: $M_\varphi = (\llbracket \varphi \rrbracket, T \cap \llbracket \varphi \rrbracket, I \cap \llbracket \varphi \rrbracket, AP, \ell_{\llbracket \varphi \rrbracket})$.
   Show that $s \in \llbracket EG \varphi \rrbracket$ iff there exists a non-trivial strongly connected component $C$ of $M_\varphi$ and $t \in C$ such that $s \rightarrow^* t$ in $M_\varphi$.

2. Deduce an algorithm to compute $\llbracket EG \varphi \rrbracket$ from $M$ and $\llbracket \varphi \rrbracket$. What is the complexity of your procedure?

**Exercise 4** (CTL+). CTL+ extends CTL by allowing boolean connectives on path formulæ, according to the following abstract syntax:

$$
\begin{align*}
f &:= \top | a | f \land g | \neg f | E \varphi | A \varphi \quad \text{(state formulæ $f, g$)} \\
\varphi &:= \varphi \land \psi | \neg \varphi | X f | f U g \quad \text{(path formulæ $\varphi, \psi$)}
\end{align*}
$$

where $a$ is an atomic proposition. The associated semantics is that of CTL*.

We want to prove that, for any CTL+ formula, there exists an equivalent CTL formula.
1. Give an equivalent CTL formula for
\[ E((a_1 U b_1) \land (a_2 U b_2)) \].

2. Generalize your translation for any formula of form
\[ E \left( \bigwedge_{i=1,...,n} (\psi_i U \psi'_i) \land G \varphi \right) . \tag{1} \]
What is the complexity of your translation?

3. Give an equivalent CTL formula for the following CTL\(^+\) formula:
\[ E(X a \land (b U c)) . \]

4. Using subformulae of form (1) and E modalities, give an equivalent CTL formula to
\[ E(X \varphi \land \bigwedge_{i=1,...,n} (\psi_i U \psi'_i) \land G \varphi') . \tag{2} \]
What is the complexity of your translation?

5. We only have to transform any CTL\(^+\) formula into (nested) disjuncts of form (2). Detail this translation for the following formula:
\[ A((F a \lor X a \lor X \neg b \lor F \neg d) \land (d U \neg c)) . \]