Homework 7

To hand in on November 22th at the beginning of the exercise session, or by mail (before 14:00) at `marie.fortin@lsv.fr`.

Answers can be written in french or in english.

Exercise 1. Fix a set of atomic propositions AP, and $\Sigma = 2^{\mathrm{AP}}$. Recall that $\sigma, \rho \in \Sigma^\omega$ are *stuttering equivalent*, written $\sigma \sim \rho$, when there exist infinite integer sequences $0 = i_0 < i_1 < \cdots$ and $0 = k_0 < k_1 < \cdots$ such that for all $\ell \geq 0$,

$$\sigma(i_\ell) = \sigma(i_\ell + 1) = \cdots = \sigma(i_{\ell+1} - 1) = \rho(k_\ell) = \rho(k_\ell + 1) = \cdots = \rho(k_{\ell+1} - 1)\,,$$

where $\sigma(i) \in \Sigma$ denotes the letter at position i in σ.

A language $L \subseteq \Sigma^\omega$ is *stutter-invariant* if for all stuttering equivalent words $\sigma, \rho \in \Sigma^\omega$, we have $\sigma \in L$ if and only if $\rho \in L$.

1. Show that if φ is an $\mathrm{LTL}(\mathrm{AP}, \mathsf{U})$ formula, then $L(\varphi) = \{\sigma \in \Sigma^\omega \mid \sigma, 0 \models \varphi\}$ is stutter-invariant.

A word $\sigma \in \Sigma^\omega$ is *stutter-free* if, for all $i \in \mathbb{N}$, either $\sigma(i) \neq \sigma(i+1)$, or $\sigma(i) = \sigma(j)$ for all $j \geq i$. Notice that if σ is stutter-free, any suffix of σ is also stutter-free.

2. Show that for all $\sigma \in \Sigma^\omega$, there exists a unique $\sigma' \in \Sigma^\omega$ such that σ' is stutter-free and $\sigma \sim \sigma'$.
3. Given $a \in \Sigma$, we write a for the formula $\bigwedge_{p \in a} p \wedge \bigwedge_{p \notin a} \neg p$. That is, $\sigma, i \models a$ if and only if $\sigma(i) = a$.
   (a) Give a formula $\psi_{a,a}$ in $\mathrm{LTL}(\mathrm{AP}, \mathsf{U})$ such that for all *stutter-free* words $\sigma \in \Sigma^\omega$, we have $\sigma, 0 \models \psi_{a,a}$ if and only if $\sigma, 0 \models a \wedge \mathsf{X}\, a$.
   (b) Let $a, b \in \Sigma$ with $a \neq b$. Give a formula $\psi_{a,b}$ in $\mathrm{LTL}(\mathrm{AP}, \mathsf{U})$ such that for all *stutter-free* words $\sigma \in \Sigma^\omega$, we have $\sigma, 0 \models \psi_{a,b}$ if and only if $\sigma, 0 \models a \wedge \mathsf{X}\, b$.
4. Let φ be any $\mathrm{LTL}(\mathrm{AP}, \mathsf{X}, \mathsf{U})$ formula. Construct by induction on φ an $\mathrm{LTL}(\mathrm{AP}, \mathsf{U})$ formula $\tau(\varphi)$ such that for all *stutter-free* words $\sigma \in \Sigma^\omega$, we have $\sigma, 0 \models \varphi$ iff $\sigma, 0 \models \tau(\varphi)$.
5. Let φ be an $\mathrm{LTL}(\mathrm{AP}, \mathsf{X}, \mathsf{U})$ formula such that $L(\varphi)$ is stutter-invariant. Show that $L(\varphi) = L(\tau(\varphi))$.