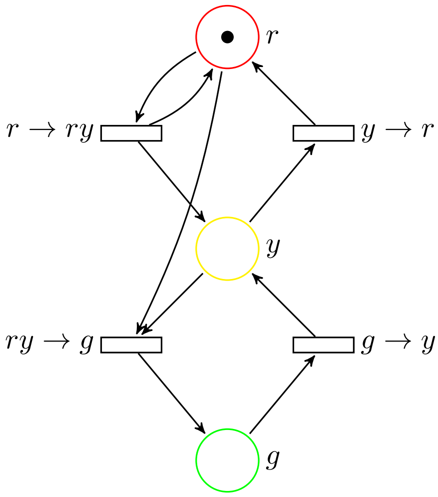
TD 12: Petri Nets

Exercise 1 (Traffic Lights)**.** Consider again the traffic lights example from the lecture notes:



1. How can you correct this Petri net to avert unwanted behaviours (like $r \to ry \to rr$) in a 1-safe manner?

2. Extend your Petri net to model two traffic lights handling a street intersection.

Exercise 2 (Producer/Consumer)**.** A producer/consumer system gathers two types of processes:

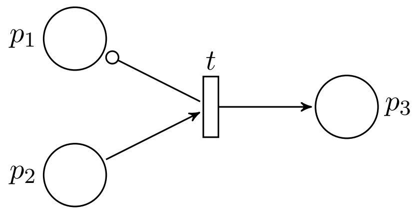
producers who can make the actions *produce* (p) or *deliver* (d), and

consumers with the actions *receive* (r) and *consume* (c).

All the producers and consumers communicate through a single unordered channel.

1. Model a producer/consumer system with two producers and three consumers. How can you modify this system to enforce a maximal capacity of ten simultaneous items in the channel?

2. An *inhibitor arc* between a place p and a transition t makes t firable only if the current marking at p is zero. In the following example, there is such an inhibitor arc between p_1 and t. A marking $(0, 2, 1)$ allows to fire t to reach $(0, 1, 2)$, but $(1, 1, 1)$ does not allow to fire t.



Using inhibitor arcs, enforce a priority for the first producer and the first consumer on the channel: the other processes can use the channel only if it is not currently used by the first producer and the first consumer.

Exercise 3 (Model Checking Petri Nets)**.** Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider as usual propositional LTL, with a set of atomic propositions AP equal to P the set of places of the Petri net. We define proposition p to hold in a marking m in $\mathbb{N}^P$ if $m(p) > 0$.

The models of our LTL formulæ are *computations* $m_0 m_1 \cdots$ in $(\mathbb{N}^P)^\omega$ such that, for all $i \in \mathbb{N}$, $m_i \to_{\mathcal{N}} m_{i+1}$ is a transition step of the Petri net $\mathcal{N}$.

1. We want to prove that state-based LTL model checking can be performed in polynomial space for 1-safe Petri nets. For this, prove that one can construct an exponential-sized Büchi automaton $\mathcal{B}_{\mathcal{N}}$ from a 1-safe Petri net that recognizes all the infinite computations of $\mathcal{N}$ starting in m_0.

2. In the general case, state-based LTL model checking is undecidable. Prove it for Petri nets with at least two unbounded places, by a reduction from the halting problem for 2-counter Minsky machines.

3. We consider now a different set of atomic propositions, such that $\Sigma = 2^{\mathrm{AP}}$, and a labeled Petri net, with a labeling homomorphism $\lambda : T \to \Sigma$. The models of our LTL formulæ are infinite words $a_0 a_1 \cdots$ in Σ^ω such that $m_0 \xrightarrow{t_0}_{\mathcal{N}} m_1 \xrightarrow{t_1}_{\mathcal{N}} m_2 \cdots$ is an execution of $\mathcal{N}$ and $\lambda(t_i) = a_i$ for all i.

   Prove that *action-based* LTL model checking can be performed in polynomial space for labeled 1-safe Petri nets.

Exercise 4 (VASS)**.** An n-dimensional *vector addition system with states* (VASS) is a tuple $\mathcal{V} = \langle Q, \delta, q_0 \rangle$ where Q is a finite set of states, $q_0 \in Q$ the initial state, and $\delta \subseteq Q \times \mathbb{Z}^n \times Q$ the transition relation. A configuration of $\mathcal{V}$ is a pair (q, v) in $Q \times \mathbb{N}^n$. An execution of $\mathcal{V}$ is a sequence of configurations $(q_0, v_0)(q_1, v_1) \cdots (q_m, v_m)$ such that $v_0 = \bar{0}$, and for $0 < i \leq m$, $(q_{i-1}, v_i - v_{i-1}, q_i)$ is in δ.

1. Show that any VASS can be simulated by a Petri net.

2. Show that, conversely, any Petri net can be simulated by a VASS.

Exercise 5 (Dickson's Lemma). A *quasi-order* $(A, \leq)$ is a set A endowed with a reflexive and transitive ordering relation $\leq$. A *well quasi order* (wqo) is a quasi order $(A, \leq)$ s.t., for any infinite sequence $a_0 a_1 \cdots$ in A^ω, there exist indices $i < j$ with $a_i \leq a_j$.

1. Let $(A, \leq)$ be a wqo and $B \subseteq A$. Show that $(B, \leq)$ is a wqo.
2. Show that $(\mathbb{N} \uplus \{\omega\}, \leq)$ is a wqo.
3. Let $(A, \leq)$ be a wqo. Show that any infinite sequence $a_0 a_1 \cdots$ in A^ω embeds an infinite increasing subsequence $a_{i_0} \leq a_{i_1} \leq a_{i_2} \leq \cdots$ with $i_0 < i_1 < i_2 < \cdots$.
4. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two wqo's. Show that the cartesian product $(A \times B, \leq_\times)$, where the product ordering is defined by $(a, b) \leq_\times (a', b')$ iff $a \leq_A a'$ and $b \leq_B b'$, is a wqo.

Exercise 6 (Coverability Graphs).

1. The construction of coverability graphs, as defined in the lecture slides, is not entirely deterministic: e.g., the order in which nodes are taken from the worklist is undefined. Give an example of a net $\mathcal{N}$ and two possible coverability graphs of $\mathcal{N}$ that are non-isomorphic to each other. In each case, indicate the order in which nodes were treated in the worklist.
2. A marking of a net $\mathcal{N}$ is said to be a *deadlock* if no transition can fire in it. Clearly, $\mathcal{N}$ contains a reachable deadlock iff the reachability graph of $\mathcal{N}$ contains a node with no outgoing edges. Can the same be said of $\mathcal{N}$ and any of its coverability graphs?