TD 11: Simulation and Bisimulation

Exercise 1 (Bisimulations). Consider the following Kripke structures:

For each couple of structures, exhibit a bisimulation relation if they are bisimilar, or a CTL* formula allowing to distinguish between them if they are not bisimilar.

Exercise 2 (Computing the Coarsest Bisimulation). Computing $\equiv$ on a single Kripke structure is very similar to the computation of a minimal DFA.

1. Design a partition refinement algorithm for computing $\equiv$, i.e. an algorithm that computes an initial relation $\equiv_0$ and refines it successively until $\equiv_k = \equiv$ for some $k$. Prove that your algorithm terminates and computes $\equiv$.

2. Apply your algorithm to the union of two bisimilar systems from the previous exercise and draw the quotiented system.

Exercise 3 (Simulations). Show that $\preceq$ is reflexive and transitive. Is it symmetric?
Exercise 4 (Simulation Quotienting). Two Kripke structures $M_1$ and $M_2$ are simulation equivalent, noted $M_1 \simeq M_2$ if $M_1 \preceq M_2$ and $M_2 \preceq M_1$. The lecture notes provide an example of two simulation equivalent but not bisimilar structures. Consider now the two following structures $M_s$ and $M_t$:

1. Which of the following relations hold: $M_s \preceq M_t$, $M_t \preceq M_s$, $M_s \simeq M_t$?

2. Construct the quotient of $(M_s \cup M_t)$ by $\simeq$. Is the resulting system bisimilar to $(M_s \cup M_t)$?

3. Prove that if $M/\simeq$ is the quotient of $M$ by $\simeq$, then $M/\simeq \preceq M$ and $M \preceq M/\simeq$.

4. Call a Kripke structure $M = (S, \rightarrow, s_0, AP, \nu)$ AP-deterministic if for each state $s$, if there exist two transitions $s \rightarrow s_1$ and $s \rightarrow s_2$ with $\nu(s_1) = \nu(s_2)$, then $s_1 = s_2$.

Show that, if two Kripke structures $M_1$ and $M_2$ are AP-deterministic, then they are bisimilar iff they are simulation equivalent.

Exercise 5 (Logical Characterization). Let us define existential CTL$^*$ as the fragment of CTL$^*$ defined by the following abstract syntax, where $p$ ranges over the set of atomic propositions $AP$:

$\varphi ::= \top | \bot | p | \neg p | \varphi \land \varphi | \varphi \lor \varphi | E \psi$  
$(state \ formulæ)$

$\psi ::= \varphi | X \psi | \psi \land \psi | \psi \lor \psi | \psi U \psi | \psi R \psi \quad \psi$  
$(path \ formulæ)$

Existential CTL$^*$ includes both LTL and existential CTL (hereafter noted ECTL), which is defined by the following abstract syntax:

$\varphi ::= \top | \bot | p | \neg p | \varphi \land \varphi | \varphi \lor \varphi | EX \varphi | E(\varphi U \varphi) | E(\varphi R \varphi) \quad \varphi$  
$(state \ formulæ)$

Let us consider two (not necessarily different) Kripke structures $M_1 = \langle S_1, T_1, I_1, AP, l_1 \rangle$ and $M_2 = \langle S_2, T_2, I_2, AP, l_2 \rangle$. We assume these structures to be total, where for any state $s$ there exists some state $s'$ such that $(s, s')$ is a transition.
1. Prove the following two statements, for any two states \( s_1 \) and \( s_2 \), and any two infinite paths \( \pi_1 \) and \( \pi_2 \) in \( M_1 \) and \( M_2 \), resp.:

(a) if \( s_1 \preceq s_2 \), then for any existential CTL\(^*\) state formula \( \varphi \), \( s_1 \models \varphi \) implies \( s_2 \models \varphi \),

(b) if \( \pi_1 = s_{0,1}s_{1,1} \cdots \) and \( \pi_2 = s_{0,2}s_{1,2} \cdots \) with \( s_{i,1} \preceq s_{i,2} \) for all \( i \) in \( \mathbb{N} \), then for any existential CTL\(^*\) path formula \( \psi \), \( \pi_1 \models \psi \) implies \( \pi_2 \models \psi \).

2. Let us consider the following relation on \( S_1 \times S_2 \):

\[
\mathcal{F} = \{(s_1, s_2) \in S_1 \times S_2 | \forall \varphi \in \text{ECTL}, s_1 \models \varphi \Rightarrow s_2 \models \varphi\}.
\]

Assuming that for all initial states \( s \) in \( I_1 \), \( \mathcal{F}(s) \cap I_2 \) is not empty, show that \( \mathcal{F} \) is a simulation between \( M_1 \) and \( M_2 \).

3. Conclude by proving the following theorem:

**Theorem 1** (Logical Characterization of Simulation). Let \( M_1 = \langle S_1, T_1, I_1, AP, l_1 \rangle \) and \( M_2 = \langle S_2, T_2, I_2, AP, l_2 \rangle \) be two total Kripke structures and \( s_1 \) and \( s_2 \) be two states of \( S_1 \) and \( S_2 \) resp. The following three statements are equivalent:

1. \( s_1 \preceq s_2 \),

2. for all existential CTL\(^*\) formulae \( \varphi \): \( s_1 \models \varphi \) implies \( s_2 \models \varphi \),

3. for all existential CTL formulae \( \varphi \): \( s_1 \models \varphi \) implies \( s_2 \models \varphi \).