**TD 10: BDDs**

**Exercise 1** (Some BDDs). Draw the BDDs for the following functions, using the order of your choice on the variables \( \{x_1, x_2, x_3\} \):

1. \( (x_1 \Leftrightarrow x_2) \lor (x_1 \Leftrightarrow x_3) \),
2. the constant sum function \( s_c(x_1, x_2, x_3) \) for \( c = 1 \): its value is 1 iff \( c = \sum_{i=1}^{3} x_i \),
3. the hidden weighted bit function \( h(x_1, x_2, x_3) \): its value is that of variable \( x_s \), where \( s = \sum_{i=1}^{3} x_i \) and \( x_0 \) is defined as 0.

**Exercise 2** (Symmetric Functions). A symmetric function of \( n \) variables has the same value for all permutations of the same \( n \) tuple of arguments. Show that a BDD for a symmetric function has at most \( \frac{n(n+1)}{2} + 1 \) nodes (when omitting the 0-node).

**Exercise 3** (Counting Solutions). Write a linear time algorithm for counting the number of solutions of a boolean function \( f \) represented by a BDD, i.e. of the number of valuations \( \nu \) s.t. \( \nu \models f \).

**Exercise 4** (An Upper Bound on the Size of BDDs). The size \( B(f) \) of a BDD for a function \( f \) is defined as the number of its nodes. Consider an arbitrary boolean function \( f \) on the ordered set \( x_1 \cdots x_n \), and consider a variable \( x_k \).

1. Show that we can bound the number of nodes labeled by \( \{x_1, \ldots, x_k\} \) by \( 2^k - 1 \).
2. How many different subfunctions on the ordered set of variables \( x_{k+1} \cdots x_n \) exist?
   Deduce another bound for the number of nodes labeled by \( \{x_{k+1}, \ldots, x_n\} \).
3. What global bound do you obtain for \( k = n - \log_2(n - \log_2 n) \)?

**Exercise 5** (Finding the Optimal Order). There are in general \( n! \) different orders for the variables \( \{x_1, \ldots, x_n\} \), and building the BDD for each of these is computationally expensive. One can nevertheless design an exponential time algorithm for finding the optimal order. Indeed, an optimal ordering on a subset \( X \) of variables does not depend on the order in which \( X' = \{x_1, \ldots, x_n\} \setminus X \) has been accessed.

1. Fix a boolean function \( f \) over variables \( \{x_1, \ldots, x_n\} \). We assume that \( f \) is provided as a BDD \( B \) for the ordering \( x_1, x_2, \ldots, x_n \).
   Given a subset \( X \) of \( \{x_1, \ldots, x_n\} \) and a variable \( x \) in \( X \), how many nodes labeled by \( x \) does any BDD \( B' \) for \( f \) has if it first treats \( X' = \{x_1, \ldots, x_n\} \setminus X \), then \( x \), and last \( X \setminus \{x\} \)? How can you compute this number on the provided BDD \( B \) for \( f \)?
2. Reduce the optimal order problem to the search of a path of minimal weight in a weighted graph with subsets of \( \{x_1, \ldots, x_n\} \) as vertices.