TD 10: BDDs

Exercise 1 (Some BDDs)**.** Draw the BDDs for the following functions, using the order of your choice on the variables $\{x_1, x_2, x_3\}$:

1. $(x_1 \Leftrightarrow x_2) \vee (x_1 \Leftrightarrow x_3)$,
2. the constant sum function $s_c(x_1, x_2, x_3)$ for $c = 1$: its value is 1 iff $c = \sum_{i=1}^{3} x_i$,
3. the hidden weighted bit function $h(x_1, x_2, x_3)$: its value is that of variable x_s, where $s = \sum_{i=1}^{3} x_i$ and x_0 is defined as 0.

Exercise 2 (Symmetric Functions)**.** A *symmetric function* of n variables has the same value for all permutations of the same n tuple of arguments.

Show that a BDD for a symmetric function has at most $\frac{n(n+1)}{2} + 1$ nodes (when omitting the 0-node).

Exercise 3 (Counting Solutions)**.** Write a linear time algorithm for counting the number of solutions of a boolean function f represented by a BDD, i.e. of the number of valuations ν s.t. $\nu \models f$.

Exercise 4 (An Upper Bound on the Size of BDDs)**.** The size $B(f)$ of a BDD for a function f is defined as the number of its nodes. Consider an arbitrary boolean function f on the ordered set $x_1 \cdots x_n$, and consider a variable x_k.

1. Show that we can bound the number of nodes labeled by $\{x_1, \ldots, x_k\}$ by $2^k - 1$.
2. How many different subfunctions on the ordered set of variables $x_{k+1} \cdots x_n$ exist? Deduce another bound for the number of nodes labeled by $\{x_{k+1}, \ldots, x_n\}$.
3. What global bound do you obtain for $k = n - \log_2(n - \log_2 n)$?

Exercise 5 (Finding the Optimal Order)**.** There are in general $n!$ different orders for the variables $\{x_1, \ldots, x_n\}$, and building the BDD for each of these is computationally expensive. One can nevertheless design an exponential time algorithm for finding the optimal order. Indeed, an optimal ordering on a subset X of variables does not depend on the order in which $X' = \{x_1, \ldots, x_n\} \backslash X$ has been accessed.

1. Fix a boolean function f over variables $\{x_1, \ldots, x_n\}$. We assume that f is provided as a BDD B for the ordering $x_1, x_2, \ldots, x_n$.

   Given a subset X of $\{x_1, \ldots, x_n\}$ and a variable x in X, how many nodes labeled by x does any BDD B' for f has if it first treats $X' = \{x_1, \ldots, x_n\} \backslash X$, then x, and last $X \backslash \{x\}$? How can you compute this number on the provided BDD B for f?

2. Reduce the optimal order problem to the search of a path of minimal weight in a weighted graph with subsets of $\{x_1, \ldots, x_n\}$ as vertices.