TD 9: Pushdown Systems

Reminder:

A *pushdown system (PDS)* is a triple $\mathcal{P} = (P, \Gamma, \Delta)$, where P is a finite set of *control states*, Γ is a finite *stack alphabet*, and $\Delta \subseteq (P \times \Gamma) \times (P \times \Gamma^*)$ is a finite set of *rules*. We write $pA \hookrightarrow qw$ when $((p, A), (q, w)) \in \Delta$. We associate with a PDS $\mathcal{P}$ and an initial configuration $c_0 \in P \times \Gamma^*$ the transition system $\mathcal{T}_\mathcal{P} = (\mathit{Con}(\mathcal{P}), \to, c_0)$, where $\mathit{Con}(\mathcal{P}) = P \times \Gamma^*$ is the set of *configurations*, and $pAw' \to qww'$ for all $w' \in \Gamma^*$ iff $pA \hookrightarrow qw \in \Delta$. We write $pw \Rightarrow p'w'$ if there is a path from pw to $p'w'$ in $\mathcal{T}_\mathcal{P}$.

Let $\mathcal{P}$ be a PDS. A *$\mathcal{P}$-automaton* is a finite automaton $\mathcal{A} = (Q, \Gamma, P, T, F)$, where the alphabet of $\mathcal{A}$ is the stack alphabet Γ, and the initial states of $\mathcal{A}$ are the control states P. It is *normalized* if there are no transitions leading into initial states. We say that $\mathcal{A}$ *accepts* the configuration pw if $\mathcal{A}$ has a path labelled by input w starting at p and ending at some final state. We denote by $\mathcal{L}(\mathcal{A})$ be the set of configurations accepted by $\mathcal{A}$. A set C of configurations is called *regular* if there is some $\mathcal{P}$-automaton $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) = C$.

Given a set C of configurations of $\mathcal{P}$, we let

$$
\begin{aligned}
\mathit{pre}^*(C) &= \{c' \mid \exists c \in C : c' \Rightarrow c\} \\
\mathit{post}^*(C) &= \{c' \mid \exists c \in C : c \Rightarrow c'\}
\end{aligned}
$$

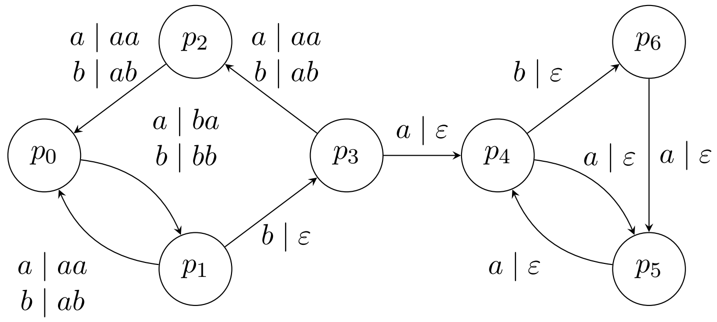
If C is regular, then so are $\mathit{pre}^*(C)$ and $\mathit{post}^*(C)$.

If $\mathcal{A}$ is a normalized $\mathcal{P}$-automaton accepting C, $\mathcal{A}$ can be transformed into an automaton accepting $\mathit{pre}^*(C)$ by applying the following saturation rule until no transition can be added:

> If $q \xrightarrow{w} r$ currently holds in $\mathcal{A}$ and $pA \hookrightarrow qw$ is a rule in $\mathcal{P}$, then add the transition (p, A, r) to $\mathcal{A}$.

The procedure for $\mathit{post}^*(C)$ is similar.

Exercise 1 (Computing $\mathit{pre}^*(C)$)**.** Consider the pushdown system represented below, with stack alphabet $\Gamma = \{a, b\}$.



Apply the algorithm described above to compute a $\mathcal{P}$-automaton accepting $pre^*(p_6 b^*)$.

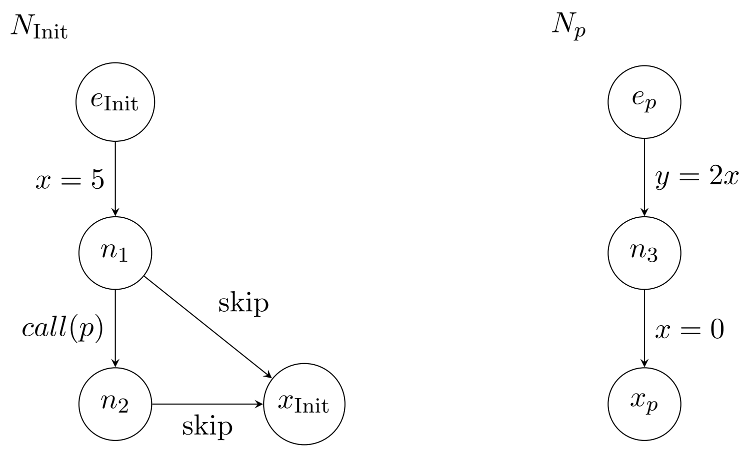
Exercise 2 (Labelled Pushdown Systems)**.** Let $\mathcal{P} = (P, \Gamma, \Delta, \Sigma)$ be a labelled pushdown system, i.e. the rules in Δ are of the form $pA \overset{a}{\hookrightarrow} qw$, where $p, q \in P$ are control locations, $A \in \Gamma$ and $w \in \Gamma^*$ are stack symbols, and additionally $a \in \Sigma$ is an *action*. For two configurations c, c' we write $c \overset{w}{\Rightarrow} c'$, where $w \in \Sigma^*$, if c can be transformed into c' by a sequence of rules whose labels yield w.

Given a regular set of configurations C, it is known how to compute $pre^*(C) = \{\, c \in Con(\mathcal{P}) \mid \exists c' \in C, w \in \Sigma^* : c \overset{w}{\Rightarrow} c' \,\}$. If C is accepted by an automaton with n states, this takes $\mathcal{O}(n^2 \cdot |\Delta|)$ time.

1. Let $L \subseteq \Sigma^*$ be a regular language and C be a regular set of configurations. We define
$$pre^*[L](C) := \{\, c \in Con(\mathcal{P}) \mid \exists c' \in C, w \in L : c \overset{w}{\Rightarrow} c' \,\}.$$
One can prove that $pre^*[L](C)$ is regular. Describe how to compute a finite automaton accepting $pre^*[L](C)$.

2. Give a bound on the amount of time it takes to compute $pre^*[L](C)$.

Exercise 3 (Data-flow Analysis)**.** We consider a problem from interprocedural data-flow analysis. A program consists of a set *Proc* of procedures that can execute and recursively call one another. The behaviour of each procedure p is described by a flow graph, an example with two procedures is shown below.



Formally, a flow graph for procedure $p \in Proc$ is a tuple $G_p = (N_p, A, E_p, e_p, x_p)$, where

- N_p are the nodes, corresponding to program locations; we denote $N := \bigcup_{p \in Proc} N_p$.

- $A = A_I \cup \{\, call(p) \mid p \in Proc \,\}$ are the actions, where A_I are *internal actions* (such as assignments etc); additionally an action can call some procedure. A is identical for all procedures.
- $E_p \subseteq N_p \times A \times N_p$ are the edges, labelled with actions from A. We denote $E := \bigcup_{p \in Proc} E_p$.
- e_p is the *entry point* of procedure p, i.e. when p is called, execution will start at e_p.
- x_p is the *exit point* of p (without any outgoing edges); when x_p is reached, p terminates and execution resumes at last call site of p.

1. Construct a labelled pushdown system with one single control location that expresses the behaviour of the procedures in *Proc*.

Suppose that the internal actions in A_I describe assignments to global variables, i.e. they are of the form $v := expr$, where v is a variable and *expr* the right-hand-side expression. If v is a variable, then $D_v \subseteq A_I$ is the set of actions that assign a value to v and $R_v \subseteq A_I$ the set of actions where v occurs on the right-hand side.

Let $Init \in Proc$ be an initial procedure and $n \in N$ a node in the flow graph. We say that variable v is *live* at n if there exists a node n' and an execution that (i) starts at e_{Init}, (ii) passes n, (iii) finally reaches n' with an action from R_v, and (iv) there is no assignment to v between n and n' in this execution. (Intuitively, this means that the value that v has at n matters for some execution; this is used in compiler construction to determine whether an optimizing compiler may "forget" the value of v at n.) For instance, in the shown example, the variable x is live at n_1 and e_p, but not in the other nodes.

1. Describe a regular language $L \subseteq A^*$ that describes the sequences of actions that can happen along such executions between n and n'.
2. Describe how, given a variable v, one can compute the set of nodes n such that v is live at n.

Exercise 4 (Multi-Pushdown Systems)**.** An n-dimensional *multi-pushdown system (n-MPDS)* is a tuple $\mathcal{M} = (P, \Gamma, (\Delta_i)_{0<i\leq n})$ where $n \geq 1$ is the number of stacks, P a finite set of control states, Γ a finite stack alphabet, and each $\Delta_i \subseteq P \times \Gamma \times P \times \Gamma^*$ is a finite transition relation. A configuration of an n-MPDS is a tuple $c = (q, w_1, \ldots, w_n)$ in $P \times (\Gamma^*)^n$. The *transition relation* $\rightarrow$ on configurations is defined as $\rightarrow = \bigcup_{0<i\leq n} \rightarrow_i$, where

$$(q, w_1, \ldots, Aw_i, \ldots, w_n) \rightarrow_i (q', w_1, \ldots, w_i' w_i, \ldots, w_n) \quad \text{iff} \quad qA \hookrightarrow q' w_i' w_i \in \Delta_i$$

1. Show that the *control state reachability problem*, i.e. given an initial configuration c in $P \times \Gamma^n$ and a control state $p \in P$, whether there exist $w_1, \ldots, w_n$ such that $c \rightarrow^* (p, w_1, \ldots, w_n)$ is undecidable as soon as $n \geq 2$.

2. Let us consider a restriction on $\to^*$: k-*bounded* runs are defined as the k-iterates $c \Rightarrow^k c'$ of the relation
$$c \to c' \qquad \text{iff} \qquad \exists i.\ c \to_i^* c'$$
i.e. a k-bounded run can be decomposed into k subruns where a single PDS is running.

   Show that the k-bounded control-state reachability problem, i.e. given an initial configuration c in $P \times \Gamma^n$ and a control state $p \in P$, whether there exist $w_1, \ldots, w_n$ such that $c \Rightarrow^k (q, w_1, \ldots, w_n)$ is decidable.