TD 5: Büchi Automata and Synchronous Büchi Transducers

Exercise 1 (Synchronous Büchi Transducers)**.** Give synchronous Büchi transducers for the following formulæ:

1. $\mathsf{SF}\, q$
2. $\mathsf{SG}\, q$
3. $p\, \mathsf{R}\, q$
4. $p\, \mathsf{S}\, q$
5. $\mathsf{G}(p \to \mathsf{F}\, q)$

Exercise 2 (Closure by Complementation)**.** The purpose of this exercise is to prove that $\mathrm{Rec}(\Sigma^\omega)$ is closed under complement. We consider for this a Büchi automaton $\mathcal{A} = (Q, \Sigma, T, I, F)$, and want to prove that its complement language $\overline{L(\mathcal{A})}$ is in $\mathrm{Rec}(\Sigma^\omega)$.

We note $q \xrightarrow{u} q'$ for q, q' in Q and $u = a_1 \cdots a_n$ in Σ^* if there exists a sequence of states $q_0, \ldots, q_n$ such that $q_0 = q$, $q_n = q'$ and for all $0 \leq i < n$, (q_i, a_{i+1}, q_{i+1}) is in T. We note in the same way $q \xrightarrow{u}_F q'$ if furthermore at least one of the states $q_0, \ldots, q_n$ belongs to F.

We define the *congruence* $\sim_\mathcal{A}$ over Σ^* by

$$u \sim_\mathcal{A} v \text{ iff } \forall q, q' \in Q,\ (q \xrightarrow{u} q' \Leftrightarrow q \xrightarrow{v} q') \text{ and } (q \xrightarrow{u}_F q' \Leftrightarrow q \xrightarrow{v}_F q')\ .$$

1. Show that $\sim_\mathcal{A}$ has finitely many congruence classes $[u]$, for u in Σ^*.
2. Show that each $[u]$ for u in Σ^* is in $\mathrm{Rec}(\Sigma^*)$, i.e. is a regular language of finite words.
3. Consider the language $K(L)$ for $L \subseteq \Sigma^\omega$

$$K(L) = \bigcup_{u,v \in \Sigma^*, [u][v]^\omega \cap L \neq \emptyset} [u][v]^\omega$$

   Show that $K(L)$ is in $\mathrm{Rec}(\Sigma^\omega)$ for any $L \subseteq \Sigma^\omega$.
4. Show that $K(L(\mathcal{A})) \subseteq L(\mathcal{A})$ and $K(\overline{L(A)}) \subseteq \overline{L(\mathcal{A})}$.
5. Prove that for any infinite word σ in Σ^ω there exist u and v in Σ^* such that σ belongs to $[u][v]^\omega$. The following theorem might come in handy when applied to couples of positions (i, j) inside σ:

Theorem 1 (Ramsey, infinite version)**.** *Let* $E = \{(i, j) \in \mathbb{N}^2 \mid i < j\}$*, and* $c : E \to \{1, \dots, k\}$ *a* k*-coloring of* E*. There exists an infinite set* $A \subseteq \mathbb{N}$ *and a color* $i \in \{1, \dots, k\}$ *such that for all* $(n, m) \in A^2$ *with* $n < m$*,* $c(n, m) = i$*.*

6. Conclude.

Exercise 3 (Closure Properties)**.** Given Büchi automata for $L_1 \subseteq \Sigma^\omega$ and $L_1 \subseteq \Sigma^\omega$, construct a Büchi automaton for

1. globally$(L_1) = \{u \in \Sigma^\omega \mid \forall u' \in \Sigma^*, u'' \in \Sigma^\omega, u = u'u'' \to u'' \in L_1\}$
2. until$(L_1, L_2) = \{uv \in \Sigma^\omega \mid u \in \Sigma^+ \wedge v \in L_2 \wedge \forall u, u'' \in \Sigma^+, u = u'u'' \to u''v \in L_1\}$