TD 4: Büchi Automata

Exercise 1 (Generalized Acceptance Condition)**.** A *generalized* Büchi automaton $\mathcal{A} = (Q, \Sigma, I, T, (F_i)_{0 \leq i < n})$ has a finite set of accepting sets F_i. An infinite run σ of $\mathcal{A}$ satisfies this generalized acceptance condition if each set F_i is visited infinitely often.

Show that for any generalized Büchi automaton, one can construct an equivalent Büchi automaton.

Exercise 2 (Rational Languages)**.** A *rational language* L of infinite words over Σ is a finite union

$$L = \bigcup X \cdot Y^\omega$$

where X is in $\mathsf{Rat}(\Sigma^*)$ and Y in $\mathsf{Rat}(\Sigma^+)$. We denote the set of *rational* languages of infinite words by $\mathsf{Rat}(\Sigma^\omega)$.

Show that $\mathsf{Rec}(\Sigma^\omega) = \mathsf{Rat}(\Sigma^\omega)$.

Exercise 3 (Deterministic Büchi Automata)**.** A Büchi automaton is *deterministic* if $|I| \leq 1$, and for each state q in Q and symbol a in Σ, $|\{(q, a, q') \in T \mid q' \in Q\}| \leq 1$.

1. Give a nondeterministic Büchi automaton for the language $L \subseteq \{a, b\}^\omega$ described by the expression $(a + b)^* a^\omega$, and a deterministic Büchi automaton for $\overline{L}$.
2. Show that there does not exist any deterministic Büchi automaton for L.
3. Let $\mathcal{A} = (Q, \Sigma, T, q_0, F)$ be a finite deterministic automaton that recognizes the language of finite words $L \subseteq \Sigma^*$. We can also interpret $\mathcal{A}$ as a deterministic Büchi automaton with a language $L' \subseteq \Sigma^\omega$; our goal here is to relate the languages of finite and infinite words defined by $\mathcal{A}$.

   Let the *limit* of a language $L \subseteq \Sigma^*$ be

   $$\overrightarrow{L} = \{w \in \Sigma^\omega \mid w \text{ has infinitely many prefixes in } L\} .$$

   Characterize the language L' of infinite words of $\mathcal{A}$ in terms of its language of finite words L and of the limit operation.

Exercise 4 (Closure by Complementation)**.** The purpose of this exercise is to prove that $\mathrm{Rec}(\Sigma^\omega)$ is closed under complement. We consider for this a Büchi automaton $\mathcal{A} = (Q, \Sigma, T, I, F)$, and want to prove that its complement language $\overline{L(\mathcal{A})}$ is in $\mathrm{Rec}(\Sigma^\omega)$.

We note $q \xrightarrow{u} q'$ for q, q' in Q and $u = a_1 \cdots a_n$ in Σ^* if there exists a sequence of states $q_0, \ldots, q_n$ such that $q_0 = q$, $q_n = q'$ and for all $0 \leq i < n$, (q_i, a_{i+1}, q_{i+1}) is in T.

We note in the same way $q \xrightarrow{u}_F q'$ if furthermore at least one of the states $q_0, \ldots, q_n$ belongs to F.

We define the *congruence* $\sim_{\mathcal{A}}$ over Σ^* by

$$u \sim_{\mathcal{A}} v \text{ iff } \forall q, q' \in Q,\ (q \xrightarrow{u} q' \Leftrightarrow q \xrightarrow{v} q') \text{ and } (q \xrightarrow{u}_F q' \Leftrightarrow q \xrightarrow{v}_F q')\ .$$

1. Show that $\sim_{\mathcal{A}}$ has finitely many congruence classes $[u]$, for u in Σ^*.

2. Show that each $[u]$ for u in Σ^* is in $\mathrm{Rec}(\Sigma^*)$, i.e. is a regular language of finite words.

3. Consider the language $K(L)$ for $L \subseteq \Sigma^\omega$

$$K(L) = \{[u][v]^\omega \mid u, v \in \Sigma^*, [u][v]^\omega \cap L \neq \emptyset\}\ .$$

   Show that $K(L)$ is in $\mathrm{Rec}(\Sigma^\omega)$ for any $L \subseteq \Sigma^\omega$.

4. Show that $K(L(\mathcal{A})) \subseteq L(\mathcal{A})$ and $K(\overline{L(A)}) \subseteq \overline{L(\mathcal{A})}$.

5. Prove that for any infinite word σ in Σ^ω there exist u and v in Σ^* such that σ belongs to $[u][v]^\omega$. The following theorem might come in handy when applied to couples of positions (i, j) inside σ:

   Theorem 1 (Ramsey, infinite version)**.** *Let $E = \{(i,j) \in \mathbb{N}^2 \mid i < j\}$, and $c : E \to \{1, \ldots, k\}$ a k-coloring of E. There exists an infinite set $A \subseteq \mathbb{N}$ and a color $i \in \{1, \ldots, k\}$ such that for all $(n,m) \in A^2$ with $n < m$, $c(n,m) = i$.*

6. Conclude.

Exercise 5 (Muller Automata)**.** A nondeterministic Muller automaton is a tuple $\mathcal{A} = (Q, \Sigma, I, T, \mathcal{F})$, where Q, Σ, I, T are as for Büchi automata and $\mathcal{F} \subseteq 2^Q$ is the acceptance condition. For a run σ of $\mathcal{A}$, denote by $\mathsf{Inf}(\sigma)$ the set of states which are visited infinitely often. A run σ is accepting if $\mathsf{Inf}(\sigma) \in \mathcal{F}$.

1. Give a *deterministic* Muller automaton for the language $(a+b)^* a^\omega$.

2. Show that for any Muller automaton $\mathcal{A}$, $L(\mathcal{A})$ is ω-regular.

3. Show that any ω-regular language is accepted by some (nondeterministic) Muller automaton.

Remark: in fact, any ω-regular language can be recognized by some *deterministic* Muller automaton.