TD 4: Büchi Automata

Exercise 1 (Generalized Acceptance Condition). A generalized Büchi automaton \( A = (Q, \Sigma, I, T, (F_i)_{0 \leq i < n}) \) has a finite set of accepting sets \( F_i \). An infinite run \( \sigma \) of \( A \) satisfies this generalized acceptance condition if each set \( F_i \) is visited infinitely often.

Show that for any generalized Büchi automaton, one can construct an equivalent Büchi automaton.

Exercise 2 (Rational Languages). A rational language \( L \) of infinite words over \( \Sigma \) is a finite union

\[
L = \bigcup X \cdot Y^\omega
\]

where \( X \) is in \( \text{Rat}(\Sigma^* \Sigma^+) \) and \( Y \) in \( \text{Rat}(\Sigma^+) \). We denote the set of rational languages of infinite words by \( \text{Rat}(\Sigma^\omega) \).

Show that \( \text{Rec}(\Sigma^\omega) = \text{Rat}(\Sigma^\omega) \).

Exercise 3 (Deterministic Büchi Automata). A Büchi automaton is deterministic if \( |I| \leq 1 \), and for each state \( q \) in \( Q \) and symbol \( a \) in \( \Sigma \), \( |\{(q, a, q') \in T \mid q' \in Q\}| \leq 1 \).

1. Give a nondeterministic Büchi automaton for the language \( L \subseteq \{a, b\}^\omega \) described by the expression \((a + b)^*a^\omega\), and a deterministic Büchi automaton for \( L \).

2. Show that there does not exist any deterministic Büchi automaton for \( L \).

3. Let \( A = (Q, \Sigma, T, q_0, F) \) be a finite deterministic automaton that recognizes the language of finite words \( L \subseteq \Sigma^* \). We can also interpret \( A \) as a deterministic Büchi automaton with a language \( L' \subseteq \Sigma^\omega \); our goal here is to relate the languages of finite and infinite words defined by \( A \).

Let the limit of a language \( L \subseteq \Sigma^* \) be

\[
\overrightarrow{L} = \{w \in \Sigma^\omega \mid w \text{ has infinitely many prefixes in } L\}.
\]

Characterize the language \( L' \) of infinite words of \( A \) in terms of its language of finite words \( L \) and of the limit operation.

Exercise 4 (Closure by Complementation). The purpose of this exercise is to prove that \( \text{Rec}(\Sigma^\omega) \) is closed under complement. We consider for this a Büchi automaton \( A = (Q, \Sigma, T, I, F) \), and want to prove that its complement language \( \overline{L(A)} \) is in \( \text{Rec}(\Sigma^\omega) \).

We note \( q \xrightarrow{u} q' \) for \( q, q' \) in \( Q \) and \( u = a_1 \cdots a_n \) in \( \Sigma^* \) if there exists a sequence of states \( q_0, \ldots, q_n \) such that \( q_0 = q, q_n = q' \) and for all \( 0 \leq i < n \), \( (q_i, a_{i+1}, q_{i+1}) \) is in \( T \).
We note in the same way $q \xrightarrow{u} q'$ if furthermore at least one of the states $q_0, \ldots, q_n$ belongs to $F$.

We define the congruence $\sim_A$ over $\Sigma^*$ by

$$u \sim_A v \text{ iff } \forall q, q' \in Q, (q \xrightarrow{u} q' \iff q \xrightarrow{v} q') \text{ and } (q \xrightarrow{u} F q' \iff q \xrightarrow{v} F q').$$

1. Show that $\sim_A$ has finitely many congruence classes $[u]$, for $u$ in $\Sigma^*$.

2. Show that each $[u]$ for $u$ in $\Sigma^*$ is in $\text{Rec}(\Sigma^*)$, i.e. is a regular language of finite words.

3. Consider the language $K(L)$ for $L \subseteq \Sigma^\omega$

$$K(L) = \{[u][v]^\omega \mid u, v \in \Sigma^*, [u][v]^\omega \cap L \neq \emptyset \}. $$

Show that $K(L)$ is in $\text{Rec}(\Sigma^\omega)$ for any $L \subseteq \Sigma^\omega$.

4. Show that $K(L(A)) \subseteq L(A)$ and $K(\overline{L(A)}) \subseteq \overline{L(A)}$.

5. Prove that for any infinite word $\sigma$ in $\Sigma^\omega$ there exist $u$ and $v$ in $\Sigma^*$ such that $\sigma$ belongs to $[u][v]^\omega$. The following theorem might come in handy when applied to couples of positions $(i,j)$ inside $\sigma$:

**Theorem 1** (Ramsey, infinite version). Let $E = \{(i,j) \in \mathbb{N}^2 \mid i < j\}$, and $c : E \to \{1, \ldots, k\}$ a $k$-coloring of $E$. There exists an infinite set $A \subseteq \mathbb{N}$ and a color $i \in \{1, \ldots, k\}$ such that for all $(n,m) \in A^2$ with $n < m$, $c(n,m) = i$.

6. Conclude.

**Exercise 5** (Muller Automata). A nondeterministic Muller automaton is a tuple $A = (Q, \Sigma, I, T, F)$, where $Q, \Sigma, I, T$ are as for B"uchi automata and $F \subseteq 2^Q$ is the acceptance condition. For a run $\sigma$ of $A$, denote by $\text{Inf}(\sigma)$ the set of states which are visited infinitely often. A run $\sigma$ is accepting if $\text{Inf}(\sigma) \in F$.

1. Give a deterministic Muller automaton for the language $(a + b)^* a^\omega$.

2. Show that for any Muller automaton $A$, $L(A)$ is $\omega$-regular.

3. Show that any $\omega$-regular language is accepted by some (nondeterministic) Muller automaton.

Remark: in fact, any $\omega$-regular language can be recognized by some deterministic Muller automaton.