Preliminaries. We recall below several definitions from the course.

- Given a Presburger formula $\varphi(x_1, \ldots, x_n)$ with $n \geq 1$ free variables, we write $\text{REL}(\varphi(x_1, \ldots, x_n))$ to denote the set of tuples (variable values) that make true the formula:
  
  $$\text{REL}(\varphi(x_1, \ldots, x_n)) \stackrel{\text{def}}{=} \{(v(x_1), \ldots, v(x_n)) \in \mathbb{N}^n : v \models \varphi, \text{valuation } v\}.$$ 

  We also write $\vec{a} \models \varphi(x_1, \ldots, x_n)$ whenever $v \models \varphi(x_1, \ldots, x_n)$ where $v : \{x_1, \ldots, x_n\} \to \mathbb{N}$ is the valuation such that $v(x_i) = \vec{a}(i)$.

- An affine counter system $S$ is a tuple of the form $(Q, n, \delta)$ such that every transition $q \xrightarrow{\varphi} q' \in \delta$, $\text{REL}(\varphi)$ is affine. Usually, $\varphi$ is encoded by a triple $(A, \vec{b}, \psi)$ such that $A \in \mathbb{Z}^{n \times n}$, $\vec{b} \in \mathbb{Z}^n$, $\psi$ has free variables among $\{x_1, \ldots, x_n\}$, and $\text{REL}(\varphi) = \{(\vec{x}, \vec{x}') \in \mathbb{N}^{2n} : \vec{x}' = A\vec{x} + \vec{b} \text{ and } \vec{x} \in \text{REL}(\psi)\}$.

- A VASS $S$ is a tuple of the form $(Q, n, \delta)$ such that every transition $t \in \delta$ is of the form $q \xrightarrow{\vec{b}} q'$ with $\vec{b} \in \mathbb{Z}^n$. Given configurations $(q, \vec{a})$, $(q', \vec{a}') \in Q \times \mathbb{N}^n$ and a transition $q \xrightarrow{\vec{b}} q' \in \delta$, we have $(q, \vec{a}) \xrightarrow{t} (q', \vec{a}') \iff \vec{a}' = \vec{a} + \vec{b}$.

- A standard counter automaton $(Q, n, \delta)$ has transitions of the form either $q \xrightarrow{\text{inc}(i)} q'$ (increment counter $i$) or $q \xrightarrow{\text{dec}(i)} q'$ (decrement counter $i$ if nonzero) or $q \xrightarrow{\text{zero}(i)} q'$ (zero-test on counter $i$).

Exercise 1. Let us consider the affine counter system below:

\[
\begin{align*}
q_1 \quad & \xrightarrow{\left(\begin{array}{cc} 1 & 0 \\
0 & 1 \end{array}\right), \left(\begin{array}{c} 3 \\
-3 \end{array}\right), x_1 < x_2} q_2 \\
q_2 \quad & \xrightarrow{\left(\begin{array}{cc} 2 & 0 \\
0 & 2 \end{array}\right), \left(\begin{array}{c} 3 \\
-3 \end{array}\right), x_1 = x_2} q_3 \\
q_3 \quad & \xrightarrow{\left(\begin{array}{cc} 1 & 0 \\
0 & 1 \end{array}\right), \left(\begin{array}{c} 1 \\
-1 \end{array}\right), x_1 = x_1} q_1
\end{align*}
\]
Design a Presburger formula \( \varphi(x_1, x_2, y_1, y_2) \) such that for every valuation \( v \), we have \( v \models \varphi \) if \( (q_1, v(x_1), v(x_2)) \overset{*}{\rightarrow} (q_3, v(y_1), v(y_2)) \), i.e. \((q_3, v(y_1), v(y_2))\) is reachable from \((q_1, v(x_1), v(x_2))\).

**Exercise 2.** Let us consider an extension of VASS by allowing extended transitions of the form \( t = q \xrightarrow{b} q' \) with \( b \in \mathbb{N}^n \) such that \((q, \bar{a}) \xrightarrow{b} (q', \bar{a}')\) iff \( \bar{a} = \bar{b} \) (equality test) and \( \bar{a}' = \bar{a} \) (update is identity).

**Question 2.1** Let \( \mathcal{V} = (Q, n, \delta) \) be an extended VASS such that the extended transitions of \( \mathcal{V} \) are exactly those below (apart from the standard transitions):

\[
q_1 \xrightarrow{\vec{b}_1} q'_1, \ldots, q_N \xrightarrow{\vec{b}_N} q'_N
\]

Show that if there is a run from \((q, \bar{x})\) to \((q', \bar{x}')\), then there is a run from \((q, \bar{x})\) to \((q', \bar{x}')\) such that the number of times extended transitions are fired is at most \(N\).

**Question 2.2** Given an initial configuration \((q, \vec{a})\), design an algorithm that computes the set below:

\[
\{ (q_i, \vec{b}_i) : i \in [1, N], (q, \vec{a}) \overset{\vec{b}_i}{\rightarrow} (q_i, \vec{b}_i) \in \mathcal{V} \}
\]

Hint: use as a subroutine an algorithm for solving the reachability problem for VASS (taken for granted).

**Question 2.3** Conclude that the reachability problem for this class of extended VASS is decidable.

**Exercise 3.**

**Question 3.1** Given \( B \geq 0 \) and \( \bar{x} \in \mathbb{N}^n \), we define the \( B \)-truncation of \( \bar{x} \), written \( \text{trunc}_B(\bar{x}) \), as a tuple in \( \mathbb{N}^n \) such that for \( i \in [1, n] \), we have \( \text{trunc}_B(\bar{x})(i) \overset{\text{def}}{=} \min(\bar{x}(i), B) \). A set \( X \subseteq \mathbb{N}^n \) is said to be simple if there are \( B \geq 0 \) and \( Y \subseteq [0, B]^n \) such that for every \( \bar{x} \in \mathbb{N}^n \), \( \bar{x} \in X \) iff \( \text{trunc}_B(\bar{x}) \in Y \). A simple guard \( \varphi \) is defined as a Presburger formula respecting the grammar below:

\[
x_i \geq k \mid x_i \leq k \mid \varphi_1 \land \varphi_2 \mid \top
\]

with \( k \in \mathbb{N}, x_i \) is a variable interpreted by a natural number in \( \mathbb{N} \) and \( \top \) is the truth constant. Let \( \varphi \) be a simple guard with free variables among \( \{x_1, \ldots, x_n\} \). Show that \( \text{REL}(\varphi) \) is a simple set, i.e. \( \varphi \) can be associated with a pair \((B, Y)\) encoding \( \text{REL}(\varphi) \).
**Question 3.2** An extended counter automata $S$ of dimension $n$ is a counter system of dimension $n$ in which the transitions are represented in the following way:

$$t = q \xrightarrow{(\varphi(x_1, \ldots, x_n), \vec{b})} q'$$

where $\varphi(x_1, \ldots, x_n)$ is a simple guard with free variables among $\{x_1, \ldots, x_n\}$ and $\vec{b} \in \mathbb{Z}^n$ (update vector). Given configurations $(q, \vec{a}), (q', \vec{a}') \in Q \times \mathbb{N}^n$, by definition $(q, \vec{a}) \xrightarrow{t} (q', \vec{a}')$ if $\varphi(x_1, \ldots, x_n)$ and $\vec{a}' = \vec{a} + \vec{b}$.

Reversal-boundedness for extended counter automata is defined as for standard counter automata: initialized extended counter automaton $(S, (q, \vec{x}))$ is $r$-reversal-bounded if for every run from $(q, \vec{x})$, every counter performs at most $r$ reversals.

Let $(S, (q_0, \vec{x}_0))$ be a reversal-bounded extended counter automata and $B_{max}$ be the maximal bound from all the bounds $B$ associated to simple guards in $S$. Let $(q_0, \vec{x}_0), (q_1, \vec{x}_1), \ldots$ be an infinite run for the extended counter automaton $S$ such that the control state $q_f$ is repeated infinitely often. Show that there are positions $l' < l$ and a set of counters $Z \subseteq [1, n]$ such that:

(a) $q_l = q_{l'} = q_f$,
(b) for $i \in Z$ and $j \in [l' + 1, l]$, $\vec{x}_j(i) - x_{j-1}(i) = 0$,
(c) for $i \in ([1, n] \setminus Z)$, we have $\vec{x}_{l'}(i) \leq \vec{x}_l(i)$,
(d) for $i \in ([1, n] \setminus Z)$ and $j \in [l' + 1, l]$, $\vec{x}_j(i) - x_{j-1}(i) \geq 0$,
(e) for $i \in ([1, n] \setminus Z)$, $\vec{x}_{l'}(i) \geq B_{max}$.

Observe that (d) implies (c).

**Question 3.3** Show that there is an infinite run from $(q_0, \vec{x}_0)$ with control state $q_f$ repeated infinitely often iff there are a finite run $(q_0, \vec{x}_0), (q_1, \vec{x}_1), \ldots, (q_l, \vec{x}_l)$, $l' < l$ and $Z \subseteq [1, n]$ such that (a)–(e) hold true.

**Question 3.4** Define a reversal-bounded extended counter automaton $S'$ such that there is an infinite run from $(q_0, \vec{x}_0)$ with $q_f$ repeated infinitely often in $S$ iff $(q_0, \vec{x}_0) \xrightarrow{t} (q_{new}, \vec{0})$ in $S'$ ($q_{new}$ is a new control state occurring in $S'$ but not in $S$).