

A Retraction Theorem for Distributed Synthesis

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Abstract

We present a general theorem for distributed synthesis problems in coordination games with ω -regular objectives of the form: If there exists a winning strategy for the coalition, then there exists an “essential” winning strategy, that is obtained by a retraction of the given one. In general, this does not lead to finite-state winning strategies, but when the knowledge of agents remains bounded, we can solve the synthesis problem. Our study is carried out in a setting where objectives are expressed in terms of events that may *not* be observable. This is natural in games of imperfect information, rather than the common assumption that objectives are expressed in terms of events that are observable to all agents. We characterise decidable distributed synthesis problems in terms of finiteness of knowledge states and finite congruence classes induced by them.

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1 Introduction

In the theory of system design and verification, the synthesis problem is formulated as a win/lose game between the system and a hypothetical opponent labelled as environment, and the solution to the problem is a winning strategy for the system. When these are games of infinite duration, and the winning condition is a set of infinite regular plays, a central theorem of the subject asserts that it can be effectively decided whether winning strategies exist for the system; moreover, when winning strategies exist, they can be realised using finite memory. A rich theory of such games has been built in the last couple of decades ([20, 27, 24, 12, 30]).

In the design of systems with multiple components that work concurrently, a similar question can be formulated. When doing so, it matters whether the components work *cooperatively* towards system goals against the adversary, or could have potentially conflicting goals despite which we would like the system to achieve its goals against the environment. Moreover, a major *raison-d’être* of such systems is *distributedness*: each component has access to partial information about the global system. This is needed for reliability and isolation of faults. Such considerations lead to the formulation of the synthesis problem as a win/lose game of imperfect information, again between the system and an environment (which may of course have global / perfect information). Moreover, we wish to synthesize strategies for each component acting independently, and hence we have a problem of distributed synthesis in imperfect information games, even in the restricted case of coordinating agents, with identical payoffs to all components. Since components act concurrently and independently it matters whether they all act *asynchronously*, each with its own clock, or *synchronously*, according to the ticks of a common global clock.

That distributed synthesis for imperfect information games is hard is well-known: the literature is replete with results showing undecidability of the winning strategy question



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([25],[9],[13],[2]). These results are in some sense generic, and have to do with ‘forking’ of information among players, which loosely amounts to uncertainty of a player about another’s information. Thus game states implicitly carry information on ‘epistemic’ states of players involving their knowledge / ignorance of each other’s states, and then about each other’s knowledge and so on; this grows unboundedly, and is a crucial source of undecidability.

Coping with such undecidability, researchers have tended to place structural constraints on the system: on the architecture of interaction among players ([19],[22]), constraining patterns of interaction to be statically determined ([21],[10]), constraining the players to a hierarchy ([18],[5]), and so on. Semantic subclasses have been studied, e.g. using a homomorphic characterisation ([3]), a recurrent certainty condition ([4]), and so on. These models have concurrent components, but no explicit communication between them; [28] identifies public vs private communication as a potential source of (un)decidability.

A striking feature of all these studies is that where decidable subclasses are obtained, they invariably result in the ‘epistemic states’ referred to above being bounded, if not in the system evolution as such, but in the construction of winning strategies. Such an observation leads us to a theoretical consideration of the question: is there a way to abstractly characterise the existence of winning strategies in terms of how these ‘epistemic states’ grow, when they are finite, bounded etc. The general problem being undecidable, there cannot be a recursive characterisation of this form; yet, we might discern a general pattern, specialising to specific bounds for structurally or otherwise constrained applications.

This is the project taken up in this paper. When we have a finite-state presentation of a game of imperfect information, with finite-state objectives, if we are told that the coalition of players, has a (distributed) winning strategy against an environment, can we identify the core of the strategy in terms of the ‘epistemic states’ encountered, and its underlying dynamic structure? We answer this by proving a **retraction theorem**: if a winning strategy exists, then a ‘small’ quotiented strategy exists as a witness. In specific cases, the retraction can be strengthened, to obtain bounds on the size of witnesses, leading to decidability. Interestingly, these bounds assert memory that is bounded as a function of bounds on epistemic states, which can we consider bounded memory in mutual information.

A natural question is: why bother? (Alternatively phrased: so what?) We consider this theorem as rudimentary methodology for distributed synthesis in imperfect information games: information sets pose a uniformity condition on strategies but at the same time cause cascading uncertainty. With finite state labelling, these mutual information states may be tamed as our ‘lateral’ retraction suggests; temporal dynamics is addressed using *progress measures*, as for instance studied in [16, 17] and [31, 14]. The interaction between these ‘lateral’ and ‘vertical’ folding (in finite state situations) is generic to these systems. We illustrate with two small examples, one that of 1-player information games, and another of hierarchies of players. However, the idea is that retractions of this kind can be constructed for a wide class of systems.

The study is carried out in a model that is as general as possible. This helps us generalize earlier results. For instance, a common assumption in distributed synthesis is that objectives are expressed in terms of events that are observable to all agents. Despite imperfect information, each agent can monitor the outcome of the interaction by updating an automaton that finally accepts if the objective is met. This is clearly against the spirit of distributedness, and our model can dispense with such assumptions. Thus the theorems include systems where objectives are expressed in terms of events that may *not* be observable.

2 The Model

We model a synchronous distributed system as a game between a *team* $N = \{1, \dots, n\}$ of players and one passive agent called *Nature*. The game is played in an infinite sequence of stages. Each player $i \in N$ has a finite set A^i of *actions* among which he can choose. We denote by $A := \times_{i \in N} A^i$ the set of action profiles — by *profile* we generally mean a tuple $x = (x^i)_{i \in N}$ of elements, one for each player in the team. The choices of Nature are called *directions* and range over a finite set D .

In each stage, every player i chooses an action $a^i \in A^i$, then Nature chooses a direction d . Together, these choices determine a *move* $\alpha = (a, d)$; we denote the set of possible moves by $\Gamma = A \times D$. A *play* is an infinite sequence $\pi \in \Gamma^\omega$ of moves, and a *history* is a finite sequence $\tau \in \Gamma^*$ of moves.

We are interested in a *finite state* presentation of the game, and hence the information revealed to player i during the play is described by a Mealy machine M^i that reads histories and outputs sequences of *observations* from a finite alphabet B^i . The machine $M^i = (Q^i, \delta^i, \beta^i, q_\varepsilon^i)$ is specified by a finite set Q^i of states, a transition function $\delta^i : Q^i \times \Gamma \rightarrow Q^i$, an output function $\beta^i : Q^i \times \Gamma \rightarrow B^i$, and an initial state q_ε^i . For an input history $\tau = \alpha_0 \alpha_1 \dots \alpha_{\ell-1}$, the *run* $\rho(\tau)$ of the machine is the sequence q_0, q_1, \dots, q_ℓ with $q_0 = q_\varepsilon$ and $q_{k+1} = \delta^i(q_k, \alpha_k)$ for all $k \leq \ell$, and its *output* $\beta^i(\tau)$ is the sequence of observations $\beta^i(q_0, \alpha_0) \beta^i(q_1, \alpha_1) \dots \beta^i(q_{\ell-1}, \alpha_{\ell-1})$. We say that two histories $\tau, \tau' \in \Gamma^*$ are indistinguishable for player i , and write $\tau \sim^i \tau'$, if the runs of M^i and τ and τ' yield the same observation sequence $\beta^i(\tau) = \beta^i(\tau')$. Clearly, this is a synchronous equivalence relation.

Overall, a distributed game is specified by a structure $\mathcal{G} = (\Gamma^*, (\sim^i)_{i \in N})$ on the tree of histories ordered by the word prefix relation with a profile of indistinguishability relations described by finite-state Mealy machines.

A *decision structure* for the game \mathcal{G} is a directed graph $S = (V, E, f, v_0)$ on a possibly infinite set of nodes V with an edge relation $E \subseteq V \times D \times V$ that describes a deterministic transition function from $V \times D$ to V , an action choice function $f : V \rightarrow A$, and an initial node v_ε . Every path $v_0 \xrightarrow{d_0} v_1 \xrightarrow{d_1} \dots \xrightarrow{d_{\ell-1}} v_\ell$ in S starting from $v_0 = v_\varepsilon$ identifies a unique history $f(v_0)d_0f(v_1)d_1 \dots f(v_{\ell-1})d_{\ell-1}$ in \mathcal{G} . We say that a history τ *follows* the decision structure S , or simply τ is in S , if τ is identified by a path in S . Conversely, every history $\tau = a_0d_0a_1d_1 \dots a_{\ell-1}d_{\ell-1}$ in \mathcal{G} corresponds in S to a unique path $v_0 \xrightarrow{d_0} v_1 \xrightarrow{d_1} \dots \xrightarrow{d_{\ell-1}} v_\ell$ from the initial node $v_0 = v_\varepsilon$. We call the sequence $v_0v_1 \dots v_\ell$ the *trace* of τ . Notice that τ follows S if, and only if, $a_k = f(v_k)$ for all $k \leq \ell$; in this case, we write $v_\varepsilon \xrightarrow{\tau} v$ for the end node $v = v_\ell$. For each player i , the indistinguishability relation \sim^i on \mathcal{G} induces a *uniformity* relation among states of S : we write $v \approx_S^i v'$ if there exist histories $\tau \sim^i \tau'$ such that $v_\varepsilon \xrightarrow{\tau} v$ and $v_\varepsilon \xrightarrow{\tau'} v'$.

A *strategy structure* is a decision structure that satisfies the following uniformity condition: if $v \approx_S^i v'$ then $f^i(v) = f^i(v')$, for all $v, v' \in V$. The uniformity relation arises as an operational consequence of epistemic indistinguishability: if two nodes are reachable by indistinguishable histories, they prescribe the same action for all pairs of histories that reach them. Notice that \approx_S^i is reflexive and symmetric, but not necessarily transitive.

Alternatively, we can view a strategy structure as a Moore machine (possibly on infinitely many states) that implements a function $S : \Gamma^* \rightarrow A$ from histories to action profiles, by assigning to each history τ , the action profile $f(v)$ prescribed at the state v reached by the path corresponding to τ in the structure S . This assignment is *information-consistent* for each player i : for any pair $\tau \sim^i \tau'$ of indistinguishable histories that follow S , the action profiles $S(\tau)$ and $S(\tau')$ agree on their player- i component. Accordingly, S can also be implemented by

a distributed profile of private Moore machines s^i each of which inputs observation sequences $\tau^i := \beta^i(\tau)$ of player i and return actions $s^i(\tau^i)$ such that $S(\tau) = (s^i(\beta^i(\tau)))_{i \in N}$ for all histories τ in S .

We are interested in strategies that enforce a specified branching-time behaviour of distributed systems. Accordingly, we define the *outcome* of a strategy S in a game \mathcal{G} to be the tree $H_S \subseteq \Gamma^*$ in S equipped with the prefix order. To specify ω -regular properties of strategy outcomes, we use tree automata $\mathcal{W} := (Q, \Delta, q_\epsilon, \Omega)$ where Q is a finite set of states, $\Delta \subseteq Q \times A \times Q^D$ is a nondeterministic transition function and $\Omega : Q \rightarrow \mathbb{N}$ is a labelling of the states with *priorities* from a finite range $\Omega(Q) \subseteq \mathbb{N}$. A *run* of the automaton \mathcal{W} on (the outcome of) a strategy S is a labelling $\rho : H_S \rightarrow Q$ of the outcome tree such that $\rho(\epsilon) = q_\epsilon$ and, for all histories $\tau \in H_S$ with $a := S(\tau)$, we have $(\rho(\tau), a, (\rho(\tau ad))_{d \in D}) \in \Delta$. The run ρ is *accepting* if, for every play $\alpha_0 \alpha_1 \dots$ through H_S , the corresponding priority sequence $\Omega(\rho(\epsilon)), \Omega(\rho(\alpha_0)), \Omega(\rho(\alpha_0 \alpha_1)), \dots$ satisfies the *parity condition*, which requests that the minimum priority appearing infinitely often in the sequence be even. We say that a joint strategy s is *winning* for \mathcal{W} , if there exists an accepting run of \mathcal{W} on the outcome tree of S .

We consider the following *distributed synthesis problem*: Given a distributed game \mathcal{G} and a specification \mathcal{W} , decide whether a finite-state winning strategy exists, and if so, construct one. Following [23], we know that:

► **Theorem 1.** *The distributed synthesis problem is undecidable.*

3 Annotations and Retractions

Faced with undecidability in general, we ask a different question: given a game \mathcal{G} and a specification \mathcal{W} , suppose that we are told that there exists a winning strategy structure. What can we infer from this? In particular, can we transform the strategy into one with fewer states, hopefully finitely many?

Let us fix a game \mathcal{G} described by a family of Mealy machines M^i , and a specification automaton \mathcal{W} . An *annotated* strategy for \mathcal{G} is a structure (S, ρ) that expands a strategy structure S with a function $\rho : V \rightarrow Q^0 \times \dots \times Q^{n-1} \times Q$ labelling every node $v \in V$ with a profile $(\rho^i(v))_{i \in N}$ of states of the Mealy machines M^i and a state $\rho_{\mathcal{W}}(v)$ of the tree automaton \mathcal{W} , as follows:

- for any history τ in S with trace $v_0 \dots v_\ell$, the sequence $\rho^i(v_0) \dots \rho^i(v_\ell)$ describes the run of M^i on τ ;
- if we consider for each history $\tau \in H_S$ the node v_τ reached by the path $v_\epsilon \xrightarrow{\tau} v_\tau$ in S , then the mapping $(\rho_{\mathcal{W}}(v_\tau))_{\tau \in H_S}$ describes a run of \mathcal{W} on H_S .

A *witness* strategy is an annotated strategy where the run described by $\rho_{\mathcal{W}}$ is accepting.

Notice that every strategy structure on a tree can be expanded with runs of the automata M^i, \mathcal{W} to obtain an annotated strategy. However, for an arbitrary strategy structure S it may be impossible to describe a run as a single-state annotation of its nodes, because computations along different paths can reach the same node of S in different states. Annotations are indeed runs of vertex-marking automata on (strategy) graphs, as studied in [1]. Nevertheless, Rabin's basis theorem [26] implies that every finite-state winning strategy can be extended – by taking the synchronised product with a certain finite-state automaton – to allow annotation with a witnessing run.

► **Theorem 2.** *Let \mathcal{G} be a distributed game with a (finite) set of directions D .*

- (i) *There exists a winning strategy for \mathcal{G} if, and only if, there exist a witness strategy on the tree D^* .*

(ii) *There exists a finite-state winning strategy for \mathcal{G} if, and only if, there exists a witness strategy on a finite set of nodes.*

Proof. The if-direction is obvious: If a strategy witness (S, ρ) exists, then the underlying strategy structure S can be readily used to solve the synthesis problem.

For the converse, suppose that there exists a winning strategy structure $S = (V, E, f, v_\varepsilon)$ for \mathcal{G} . Consider the tree unravelling S' of S , that is, the strategy structure on D^* in which the nodes π correspond to paths in S and the action choice $f'(\pi) := f(v_\pi)$ is inherited from the end node $v_\varepsilon \xrightarrow{\pi} v_\pi$. Since S is a winning strategy, the specification automaton \mathcal{W} has an accepting run $\rho_{\mathcal{W}} : H_S \rightarrow Q$ on the tree of histories H_S that follow S . Further, for each player i , the runs of the deterministic Mealy machine M^i on all histories that follow S induce a labelling $\rho^i : H_S \rightarrow Q^i$. For every node π in S' , set $\rho'(\pi) = \rho(\tau_\pi)$ for the history τ_π identified by the strategy path π . Then, (S', ρ') is a witness strategy on D^* .

In case the witness S at the outset is actually a finite strategy structure, the language of witnessing annotations (S', ρ') on the associated strategy tree S' is regular, and by Rabin's basis theorem ([30]), it contains a regular tree. This can be turned into an annotation of the finite-state strategy with the same unravelling. ◀

Our vehicle for moving from a given strategy structure to one with fewer states are particular maps on the strategy nodes. For a strategy structure S and a map $h : V \rightarrow V$, we define the image $h(S)$ to be the decision structure $(\hat{V}, \hat{E}, \hat{f}, \hat{v}_\varepsilon)$ on a subset $\hat{V} \subseteq V$ of nodes with a new transition relation $\hat{E} := \{(u, d, v) \mid (h(u), d, v) \in E\}$, choice function $\hat{f}(v) := f(h(v))$, and initial state $\hat{v}_\varepsilon = h(v_\varepsilon)$; we restrict \hat{V} to the set of nodes reachable from \hat{v}_ε via \hat{E} . Intuitively, the map folds a decision structure by redirecting the decision at a node v to the node $h(v)$ and continuing the play from there onwards. This way of performing surgery on strategies is used frequently, for instance, to show that memoryless strategies are sufficient for winning parity games with perfect information (see, e.g., [11]).

A map h is a *retraction* for an annotated strategy (S, ρ) if

- $\rho(v) = \rho(h(v))$ for all $v \in V$, and
- $\hat{f}^i(u) = \hat{f}^i(v)$ for every pair of nodes $u \approx_{S(h)}^i v$, for each player i .

The *retract* $h(S, \rho)$ is the image $h(S)$ expanded with the annotation ρ on the nodes in its domain \hat{V} .

The constraints on a retraction to preserve the annotation with states of the observation automata and to respect the uniformity relation ensure that the decision structure obtained as an image is indeed a strategy structure.

► **Lemma 3.** *Let (S, ρ) be an annotated strategy for a game \mathcal{G} . If h is a retraction for (S, ρ) , then the retract $h(S, \rho)$ is also an annotated strategy for \mathcal{G} .*

We are particularly interested in retractions h that are *conservative* in the sense that if (S, ρ) is a witness then the retract $h(S, \rho)$ is also a witness.

Safety winning conditions are ω -regular conditions described by automata that accept a run only if all the occurring states belong to a designated safe subset; they can be described by parity automata with only two priorities. Since retractions preserve the annotation of strategy nodes with states of the specification automaton, it immediately follows that retractions are conservative for strategies in games with safety conditions.

► **Lemma 4.** *Let \mathcal{G} be a distributed game with a safety winning condition. Then, every retraction for a strategy annotation for \mathcal{G} is conservative.*

In contrast, liveness properties of strategies may be hurt by arbitrary retractions.

3.1 Progress measures

We are making some progress in our journey towards identifying the “essential” core of the winning strategy, but we are far from done as yet. The next notion we need is that of *progress measures*, as introduced by Klarlund in his thesis [15], which yield a local representation of an ordering that assigns a value to every state of a system, such that by following transitions that reduce the value, a specified property is satisfied in the limit. We use such measures to identify how we might equate states across the temporal dimension as the strategy evolves over time, such that the parity condition is preserved.

For a parity condition $\Omega : Q \rightarrow \{0, \dots, r-1\}$ with r priorities, we consider a standard measure that ranges over r -tuples of ordinals. Given two tuples $x, y \in \text{On}^r$ and a priority k , we write $x \succ_k y$ if (x_0, \dots, x_k) is lexicographically greater than (y_0, \dots, y_k) .

A *parity progress measure* on an annotated strategy (S, ρ) is a function $\mu : V \rightarrow \text{On}^r$ such that for each transition (u, d, v) in S , we have

- if $k = \Omega(\rho_{\mathcal{W}}(u))$ is even, then $\mu(u) \succeq_k \mu(v)$, and
- if $k = \Omega(\rho_{\mathcal{W}}(u))$ is odd, then $\mu(u) \succ_k \mu(v)$.

It is well known that such progress measures describe winning strategies in parity games [8, 17, 14].

► **Theorem 5.** *An annotated strategy (S, ρ) is a witness if, and only if, there exists a parity progress measure on (S, ρ) .*

Proof. To see that whenever there exists a parity progress measure for an annotated strategy S , the annotation $\rho_{\mathcal{W}}$ describes an accepting run, note that each of the preorders \succeq_k is well-founded (there are no infinite descending chains), so their lexicographic product \succeq is also well-founded. Now suppose that, for some infinite path in $(S, \rho_{\mathcal{W}})$, the least priority k that occurs infinitely often is odd. Since, with every transition (u, d, w) from a node of priority $k = \Omega(\rho_{\mathcal{W}}(u))$ in the strategy, the measure μ decreases in \succeq_k and thus in \succeq , this implies that we have an infinite descending chain in \succeq – a contradiction. Accordingly, the run described by $\rho_{\mathcal{W}}$ satisfies the parity condition.

Conversely, let us assume that the given annotation (S, ρ) describes an accepting run. We follow a procedure described by Grädel and Walukiewicz [11] to define a progress measure $\mu : V \rightarrow \text{On}^r$: For each odd priority k , consider the sequence $(Z_\alpha)_{\alpha \in \text{On}}$ of sets, where Z_0 consists of nodes in S from which every path either never reaches priority k , or it reaches a smaller priority before reaching k ; for every ordinal α , the set Z_α consists of all nodes $u \in V$ such that, if there is a path from u to a node v of priority k , then all successors of v belong to $\cup_{\beta < \alpha} Z_\beta$. Finally we set $\mu_k(v)$ to be the least ordinal β such that $v \in Z_\beta$. For even priorities k , the component $\mu_k(v)$ is set to zero. One can now verify that the mapping μ defined in this way is a progress measure. ◀

We say that a retraction $h : V \rightarrow V$ is μ -*monotone*, if $\mu(v) \succeq_k \mu(h(v))$ for every node $v \in V$ of priority $k = \Omega(\rho_{\mathcal{W}}(v))$. In addition to preserving runs, monotone retractions preserve progress measures.

► **Lemma 6.** *Let (S, ρ) be an witness strategy with a parity progress measure μ . Then, any μ -monotone retraction for (S, ρ) is conservative.*

Proof. Let h be a μ -monotone retraction for a witness strategy with a parity progress measure as in the statement. We show that μ is a parity progress measure for the retract $h(S)$ as well: By definition of the retraction, for every transition (u, d, v) in the image $h(S)$, there is a transition $(h(u), d, v) \in E$. Moreover, the source nodes of the two transition have the

same ρ -annotation, and hence the same priority k . Since μ is a progress measure on S , we have $\mu(h(u)) \succeq_k \mu(v)$, for the priority On on the other hand, by μ -monotonicity of h , we have $\mu(u) \succeq_k \mu(h(u))$. Therefore, $\mu(u) \succeq_k \mu(v)$. In case the priority k of $\rho_{\mathcal{W}}(u) = \rho_{\mathcal{W}}(h(u))$ is odd, the ordering is strict. Accordingly, μ is a parity progress measure for the retract $h(S)$ which implies, in particular, that $h(S)$ is a witness. \blacktriangleleft

We can easily verify that μ -monotone retractions are closed under composition.

► **Lemma 7.** *For a game \mathcal{G} , suppose there exists a witnessing strategy (S, ρ) with a parity progress measure μ . Let g and h be μ -monotone retractions for (S, ρ) and $h(S, \rho)$, respectively. Then the composition $g \circ h$ is a μ -monotone retraction for (S, ρ) .*

4 Compacting Retractions

Our objective is to retract strategies into smaller ones, hopefully of finite size. Towards this, we introduce the notion of **distributed states**, which we understand as the atoms of a strategy annotation on which retractions will operate.

Let us fix an annotated strategy (S, ρ) . Towards defining distributed states, it is convenient to include the profile of uniformity relations \approx^i into the signature, and to drop the initial state. Thus, we shall view the strategy as a structure $S = (V, E, (\equiv^i)_{i \in N}, \rho)$. Further, we define the relation $\approx := \cup_{i \in N} \approx^i$.

Now, a distributed state, or shortly *d-state*, is a structure $\kappa = (V_\kappa, E, (\approx^i)_{i \in N}, \rho)$, induced in S by a subset $V_\kappa \subseteq V$ that forms a maximal \approx -connected component. Naturally, E , \approx^i and ρ are the relations of S restricted to V_κ . We denote the set of all d-states of the structure S by \mathcal{K}_S and the isomorphism relation among them by \cong_S . Then, \mathcal{K}_S / \cong_S is the quotient of \mathcal{K}_S by \cong_S and $[\kappa]_{\cong_S}$ is the equivalence class that contains the d-state κ .

Note that, as defined, it is hardly clear when a d-state is finite. *A priori* the maximal \approx -connected components of S would be infinite. This raises the question: when are d-states finite, and when they are, how does this impact decidability of distributed synthesis.

The notion of a d-state reveals three parameters for the size of a witnessing strategy: the number of connected nodes in single d-state, the number of isomorphic d-states in one \cong -class, and the index of \cong , that is, the number of non-isomorphic d-states in S . For a witnessing strategy to be finite, all these parameters must be finite.

In the rest of the section we show that, if all d-states in a witness are of finite size, then, for the purposes of decidability, the index of \cong_S , is the only relevant parameter. First, we show that for any such witness, we can pick an arbitrary \cong -class and retract it into a finite one, without introducing new classes or enlarging the existing ones.

► **Lemma 8.** *Let S be an witness strategy with a parity progress measure μ . Then, for any d-state $\kappa \in \mathcal{K}_S$ of finite size, there exists a μ -monotone retraction h such that $[\kappa]_{\cong_{h(S)}}$ is a finite set in the retract $h(S)$ and $\mathcal{K}_{h(S)} \subseteq \mathcal{K}_S$.*

Proof. Let $X := [\kappa]_{\cong_S}$ be the \cong -equivalence class of κ in S . For each d-state $x \in X$, consider an isomorphism $\pi_x : V_\kappa \rightarrow V_x$ from κ , and let $\mu_x : V_x \rightarrow \text{On}^r$ be the restriction of the parity progress measure μ to the domain of x . Now consider the point-wise ordering on X which puts $x > y$ if $\mu_x \circ \pi_x(v) \succ \mu_y \circ \pi_y(v)$ for each node $v \in V_\kappa$ in the well-founded lexicographic order \succ on On^r . Since κ is finite, by Dickson's Lemma, it follows that $>$ is a well-quasi order. Now, construct the retraction h that maps every d-state in X , to the minimum $>$ -comparable d-state X and fixes the nodes of any d-state that is not in X . Since $>$ is a well-quasi order, the set of incomparable elements is finite. Therefore $[\kappa]_{\cong_{h(S)}}$ is a finite set. As h fixes every

d -state that is not isomorphic to κ , we can conclude that $h(S)$ contains only $\cong_{h(S)}$ -classes of S and none of them increase its size. \blacktriangleleft

If a strategy witness is of finite \cong -index and its d -states are all finite, we can apply the above theorem successively to retract every \cong -class into a finite one, thus finally obtaining a finite-state strategy. This is the our main result.

► **Theorem 9.** *Let \mathcal{G} be a distributed game with an arbitrary ω -tree regular winning condition. If there exists a witness for \mathcal{G} in which all d -states are finite and the \cong -index is finite as well, then there exists a finite winning strategy for \mathcal{G} .*

For a given class \mathcal{C} of games, the Retraction Theorem 9 can be used as follows. We set out by considering tree-shaped strategies for the game instances $\mathcal{G} \in \mathcal{C}$. Note that in games where the observations and the winning condition are specified by finite-state automata, for every tree-shaped strategy structure S , the d -states $\kappa \in \mathcal{K}_S$ are finite, since there are only finitely many histories of the same length. Next, we look at d -cells κ that may appear in tree strategies S with a progress measure μ and construct partial retractions $h : \kappa \rightarrow \kappa$ that are μ -monotone and image finite; we speak of *horizontal* retractions, because each h maps any node of S to a node of the same depth in the strategy tree. If we succeed to construct horizontal imaga-finite retractions, we can apply Theorem 9 – which, intuitively, states that every \cong -class of a given strategy annotation can be compacted to a finite one via *vertical* retractions – to conclude that whenever a game in \mathcal{C} admits a winning strategy, it admits a finite-state winning strategy. If, additionally, the construction of horizontal and vertical retractions for a specific class \mathcal{C} allows to derive recursive bounds on the size and number the d -states in the retract, we obtain an effective procedure for solving the synthesis problem.

More generally, the theorem may be used with retractions that are not composed of horizontal and vertical mappings, and also with progress measures other than parity progress measures.

5 Applications

Theorem 9 generalises the perfect-information construction developed in [3] for the case of games where the winning condition is *observable*. In contrast to our setting where the winning condition automaton \mathcal{W} depends on the actions of all players, observable winning conditions correspond to the special case where the runs of \mathcal{W} depend only on *public* observations. More precisely, \mathcal{W} corresponds to an automaton that reads aggregations $c(b)$ of the observations b^i output by the Mealy automata M^i such that the values $c(b) \neq c(b')$ are different for two profiles b and b' only if $b^i \neq b'^i$ differ for all players i . In other words, for any observation profile b , the value $c(b)$ relevant for acceptance is common knowledge among the players.

The central result in [3] shows that there exists a uniform mapping $f : \Gamma^* \rightarrow \Gamma^*$ on the histories of the game that induces a retraction for any strategy annotation, and moreover guarantees that any two homomorphically equivalent d -states have the same image under f . For games with linear-time winning conditions, which can be determinised and thus yield a canonical annotation, this leads to retracts in which every homomorphism equivalence type of a d -state appears at most once. For games with finitely many d -states, up to homomorphic equivalence, this provides an effective solution to the synthesis problem.

Our setting is more general than the one of [3] in two respects: winning conditions are formulated as tree properties and, more importantly, they may be unobservable. Since tree automata recognising the winning condition cannot be determinised, there is no canonical run (even in a larger sense, see [6]), hence our construction relies on fixing a strategy and

a run. Due to non-observability of the condition, we also need to fix a progress measure – here we opted for parity progress measures, for simplicity; other progress measure work as well and may allow a better analysis. As a consequence of these arbitrary choices, it is not immediate to obtain general algorithmic results. Nevertheless, our framework can be used as a general tool to analyse specific game classes.

For instance, for classes of games in which the size of d-states is bounded by the input instance, Theorem 9 yields an upper bound on the size of a minimal winning strategy and thus provides a procedure for deciding whether a winning strategy exists, and for constructing one if this is the case. Even if such a procedure would be highly inefficient – particularly, as it relies on Dickson’s lemma – the framework allows to identify decidable instances.

We present two examples to illustrate this approach. Both refer to slight variations of a standard setting. Our first example is on a game between just one player with imperfect information against the environment. In the literature, such games have usually been considered with observable winning conditions ([29, 7]), which in our setting correspond to the situation where the winning condition automaton reads the output of the Mealy observation automaton, rather than moves that include information about the moves of Nature. Moreover, our example refers to branching-time specifications instead of the more classical linear-time conditions. Our second example refers to a multi-player game derived from the perfect-information setting by introducing an observation delay.

5.1 One-player games with hidden objectives

Let us consider a game \mathcal{G} for a single player 0 against Nature with a winning condition specified by an ω -tree automaton \mathcal{W} . We will show that whenever there exists a winning strategy for \mathcal{G} , there also exists a finite-state winning strategy. This is not a new result, it is well known for linear-time winning conditions and not surprising in the branching time setting.

Let us assume that the player has a winning strategy, possibly on an infinite set of states. Then, there exists a tree-shaped witness $\mathcal{S} = (V, E, v_\varepsilon, \rho)$ with a distinct node for every history that follows S . Further, by Theorem 5, there exists a parity progress measure μ on \mathcal{S} . Our aim is to construct a retract $h(\mathcal{S})$ with finitely many d-states, up to isomorphism, and then to apply Theorem 9.

Note that on any tree-shaped strategy structure, the indistinguishability \sim and the uniformity relation \cong coincide. Since S is finitely branching, there are only finitely many indistinguishable histories of the same length. Hence, each d-state of S is finite. As the domain of S is countable, we can enumerate the d-states as $\kappa_0, \kappa_1, \dots$

To define the mapping $h : V \rightarrow V$, we consider the d-states in this order. In the stage corresponding to a d-state $\kappa = \kappa_\ell$, we define for every label $q \in Q^1 \times Q_{\mathcal{W}}$ that appears on a node of V_κ the set $U_q := \{u \in V_\kappa \mid \rho(u) = q\}$ and pick the \succeq -least element $u_{\min} \in U_q$ with respect to the progress measure μ . Then, we map $h(u) := u_{\min}$ for all nodes $u \in U_q \setminus \cup_{j < \ell} V_{\kappa_j}$ that were not previously assigned.

The mapping h defined in this way is a retraction. All d-states in the retract are finite – each label appears at most once, hence, their size is at most $(|Q^1| \times |Q|)$. Accordingly, there are finitely many d-states, up to isomorphism. By Theorem 9, we can thus conclude that every solvable one-player game with imperfect information admits a finite-state winning strategy.

For the particular case of observable linear-time winning conditions, our construction yields the standard powerset construction for solving one-player games with imperfect information (see, e.g., [7]).

5.2 Coordination games with observation lags

Our second example involves a team of two players that play a parity game against Nature. The setting is standard, we assume that the players move in turns and receive perfect information about the current state, the only twist is that each of them may receive his information with a delay that is nondeterministically chosen by Nature within a bounded time window of up to 3 rounds (independently for the two players). Such a game model is more general than that of concurrent games.

Why are such games of interest? There is a spectrum of game models between the extremes of perfect information, where every player knows the global game state, and that of imperfect information, in which a player may remain perpetually uncertain in his knowledge of global state, or that of other players. A natural instance of such in-between games would be one with bounded imperfection, whereby every player receives perfect information about the global state, up to a bounded delay.

While the detailed formalisation of these games requires some redefinitions in our model, it is easy to get the overall idea of how the techniques developed in this paper can be applied, and we sketch the idea below.

Consider any witness strategy for the 3-delay-game, where the underlying strategy structure is a tree. Clearly such a witness always exists.

We now claim that the tree witness has all d-states of size at most $|A|^3$. To see this, observe that for any two histories τ, τ' in the witness strategy at depth ℓ , if their least common ancestor is at a depth less than $\ell - 3$, then they are distinguishable. Therefore, every \approx -connected component at depth ℓ is made up of histories which have their least common ancestor at depth greater than or equal to $\ell - 3$. Since every node has branching factor at most $|\Gamma|$, the d-states have size at most $|\Gamma|^3$.

Thus, we have only finitely many non-isomorphic d-states in the witness tree and hence by our earlier theorem, there is a finite-state winning strategy as well. Note that this holds for any bounded delay in receiving perfect information in coordination games.

6 Discussion

We have suggested that distributed synthesis in the context of finite state synchronous coordination games can be studied via retractions on winning strategies. The central idea is that retractions yield finite-state winning strategies when d-states are themselves finite, and the induced congruence classes can be bounded. While decision procedures in general work with some form of quotienting (as in the case of *filtrations* employed in modal logic), imperfect information games bring in the extra dimension of d-states potentially growing unboundedly. What is offered here is a technique for combining the two.

That retractions can be composed is easy to see, and hence we can hope to build structure in strategies via retractions, starting from abstract ones that realise a limited objective, and refining them successively. Progress measures then would need to be finer as well, as the applications demand. The admittedly limited examples presented here already suggest that there are many applications ahead.

The main question is a structural characterisation of the “largest” class of games for which retractions yield finite state winning strategies, and decidability of the existence of winning strategies. Another natural question is the characterisation of when memoryless winning strategies exist. The classification of decidable cases driven by practical applications (from the viewpoint of system design and verification) is perhaps more urgent.

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