Automates d'arbre

TD n°4 : Hedge Automata, alternating word automata and membership

Exercise 1 :

Let $\Sigma = \{a, b, c\}$. Define a NFHA $\mathcal{A}$ whose language is the trees which have two nodes labelled by b whose greatest common ancestor is labelled by c.

Solution:

$Q = \{q_\bot, q_\top, q_b\}$, $F = \{q_\top\}$ and $\Delta =$

- ⋆ $b(q_\bot^*) \longrightarrow q_b$
- ⋆ $\_(q_\bot^* q_b q_\bot^*) \longrightarrow q_b$
- ⋆ $c(q_\bot^* q_b q_\bot^* q_b q_\bot^*) \longrightarrow q_\top$
- ⋆ $\_(q_\bot^* q_\top q_\bot^*) \longrightarrow q_\top$
- ⋆ $\_(q_\bot^*) \longrightarrow q_\bot$

If $\mathcal{X}$ is a set of propositional variables, let $\mathbb{B}(\mathcal{X})$ be the set of positive propositional formulae on $\mathcal{X}$, i.e., formulae generated by the grammar $\phi ::= \bot \mid \top \mid x \in X \mid \phi \vee \phi \mid \phi \wedge \phi$.
A AWA (Alternating Word Automata) is a tuple $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ where Σ is a finite set (alphabet), Q is a finite set (of states), $Q_0 \subseteq Q$ (initial states), $Q_f \subseteq Q$ (final states) and δ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ on a word w is a tree t labelled by $Q \times \mathbb{N}$ such that :

— if $w = \varepsilon$, then $t = (q_0, 0)$ with $q_0 \in Q_0$.

— if $w = a.w'$, then $t = (q_0, k)(t_1, \dots, t_n)$ where k is the length of w, $q_0 \in Q_0$ and such that for all i, t_i is a run of w' on $(Q, \Sigma, \{q_i\}, Q_f, \delta)$ for some q_i satisfying $\{q_1, \dots, q_n\} \models \delta(q_0, a)$.

We say that a run is accepting if every leaf of the form $(q, 0)$ satisfies that $q \in Q_f$. Notice that a run may have leaves of the form (q, i) with $i > 0$ if $\varnothing \models \delta(q_0, a)$. Those leaves are considered as 'success' leaves in this semantic. The language of a AWA is the set of words which have an accepting run.

Exercise 2 :

1) Assume given a simple AWA $\mathcal{A}$, i.e., an AWA for which the transition function is with values in formulae of the form $q \wedge q'$, $q \vee q'$, $\top$ or $\bot$ with q, q' states. Given a word w, construct a top-down NFTA $\mathcal{A}_w$, in time $O(|w|.|\mathcal{A}|)$, such that $w \in L(\mathcal{A})$ iff $L(\mathcal{A}_w) \neq \varnothing$.
hint : If $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$, you will construct a top-down NFTA of the form $(Q', \Sigma', Q'_0, \Delta')$ with $\Sigma' = \{\wedge(2), \vee(1), \top(0), succ(0)\}$ and $Q' = \{0, 1, \dots, |w|\} \times Q$. The general idea is to have a correspondence between accepting runs of w in $\mathcal{A}$ and accepting runs in $\mathcal{A}_w$.
2) Generalize this construction (with the same complexity) to general AWA.
hint : The alphabet will be the same, but Q' will be something like

$$\{0, \dots, |w|\} \times (Q \sqcup \bigsqcup_{(a,q) \in \Sigma \times Q} Pos(\delta(q, a))$$

where $Pos(\phi)$ is the set of positions of the formula ϕ seen as a tree. You may use ε-transitions.

3) Deduce that membership in AWA is solvable in time $O(|w|.|\mathcal{A}|)$.
4) Prove that this problem is P-hard by reducing emptiness in NFTA.

Solution:

Exercise 3 :

1) Prove that the membership problem for a NFHA whose horizontal languages are given by (non necessarily deterministic) word automata is in PTIME. Prove that in the case where the NFHA is deterministic and every horizontal languages are given by deterministic automata, then your algorithm is in linear time.
2) Prove that membership for DFHA where horizontal languages are given by AWA is in PTIME.
3) Let Φ be a propositional formula in CNF with n variables $x_1, ..., x_n$. Construct, in polynomial time, an AWA $\mathcal{A}_\Phi$ whose language is $\{w \in \{0,1\}^n \mid w \models \Phi \text{ i.e. } \Phi_{[x_i \leftarrow w_i]} = \top\}$.
4) Deduce that membership for NFHA where horizontal languages are given by AWA is NP-complete.

Solution:

1) See section 8.5 of TATA.
2) Similar to 1).
3) Let $\Phi = \bigwedge_{j=1}^{m} C_j$ with C_j clauses. The expected AWA is the following :
   - $Q = \{q_0\} \cup \{q_j^{\alpha,k} \mid 1 \leq j \leq m, 1 \leq k \leq n, \alpha \in \{0,1\}\}$
   - $I = \{q_0\}$
   - $F = \{q_j^{1,n} \mid 1 \leq j \leq m\}$
   - $\Delta =$
     - ⋆ $q_0, 1 \longrightarrow \bigwedge_{j \mid x_1 \in C_j} q_j^{1,1} \wedge \bigwedge_{j \mid x_1 \notin C_j} q_j^{0,1}$
     - ⋆ $q_0, 0 \longrightarrow \bigwedge_{j \mid \neg x_1 \in C_j} q_j^{1,1} \wedge \bigwedge_{j \mid \neg x_1 \notin C_j} q_j^{0,1}$
     - ⋆ $q_j^{1,k}, \_ \longrightarrow q_1^{1,k+1}$ for all $k < n$ and all j
     - ⋆ $q_1^{0,k}, 1 \longrightarrow q_j^{1,k+1}$ for all j and all $k < n$ such that $x_k \in C_j$
     - ⋆ $q_1^{0,k}, 1 \longrightarrow q_j^{0,k+1}$ for all j and all $k < n$ such that $x_k \notin C_j$
     - ⋆ $q_1^{0,k}, 0 \longrightarrow q_j^{1,k+1}$ for all j and all $k < n$ such that $\neg x_k \in C_j$
     - ⋆ $q_1^{0,k}, 0 \longrightarrow q_j^{0,k+1}$ for all j and all $k < n$ such that $\neg x_k \notin C_j$
4)
   - in NP : first, guess a run i.e. a coloring ρ by states of your tree. Second, check this is an accepting run i.e. for all position p of t, check that there exists a transition of the form $t(p)(L) \longrightarrow \rho(p)$, that $\rho(p.1)...\rho(p.k) \in L$ (which can be done in P by exercise 2) and $\rho(\epsilon)$ is a final state.
   - NP-hard : we reduce SAT. Let Φ in CNF. We construct $\tilde{\mathcal{A}}_\Phi$ this way : $Q = \{0, 1, q_f\}$, $F = \{q_f\}$, $\Sigma = \{@, \#\}$ and $\Delta = \{\#(\epsilon) \longrightarrow 1, \#(\epsilon) \longrightarrow 0, a(\mathcal{A}_\Phi) \longrightarrow q_f\}$ where $\mathcal{A}_\Phi$ is from question 3. Then Φ is satisfiable iff $a(\#, ..., \#) \in L(\tilde{\mathcal{A}}_\Phi)$.