Automates d'arbre

TD n°3 : WSkS

Exercise 1 :

Produce formulae of WSkS for the following predicates :

- the set X has exactly two elements.
- the set X contains at least one string beginning with a 1.
- $x \leq_{lex} y$ where $\leq_{lex}$ is the lexicographic order on $\{1, ...k\}^*$.
- given a formula of WSkS ϕ with one free first-order variable, produce a formula of WSkS expressing that there is an infinity of words on $\{1, ..., k\}^*$ satisfying ϕ.
- the set X has an even number of elements.

Solution:

- $$|X| \leq 2 \doteq \forall Y. Y \subseteq X \Rightarrow (Y = \varnothing \vee Sing(Y) \vee Y = X)$$
  $$|X| \geq 2 \doteq \exists x, y. x \neq y \wedge x \in X \wedge y \in X$$
  $$|X| = 2 \doteq |X| \leq 2 \wedge |X| \geq 2$$
- $$X \cap 1.\Sigma^* \neq \varnothing \doteq \exists x. x \in X \wedge 1 \leq x$$
- $$x \leq_{lex} y \doteq x \leq y \vee (\exists z. \bigvee_{i<j\leq k} (z.i \leq x \wedge z.j \leq y))$$
- $$X \models \phi \doteq \forall x, x \in X \Rightarrow \phi(x)$$
  $$\phi \text{ satisfied by an infinity of words } \doteq \forall X,\ X \models \phi \Rightarrow \exists Y,\ X \subsetneq Y \wedge Y \models \phi$$
- $$y = x +_X 1 \doteq x \in X \wedge y \in X \wedge x \leq_{lex} y \wedge \neg\exists z. (z \in X \wedge x \lneq_{lex} z \wedge z \lneq_{lex} y)$$
  $$y = x +_X 2 \doteq \exists z. z = x +_X 1 \wedge y = z +_X 1$$
  $$x = X_{max} \doteq x \in X \wedge \forall y. (y \in X \Rightarrow y \leq_{lex} x)$$
  $$x = X_{min} \doteq x \in X \wedge \forall y. (y \in X \Rightarrow x \leq_{lex} y)$$
  $$|X| \text{ is even} \doteq \exists Y. (\exists x. x = X_{min} \wedge x \in Y) \wedge (\exists y. y = X_{max} \wedge y \notin Y) \wedge$$
  $$\forall z. (z \in Y \Rightarrow \exists w. (w = z +_X 2 \wedge w \in Y))$$

Exercise 2 :

Prove that the predicate $x = 1y$ is not definable in WSkS.

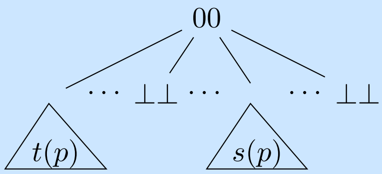
Solution:

We use the equivalence with recognizable tree languages. So we have to prove that $L = \{tra(x, y) \mid x = 1.y\}$ is not recognizable. Using the translation, we see that

$$L \cap \{t_i\sigma \mid t_i = 00(i\bot(x_1, ..., x_k), y_2, ..., y_k), i \in \{0, 1\}, \sigma \text{ closed substitution}\}$$

$$= \{tra(x,y) \mid x = 1.y \land y \in \{2,...,k\}.\{1,...,k\}^*\} = L'$$

So it is enough to prove that L' is not recognizable. Now elements of L' are of the form :



with $p \in \{2,...,k\}.\{1,...,k\}^*$, t and s injective and the height of t and s strictly increasing with p. You can reason by contradiction using the pumping lemma : for p large enough, using the pumping lemma, you can iterate a piece of $t(p)$ without touching $s(p)$ (or vice versa) while staying in L' which is absurd by injectivity.

Exercise 3 :

Let $n \in \mathbb{N}^*$, $n = \prod_{i=0}^{\infty} p_i^{e_i}$ its prime factorization and $e_i = \sum_{j=0}^{\infty} b_{i,j} 2^j$ the binary representation of e_i. We code n in WS2S by the set $S_n = \{1^i 2^j \mid b_{i,j} = 1\}$. Produce formulae of WS2S for the predicates $X = S_n$, $\exists n. X = S_n$ and $\exists n, m. X = S_n \land Y = S_m \land Z = S_{nm}$. Deduce that the first order theory of integers with multiplication and equality is decidable.

Solution:

—

- $\star$

$$x \in 1^\star.2^\star \doteq \exists z.\, z \leq x \land \forall Y.\, (z \in Y \land \forall y.\, (y.1 \in Y \Rightarrow y \in Y)) \Rightarrow \epsilon \in Y$$
$$\land \forall X.\, (x \in X \land \forall w.\, (w.2 \in X \Rightarrow w \in X)) \Rightarrow z \in X$$

- $\star$

$$\exists n.\, X = S_n \doteq X \subseteq 1^\star.2^\star \setminus \{\epsilon\} \doteq \forall x.\, x \in X \Rightarrow x \in 1^\star.2^\star \land x \neq \epsilon$$

—
$$X = S_n \doteq \forall x.\, (x \in X \Leftrightarrow \bigvee_{(i,j) \mid b_{i,j}=1} x = 1^i.2^j)$$

—

$$\exists n, m.\, X = S_n \land Y = S_m \land Z = S_{nm} \doteq (\exists n, X = S_n) \land (\exists m, Y = S_m) \land (\exists p, Z = S_p) \land$$
$$\exists R.\, \neg(\exists x.\, x \in 1^\star \land x \in R) \land \forall x.$$
$$(((x \in X \land x \in Y \land x \in R) \Rightarrow (x \in Z \land x.2 \in R)) \land$$
$$((x \in X \land x \notin Y \land x \in R) \Rightarrow (x \notin Z \land x.2 \in R)) \land$$
$$((x \in X \land x \notin Y \land x \notin R) \Rightarrow (x \in Z \land x.2 \notin R)) \land$$
$$((x \in X \land x \in Y \land x \notin R) \Rightarrow (x \notin Z \land x.2 \in R)) \land$$
$$((x \notin X \land x \in Y \land x \in R) \Rightarrow (x \notin Z \land x.2 \in R)) \land$$
$$((x \notin X \land x \in Y \land x \notin R) \Rightarrow (x \in Z \land x.2 \notin R)) \land$$
$$((x \notin X \land x \notin Y \land x \in R) \Rightarrow (x \in Z \land x.2 \notin R)) \land$$
$$((x \notin X \land x \notin Y \land x \notin R) \Rightarrow (x \notin Z \land x.2 \notin R)))$$

— by induction on the size of the formula. Watch out, $\exists x. ...$ should be transform into $\exists X.\, (\exists n.\, X = S_n) \land ...$ (idem for $\forall$), because there are sets that do not represent integers.

Exercise 4 :

Let L be a recognizable language on $\mathcal{F} = \{f_1, \dots, f_n\}$. Let ϕ be a WSkS formula defining L, i.e., a formula with $n+1$ second order variables X, X_1, ..., X_n such that for every $t \in T(\mathcal{F})$,

$$Pos(t), Pos_{f_1}(t), \dots, Pos_{f_n}(t) \models \phi \text{ iff } t \in L$$

where $Pos_{f_i}(t)$ is the set of positions of t labelled by f_i.
We want to prove that the language $\bar{L}$ of trees t such that for every decomposition $t = C[t']$ there is a decomposition $t' = C'[t'']$ with $C'[C[t'']] \in L$ is also recognizable. From ϕ, construct a WSkS formula $\bar{\phi}$ defining $\bar{L}$.

Solution:

We want to construct a formula $\bar{\phi}$ such that for every $t \in T(\mathcal{F})$,

$$t \in \bar{L} \text{ iff } Pos(t), Pos_{f_1}(t), \dots, Pos_{f_n}(t) \models \bar{\phi}.$$

For a tree t and positions $p \leq p'$ of t write p'' such that $p' = p.p''$ and $t_{p,p'}$ the tree :

$$t_{|p}[t[t_{|p'}]_p]_{p''}$$

Remember that $t_{|p}$ is the subtree of t at position p and $t[s]_p$ is the tree obtained by replacing the subtree at position p of t by s. Intuitively, $t_{p,p'}$ is the tree $C'[C[t'']]$ obtained by the decomposition $t = C[t']$ where t' is the subtree at position p of t and $t' = C'[t'']$ where t'' is the subtree at position p'' of t' (or equivalently, the subtree at position p' of t). $\bar{\phi}$ will have the following form :

$$\bar{\phi}(X, X_1, \dots, X_n) = \forall x. \exists y. x \in X \wedge y \in X \wedge x \leq y \wedge \tilde{\phi}(X, X_1, \dots, X_n, x, y)$$

where $\tilde{\phi}$ satisfies the following property : for every $t \in T(\mathcal{F})$ and every positions $p \leq p'$ of t,

$$Pos(t), Pos_{f_1}(t), \dots, Pos_{f_n}(t), p, p' \models \tilde{\phi} \text{ iff } Pos(t_{p,p'}), Pos_{f_1}(t_{p,p'}), \dots, Pos_{f_n}(t_{p,p'}) \models \phi.$$

Modulo renaming, we can assume that x and y do not appear in ϕ. $\tilde{\phi}$ will be defined by induction on the size of ϕ :

- ϕ is of the form $z = z'$ with z, z' first-order variables : $\tilde{\phi}$ is $z = z'$;
- ϕ is of the form $z = z'.i$ with z, z' first-order variables and $1 \leq i \leq k$: that is the interesting case. $\tilde{\phi}$ is then

$$(z'.i = x \Rightarrow z = y) \wedge (z'.i = y \Rightarrow z = \varepsilon) \wedge ((z'.i \neq y) \wedge (z'.i \neq x)) \Rightarrow z = z'.i \ ;$$

- ϕ is of the form $z \in Z$ with z first-order variable and Z second-order variable : $\tilde{\phi}$ is $z \in Z$;
- ϕ is of the form $\exists z.\phi'$: $\tilde{\phi}$ is $\exists z.\tilde{\phi}'$;
- ϕ is of the form $\exists Z.\phi'$: $\tilde{\phi}$ is $\exists Z.\tilde{\phi}'$;
- ϕ is of the form $\phi' \vee \phi''$: $\tilde{\phi}$ is $\tilde{\phi}' \vee \tilde{\phi}''$;
- ϕ is of the form $\neg\phi'$: $\tilde{\phi}$ is $\neg\tilde{\phi}'$.

$\tilde{\phi}$ has the same free variables as ϕ, plus two extra x and y. For words w, p, p' in $\{1, \dots, k\}$, with $p \leq p'$ (so $p' = p.p''$ for some p''), define $[w, p, p']$ the following word :

- if $p \not\leq w$, then $[w, p, p'] = p''.w$;
- if $p \leq w$, so $w = p.w'$, and $p' \not\leq w$, then $[w, p, p'] = w'$;
- if $p' \leq w$, so $w = p'.w'$ then $[w, p, p'] = p''.p.w'$.

For S a set of words, define $[S, p, p'] = \{[w, p, p'] \mid w \in S\}$. Observe that $[Pos(t), p, p'] = Pos(t_{p,p'})$ and $[Pos(t_{f_i}), p, p'] = Pos(t_{p,p'})$. Observe also that $[\_, p, p']$ is a bijection from $\{1, \dots, k\}^\star$ to $\{1, \dots, k\}^\star$. We prove the following invariant by induction on the size of ϕ : for every formula ϕ with free variables x_1, ..., x_r, X_1, ..., X_l, for every words w_1, ..., w_r, sets of words S_1, ..., S_l, and words $p \leq p'$:

$$w_1, \dots, w_r, S_1, \dots, S_l, p, p' \models \tilde{\phi} \text{ iff } [w_1, p, p'], \dots, [w_r, p, p'], [S_1, p, p'], \dots, [S_l, p, p'] \models \phi$$

— $z = z'$: $w_1 = w_2$ iff $[w_1, p, p'] = [w_2, p, p']$ since $[\_, p, p']$ is an injective function.
— $z = z'.i$: do the case distinction and draw a picture.
— $z \in Z$: $w_1 \in S_1$ iff $[w_1, p, p'] \in [S_1, p, p']$ by definition of $[S_1, p, p']$ and injectivity.
— $\exists z.\phi'$: if

$$w_1, \ldots, w_r, S_1, \ldots, S_l, p, p' \models \tilde{\phi} = \exists z.\tilde{\phi}'$$

then there is a word w such that

$$w_1, \ldots, w_r, w, S_1, \ldots, S_l, p, p' \models \tilde{\phi}'.$$

So by induction hypothesis,

$$[w_1, p, p'], \ldots, [w_r, p, p'], [w, p, p'], [S_1, p, p'], \ldots, [S_l, p, p'] \models \phi'$$

and then

$$[w_1, p, p'], \ldots, [w_r, p, p'], [S_1, p, p'], \ldots, [S_l, p, p'] \models \phi = \exists z.\phi'.$$

Conversely, if

$$[w_1, p, p'], \ldots, [w_r, p, p'], [S_1, p, p'], \ldots, [S_l, p, p'] \models \phi = \exists z.\phi'$$

then there is a word w such that

$$[w_1, p, p'], \ldots, [w_r, p, p'], w, [S_1, p, p'], \ldots, [S_l, p, p'] \models \phi'.$$

Since $[\_, p, p']$ is surjective, there is a word w' such that $[w', p, p'] = w$ and by induction hypothesis :

$$w_1, \ldots, w_r, w', S_1, \ldots, S_l, p, p' \models \tilde{\phi}'$$

and so

$$w_1, \ldots, w_r, S_1, \ldots, S_l, p, p' \models \tilde{\phi} = \exists z.\tilde{\phi}'.$$

— $\exists Z.\phi'$: similar.