Exercise 1:

We consider the (GII) problem (ground instance intersection):

**Instance**: A term in \(T(F,X)\) and \(A\) a NFTA

**Question**: Is there at least one ground instance of \(t\) accepted by \(A\)?

1) Suppose that \(t\) is linear. Prove that (GII) is P-complete.
   * hint: You may use ideas from exercise 3 of TD1. For the hardness, reduce the emptiness problem.

2) Suppose that \(A\) is deterministic. Prove that (GII) is NP-complete.
   * hint: For the hardness, reduce (SAT).

3) Prove that (GII) is EXPTIME-complete.
   * hint: For the hardness, reduce the intersection non-emptiness problem.

4) Deduce that the complement problem:
   **Instance**: A term in \(T(F,X)\) and linear terms \(t_1, \ldots, t_n\)
   **Question**: Is there a ground instance of \(t\) which is not an instance of any \(t_i\)?
   is decidable.

Solution:

1) in P: use a construction similar to exercise 4 TD1, intersect with \(A\) and test the non-emptiness.
   * P-hard: testing the emptiness of \(A\) is equivalent to testing (GII) on \(A\) and a variable.

2) in NP: guess for each variable an accessible state of \(A\) and verify that you can complete this to an accepting run by running the automata.
   * NP-hard: We reduce (SAT): let \(F = \{\neg(1), \vee(2), \wedge(2), \bot(0), \top(0)\}\) and \(A_{SAT}\) the DFTA with \(Q = \{q_\top, q_\bot\}\), \(F = \{q_\top\}\) and \(\Delta = \)
   * \(\star \bot \rightarrow q_\bot\)
   * \(\star \top \rightarrow q_\top\)
   * \(\neg(q_\alpha) \rightarrow q_{\neg\alpha}\)
   * \(\vee(q_\alpha, q_\beta) \rightarrow q_{\alpha\lor\beta}\)
   * \(\wedge(q_\alpha, q_\beta) \rightarrow q_{\alpha\land\beta}\)
   The language of \(A_{SAT}\) is the set of closed valid formulae.

   Let \(\phi\) a CNF formula, \(\phi = \bigwedge_{i=1}^n c_i\) where \(c_i\) are clauses. Define \(t_{c_i}\) by induction on the size of \(c_i\):
   * if \(c_i = x_j\), \(t_{c_i} = x_j\)
   * if \(c_i = \neg x_j\), \(t_{c_i} = \neg(x_j)\)
   * if \(c_i = x_j \lor c'_i\), \(t_{c_i} = \lor(x_j, t_{c'_i})\)
   * if \(c_i = \neg x_j \lor c'_i\), \(t_{c_i} = \lor(\neg(x_j), t_{c'_i})\)

   Then \(t_\phi = \bigwedge(t_{c_1}, \bigwedge(t_{c_2}, \ldots, \bigwedge(t_{c_{n-1}}, t_{c_n})\ldots))\). \(\phi\) is satisfiable iff a closed instance of \(t_\phi\) is recognized by \(A_{SAT}\).

3) in EXP: for each coloring of \(t\) by states (exponentially many):
   * check that the coloring of every occurrence of a variable is an accessible state (in P)
   * check that the coloring corresponds to an accepting run (in P)
Exercise 2:

A bottom-up tree transducer (NUTT) is a tuple set (of states), \( F \) and \( \Delta \) are finite ranked sets (of input and output), \( Q \) is a finite set of rules of the form:

- \( q(x_1, \ldots, x_n) \rightarrow q'(u) \) where \( u \in T(F', \{x_1, \ldots, x_n\}) \)
- \( q(x_1) \rightarrow q'(u) \) where \( u \in T(F', \{x_1\}) \).

We say that \( \Delta' = \{ q(n(h(x_1, \ldots, x_n)) \rightarrow h(q_1(x_1), \ldots, q_n(x_n)) \mid \text{for } q_k \in I_k \} \)

Then \( L(A_1) \cap \ldots \cap L(A_n) \neq \emptyset \) iff \( t \) has a closed instance in \( L(\bar{A}) \).

4) Use question 3 and exercise 4 of TD1.

Solution:

1) \( Q = \{ q \}, Q_f = \{ Q \} \) and \( \Delta = \)

- \( f(q_1(x_1), \ldots, q_n(x_n)) \rightarrow q(u) \) where \( f \in F \) and \( u \in T(F', \{x_1, \ldots, x_n\}) \)
- \( q(x_1) \rightarrow q'(u) \) where \( u \in T(F', \{x_1\}) \).

We say that \( U \) is linear when the right side of the rules of \( \Delta \) are. This defines a rewrite system \( \rightarrow_U \) on \( T(F \cup F' \cup Q) \). The relation induced by \( U \) is then \( R(U) = \{ (t, t') \mid t \in T(F), t' \in T(F'), t \rightarrow_U q(t'), q \in Q_f \} \).

1) Prove that tree morphisms are a special case of NUTT that is if \( \mu : T(F) \rightarrow T(F') \) is a morphism, then there exists a NUTT \( U_\mu \) such that \( R(U_\mu) = \{ (t, \mu(t)) \mid t \in T(F) \} \). Be sure that if \( \mu \) is linear then \( U_\mu \) is too.

2) Prove that the domain of a NUTT \( U \), that is \( \{ t \in T(F) \mid \exists t' \in T(F'), (t, t') \in U \} \), is recognizable.

3) Prove that the image of a recognizable tree language \( L \) by a linear NUTT \( U \), that is \( \{ t' \in T(F') \mid \exists t \in L, (t, t') \in U \} \), is recognizable.
Exercise 3:

1) We can see the set of runs of an NFTA $A = (Q, F, Q_f, \Delta)$ as a tree language on $F \times Q = \{(f, q)(n) \mid f(n) \in F, q \in Q\}$ as the smallest set $Run(A)$ included in $T(F \times Q)$ such that:

- if $a \rightarrow q \in \Delta$, then $(a, q) \in Run(A)$
- if $f(q_1, ..., q_n) \rightarrow q \in \Delta$ and $t_1, ..., t_n \in Run(A)$ with $t_i(\epsilon) = (\_, q_i)$ then $(f, q)(t_1, ..., t_n) \in Run(A)$.

Then the set of accepting runs can be seen as $Acc(A) = \{t \in Run(A) \mid t(\epsilon) = (\_, q), q \in Q_f\}$.

Prove that $Acc(A)$ is in the smallest class $Stab$ of sets which contains all the $T(F)$ for any finite ranked set $F$ and which is stable by image of linear morphisms and inverse image of morphisms. For example, you should be able to prove that $Acc(A) = \beta^{-1}(\gamma(\delta^{-1}(T(F'))))$ where $\gamma$ is linear.

2) Deduce that $Stab = Rec$.

Solution:

1) We define $\beta, \gamma$ and $\delta$ this way:

- $\delta : T(F_A) \longrightarrow T(F_{\bot})$ where $F_{\bot} = F \cup \{q(1) \mid q \in F\} \cup \{\bot(0)\}$ and $F_A = \{(f, q_1, ..., q_n, q)(n) \mid f(n) \in F, q_i \in q\} \cup \{q(1) \mid q \in Q\}$ such that:
  
  * $q(x) \rightarrow q(x)$ if $q \in F$
  * $q(x) \rightarrow \bot$ if $q \notin F$
  * $(f, q_1, ..., q_n, q)(x_1, ..., x_n) \rightarrow f(x_1, ..., x_n)$ if $f(q_1, ..., q_n) \rightarrow q \in \Delta$
  * $(f, q_1, ..., q_n, q)(x_1, ..., x_n) \rightarrow \bot$ else

- $\gamma : T(F_A) \longrightarrow T(F_Q)$ linear where $T(F_Q) = F \cup \{q(1) \mid q \in Q\}$ such that:
  
  ** $q(x) \rightarrow q(x)$
  ** $(f, q_1, ..., q_n, q)(x_1, ..., x_n) \rightarrow q(f(q_1(x_1), ..., q_n(x_n)))$

- $\beta : T(F \times Q) \longrightarrow T(F_Q)$ such that:
  
  *** $(f, q)(x_1, ..., x_n) \rightarrow q(f(x_1, ..., x_n))$

Then $Acc(A) = \beta^{-1}(\gamma(\delta^{-1}(T(F_{\bot} \setminus \bot))))$.

2) $Stab \subseteq Rec$ : $Rec$ is stable under inverse image, linear image and contains all the $T(F)$.

$Stab \subseteq Rec$ : Let $L \in Rec$ and $A$ a NFTA recognizing $L$. Define $\alpha : T(F \times Q) \longrightarrow T(F)$ linear such that:

*** $(f, q)(x_1, ..., x_n) \rightarrow f(x_1, ..., x_n)$

Then $L = \alpha(Acc(A))$ and by 1), $L \in Stab$. 