Automates d'arbre

TD n°2 : Decision problems & tree homomorphisms

Exercise 1 :

We consider the **(GII)** problem (ground instance intersection) :

Instance : t a term in $T(\mathcal{F}, \mathcal{X})$ and $\mathcal{A}$ a NFTA

Question : Is there at least one ground instance of t accepted by $\mathcal{A}$?

1) Suppose that t is linear. Prove that **(GII)** is P-complete.
   hint : You may use ideas from exercise 3 of TD1. For the hardness, reduce the emptiness problem.
2) Suppose that $\mathcal{A}$ is deterministic. Prove that **(GII)** is NP-complete.
   *hint : for the hardness, reduce **(SAT)**.*
3) Prove that **(GII)** is EXPTIME-complete.
   hint : for the hardness, reduce the intersection non-emptiness problem.
4) Deduce that the complement problem :
   Instance : t a term in $T(\mathcal{F}, \mathcal{X})$ and linear terms t_1, ..., t_n
   Question : Is there a ground instance of t which is not an instance of any t_i ?
   is decidable.

Solution:

1) in P : use a construction similar to exercise 4 TD1, intersect with $\mathcal{A}$ and test the non-emptiness.
   P-hard : testing the emptiness of $\mathcal{A}$ is equivalent to testing **(GII)** on $\mathcal{A}$ and a variable.
2) in NP : guess for each variable an accessible state of $\mathcal{A}$ and verify that you can complete this to an accepting run by running the automata.
   NP-hard : We reduce **(SAT)** : let $\mathcal{F} = \{\neg(1), \vee(2), \wedge(2), \bot(0), \top(0)\}$ and $\mathcal{A}_{SAT}$ the DFTA with $Q = \{q_\top, q_\bot\}$, $F = \{q_\top\}$ and $\Delta =$
   - ⋆ $\bot \longrightarrow q_\bot$
   - ⋆ $\top \longrightarrow q_\top$
   - ⋆ $\neg(q_\alpha) \longrightarrow q_{\neg\alpha}$
   - ⋆ $\vee(q_\alpha, q_\beta) \longrightarrow q_{\alpha\vee\beta}$
   - ⋆ $\wedge(q_\alpha, q_\beta) \longrightarrow q_{\alpha\wedge\beta}$

   The language of $\mathcal{A}_{SAT}$ is the set of closed valid formulae.
   Let ϕ a CNF formula, $\phi = \bigwedge_{i=1}^{n} c_i$ where c_i are clauses. Define t_{c_i} by induction on the size of c_i :
   - if $c_i = x_j$, $t_{c_i} = x_j$
   - if $c_i = \neg x_j$, $t_{c_i} = \neg(x_j)$
   - if $c_i = x_j \vee c'_i$, $t_{c_i} = \vee(x_j, t_{c'_i})$
   - if $c_i = \neg x_j \vee c'_i$, $t_{c_i} = \vee(\neg(x_j), t_{c'_i})$

   Then $t_\phi = \wedge(t_{c_1}, \wedge(t_{c_2}, ..., \wedge(t_{c_{n-1}}, t_{c_n})...))$. ϕ is satisfiable iff a closed instance of t_ϕ is recognized by $\mathcal{A}_{SAT}$.
3) in EXP : for each coloring of t by states (exponentially many) :
   - check that the coloring of every occurrence of a variable is an accessible state (in P)
   - check that the coloring corresponds to an accepting run (in P)

— for every variable, let $\{q_1, ..., q_n\}$ be the set of the colorings of all occurrence of x. Check that $L(\mathcal{A}_{q_1}) \cap ... \cap L(\mathcal{A}_{q_n})$ is non empty where A_q is the NFTA obtained from $\mathcal{A}$ by changing the set of final states to $\{q\}$ (in P)

EXP-hard : We reduce intersection non-emptiness : let $(A_k = (Q_k, \mathcal{F}, I_k, \Delta_k))_{k \in \{1,...,n\}}$ a finite sequence of top-down NFTA (we can transform a bottom-up NFTA to a top-down one in polynomial time). We suppose that all the Q_k are disjoint. Define :

— $\mathcal{F}' = \mathcal{F} \cup \{h(n)\}$
— $t = h(x, ..., x)$
— $\tilde{\mathcal{A}} = (\bigsqcup Q_k \sqcup \{q_0\}, \mathcal{F}', \{q_0\}, \Delta' \sqcup \bigsqcup \Delta_k)$ where

$$\Delta' = \{q_0(h(x_1, ..., x_n)) \longrightarrow h(q_1(x_1), ..., q_n(x_n)) \mid \; for\ q_k \in I_k\}$$

Then $L(\mathcal{A}_1) \cap ... \cap L(\mathcal{A}_n) \neq \varnothing$ iff t has a closed instance in $L(\tilde{A})$.

4) Use question 3 and exercise 4 of TD1.

Exercise 2 :

A bottom-up tree transducer (NUTT) is a tuple $U = (Q, \mathcal{F}, \mathcal{F}', Q_f, \Delta)$ where Q is a finite set (of states), $\mathcal{F}$ and $\mathcal{F}'$ are finite ranked sets (of input and output), $Q_f \subseteq Q$ (final states) and Δ is a finite set of rules of the form :

- $f(q_1(x_1), ..., q_n(x_n)) \to q(u)$ where $f \in \mathcal{F}$ and $u \in T(\mathcal{F}', \{x_1, ..., x_n\})$
- $q(x_1) \to q'(u)$ where $u \in T(\mathcal{F}', \{x_1\})$.

We say that U is linear when the right side of the rules of Δ are. This defines a rewrite system $\to_U$ on $T(\mathcal{F} \cup \mathcal{F}' \cup Q)$. The relation induced by U is then $\mathcal{R}(U) = \big\{(t, t') \mid t \in T(\mathcal{F}),\, t' \in T(\mathcal{F}'),\, t \to_U^* q(t'),\, q \in Q_f\big\}$.

1) Prove that tree morphisms are a special case of NUTT that is if $\mu : T(\mathcal{F}) \longrightarrow T(\mathcal{F}')$ is a morphism, then there exists a NUTT U_μ such that $\mathcal{R}(U_\mu) = \big\{(t, \mu(t)) \mid t \in T(\mathcal{F})\big\}$. Be sure that if μ is linear then U_μ is too.
2) Prove that the domain of a NUTT U, that is $\big\{t \in T(\mathcal{F}) \mid \exists t' \in T(\mathcal{F}'),\, (t, t') \in U\big\}$, is recognizable.
3) Prove that the image of a recognizable tree language L by a linear NUTT U, that is $\big\{t' \in T(\mathcal{F}') \mid \exists t \in L,\, (t, t') \in U\big\}$, is recognizable.

Solution:

1) $Q = \{q\}$, $Q_f = \{Q\}$ and $\Delta =$
   ⋆ $f(q(x_1), ..., q(x_n)) \longrightarrow q(\mu(f)(x_1, ..., x_n))$ linear when μ is
2) $Q = Q_U$, $F = F_U$ and $\Delta =$
   ⋆ $f(q_1, ..., q_n) \longrightarrow q$ if there exists u such that $f(q_1(x_1), ..., q_n(x_n)) \longrightarrow q(u) \in \Delta_U$
   ⋆ $q \longrightarrow q'$ if there exists u such that $q(x_1) \longrightarrow q'(u) \in \Delta_U$
3) Let U a NUTT and $\mathcal{A}$ a NFTA on $\mathcal{F}$. For every pair of rules $r = f(q_1(x_1), ..., q_n(x_n)) \longrightarrow q(u) \in \Delta_U$ and $r' = f(q'_1, ..., q'_n) \longrightarrow q' \in \Delta_{\mathcal{A}}$, we define :
   — $Q^{r,r'} = \{q_p^{r,r'} \mid p \in Pos(u)\}$
   — $\Delta^{r,r'} =$
     ⋆ $g(q_{p.1}^{r,r'}, ..., q_{p.k}^{r,r'}) \longrightarrow q_p^{r,r'}$ for $p \in Pos(u)$ such that $u(p) = g \in \mathcal{F}'$
     ⋆ $(q_i, q'_i) \longrightarrow q_p^{r,r'}$ if $u(p) = x_i$ (linearity assure that we only have one of this kind for every i)
     ⋆ $q_\epsilon^{r,r'} \longrightarrow (q, q')$

   For every rule $r = q(x) \longrightarrow q'(u) \in \Delta_U$, we define :
   — $Q^r = \{q_p^r \mid p \in Pos(u)\} \times Q_{\mathcal{A}}$
   — $\Delta^r =$
     ⋆ $g((q_{p.1}^r, q''), ..., (q_{p.k}^r, q'')) \longrightarrow (q_p^r, q'')$ for $p \in Pos(u)$ such that $u(p) = g \in \mathcal{F}'$ and $q'' \in Q_{\mathcal{A}}$
     ⋆ $(q, q'') \longrightarrow (q_p^r, q'')$ if $u(p) = x$ and $q'' \in Q_{\mathcal{A}}$ (linearity assure that we only have one of this kind)
     ⋆ $(q_\epsilon^r, q'') \longrightarrow (q, q'')$

Then this NFTA works :

$$\tilde{Q} = Q_U \times Q_{\mathcal{A}} \cup \bigcup_{(r,r')} Q^{r,r'} \cup \bigcup_r Q^r$$

$$\tilde{F} = F_U \times F_{\mathcal{A}}$$

$$\tilde{\Delta} = \bigcup_{(r,r')} \Delta^{r,r'} \cup \bigcup_r \Delta^r$$

Exercise 3 :

1) We can see the set of runs of an NFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ as a tree language on $\mathcal{F} \times Q = \{(f,q)(n) \mid f(n) \in \mathcal{F},\ q \in Q\}$ as the smallest set $Run(\mathcal{A})$ included in $T(\mathcal{F} \times Q)$ such that :
   - if $a \to q \in \Delta$, then $(a,q) \in Run(\mathcal{A})$
   - if $f(q_1, ..., q_n) \to q \in \Delta$ and $t_1, ..., t_n \in Run(\mathcal{A})$ with $t_i(\epsilon) = (\_, q_i)$ then $(f,q)(t_1, ..., t_n) \in Run(\mathcal{A})$.

   Then the set of accepting runs can be seen as $Acc(\mathcal{A}) = \{t \in Run(\mathcal{A}) \mid t(\epsilon) = (\_, q),\ q \in Q_f\}$.

   Prove that $Acc(\mathcal{A})$ is in the smallest class **Stab** of sets which contains all the $T(\mathcal{F})$ for any finite ranked set $\mathcal{F}$ and which is stable by image of linear morphisms and inverse image of morphisms. For example, you should be able to prove that $Acc(\mathcal{A}) = \beta^{-1}(\gamma(\delta^{-1}(T(\mathcal{F}'))))$ where γ is linear.
2) Deduce that **Stab** = **Rec**.

Solution:

1) We define β, γ and δ this way :
   — $\delta : T(\mathcal{F}_{\mathcal{A}}) \longrightarrow T(\mathcal{F}_{\perp})$ where $\mathcal{F}_{\perp} = \mathcal{F} \cup \{q(1) \mid q \in F\} \cup \{\perp(0)\}$ and $\mathcal{F}_{\mathcal{A}} = \{(f, q_1, ..., q_n, q)(n) \mid f(n) \in \mathcal{F}, q, q_i \in q\} \cup \{q(1) \mid q \in Q\}$ such that
   - $\star$ $q(x) \mapsto q(x)$ if $q \in F$
   - $\star$ $q(x) \mapsto \perp$ if $q \notin F$
   - $\star$ $(f, q_1, ..., q_n, q)(x_1, ...x_n) \mapsto f(x_1, ..., x_n)$ if $f(q_1, ..., q_n) \longrightarrow q \in \Delta$
   - $\star$ $(f, q_1, ..., q_n, q)(x_1, ...x_n) \mapsto \perp$ else

   — $\gamma : T(\mathcal{F}_{\mathcal{A}}) \longrightarrow T(\mathcal{F}_Q)$ linear where $T(\mathcal{F}_Q) = \mathcal{F} \cup \{q(1) \mid q \in Q\}$ such that :
   - $\star\star$ $q(x) \mapsto q(x)$
   - $\star\star$ $(f, q_1, ..., q_n, q)(x_1, ..., x_n) \mapsto q(f(q_1(x_1), ..., q_n(x_n)))$

   — $\beta : T(\mathcal{F} \times Q) \longrightarrow T(\mathcal{F}_Q)$ such that :
   - $\star\star\star$ $(f,q)(x_1, ..., x_n) \mapsto q(q(f(x_1, ..., x_n)))$

   Then $Acc(\mathcal{A}) = \beta^{-1}(\gamma(\delta^{-1}(T(\mathcal{F}_{\perp} \setminus \perp))))$.
2) **Stab** $\subseteq$ **Rec** : **Rec** is stable under inverse image, linear image and contains all the $T(\mathcal{F})$.

   Stab $\subseteq$ **Rec** : Let $L \in$ **Rec** and $\mathcal{A}$ a NFTA recognizing L. Define $\alpha : T(\mathcal{F} \times Q) \longrightarrow T(\mathcal{F})$ linear such that :

   $\star\star\star\star$ $(f,q)(x_1, ..., x_n) \mapsto f(x_1, ..., x_n)$

   Then $L = \alpha(Acc(\mathcal{A}))$ and by 1), $L \in$ **Stab**.