Exercise 1:
Let $F = \{ f(2), g(1), a(0) \}$. Give a DFTA and a top-down DFTA for the set $G(t)$ of ground instances of the term $t = f(f(a, x), g(y))$ which is defined by:

$$G(t) = \left\{ f(f(a, u), g(v)) \mid u, v \in T(F) \right\}$$

Exercise 2:
Are the following tree languages recognizable (by a bottom-up FTA)?
- $F = \{ f(2), a(0) \}$ and $L = \{ f(t, t) \mid t \in T(F) \}$.
- $F = \{ g(1), a(0) \}$ and $L$ the set of ground terms of even height.
- $F = \{ f(2), g(1), a(0) \}$ and $L$ the set of ground terms of even height.

Exercise 3:
1) Let $E$ be a finite set of linear terms on $T(F, X)$. Prove that $Red(E) = \{ C[t\sigma] \mid C \in C(F), t \in E, \sigma$ ground substitution $\}$ is recognizable.
2) Prove that if $E$ contains only ground terms, then one can construct a DFTA recognizing $Red(E)$ whose number of states is at most $n + 2$, where $n$ is the number of nodes of $E$.

Exercise 4:
Let $F = \{ f(2), a(0), b(0) \}$.
1) Let $L_1$ be the smallest set such that:
   - $f(a, b) \in L_1$
   - $t \in L_1 \Rightarrow f(a, f(t, b)) \in L_1$
   Prove that $L_1$ is recognizable.
2) Prove that $L_2 = \{ t \in T(F) \mid |t|_{a} = |t|_{b} \}$ is not recognizable.
3) Let $L$ be recognizable on $F$ and $C(L)$ be the closure of $L$ by the congruence generated by the equation $f(x, y) = f(y, x)$. Prove that $C(L)$ is recognizable.
4) Let $L$ be recognizable on $F$ and $AC(L)$ be the closure of $L$ by the congruence generated by the equations $f(x, y) = f(y, x)$ and $f(x, f(y, z)) = f(f(x, y), z)$. Prove that $AC(L)$ is not recognizable in general.
5) Let $L$ be recognizable on $F$ and $A(L)$ be the closure of $L$ by the congruence generated by the equation $f(x, f(y, z)) = f(f(x, y), z)$. Prove that $A(L)$ is not recognizable in general.

Exercise 5:
Let $F = \{ f(2), g(1), a(0) \}$. Give a DFTA and a top-down NFTA for the set $M(t)$ of terms which have a ground instance of the term $t = f(a, g(x))$ as a subterm, ie. $M(t) = \{ C[f(a, g(u))] \mid C \in C(F), u \in T(F) \}$.

Prove that $M(t)$ is not recognizable by a top-down DFTA or even a finite union of languages recognizable by a top-down DFTA.

Hint: you can use without proof the following fact (prove it if you have time): let $t$ be a tree. The path language $\pi(t)$ is defined by:

- if $t$ is a constant, $\pi(t) = \{ t \}$
- if $t = f(t_1, ..., t_n)$, $\pi(t) = \bigcup_{i=1}^{n} \{ fiw \mid w \in \pi(t_i) \}$

Let $L$ be a tree language. The path language of $L$ is $\pi(L) = \bigcup_{t \in L} \pi(t)$. The path closure of $L$ is defined by $\text{pathclosure}(L) = \{ t \mid \pi(t) \subseteq \pi(L) \}$. $L$ is recognizable by a top-down DFTA iff $L$ is recognizable and path closed, ie. $L = \text{pathclosure}(L)$.

*taken from *Tree Automata Techniques and Applications*