Techniques de réécriture

TD n°4 : Confluence & completion

Exercise 1:
Compute the critical pairs of the following rewrite systems. Which one are locally confluent?
1. \( f(g(f(x))) \rightarrow x, f(g(x)) \rightarrow g(f(x)) \)
2. \( 0 + y \rightarrow y, x + 0 \rightarrow x, s(x) + y \rightarrow s(x + y), x + s(y) \rightarrow s(x + y) \)
3. \( f(x, x) \rightarrow a, f(x, g(x)) \rightarrow b \)
4. \( f(f(x, y), z) \rightarrow f(x, f(y, z)), f(x, 1) \rightarrow x \)

Exercise 2:
Let \( P = (\alpha_i, \beta_i)_{1 \leq i \leq n} \) be an instance of PCP. Define \( R(P) = \{ A \rightarrow f(\alpha_i(\epsilon), \beta_i(\epsilon)), f(x, y) \rightarrow f(\alpha_i(x), \beta_i(y)), f(x, x) \rightarrow B, f(x, y) \rightarrow A \} \) on \( F = \{ f(2), A(0), B(0), 0(1), 1(1), \epsilon(0) \} \).
1. Prove that \( P \) has a solution iff \( A \rightarrow^* B \).
2. Deduce that confluence is undecidable.

Algorithm 1 Basic completion procedure

Require: A finite set \( E \) of identities and a reduction order \( > \)
Ensure: A finite convergent rewrite system \( R \) equivalent to \( E \) if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully
1: if there exists \((s, t) \in E \) such that \( s \neq t, s \not> t \) and \( t \not> s \) then
2: terminates with output FAIL
3: else
4: \( i := 0 \)
5: \( R_0 := \{(l, r) | (l, r) \in E \cup E^{-1} \land l > r \} \)
6: end if
7: repeat
8: \( R_{i+1} := R_i \)
9: for all \((s, t) \in CP(R_i)\) do
10: Reduce \( s \) and \( t \) to some \( R_i \)-normal forms \( \tilde{s} \) and \( \tilde{t} \)
11: if \( \tilde{s} \neq \tilde{t} \wedge \tilde{s} \not> \tilde{t} \wedge \tilde{t} \not> \tilde{s} \) then
12: terminates with output FAIL
13: end if
14: if \( \tilde{s} > \tilde{t} \) then
15: \( R_{i+1} := R_{i+1} \cup \{(\tilde{s}, \tilde{t})\} \)
16: end if
17: if \( \tilde{t} > \tilde{s} \) then
18: \( R_{i+1} := R_{i+1} \cup \{(\tilde{t}, \tilde{s})\} \)
19: end if
20: end for
21: \( i := i + 1 \)
22: until \( R_i = R_{i+1} \)
23: return \( R_i \)
Exercise 3:
We are considering this basic completion procedure.
1. Prove that this procedure is correct by showing it consists in a strategy for applying some rules from the completion procedure seen in the course.
2. Which rules are not used?
3. What can you say about $\bigcup_{i\in\mathbb{N}} R_i$ if the procedure does not terminate?

Exercise 4:
Apply the basic completion procedure on the following set of identities, with the suitable reduction order:
1. $\{ (x \ast (y + z), (x \ast y) + (x \ast z)), ((u + v) \ast w, (u \ast w) + (v \ast w)) \}$ and the LPO with $\ast > +$.
2. $\{ (x + 0, x), (x + s(y), s(x + y)) \}$ and the KBO with $s > +$ and weight 1 for all variables and symbols.
3. $\{ (f(g(f(x))), x) \}$ and the LPO with $f > g$.

Algorithm 2 Huet’s completion procedure

Require: A finite set $E$ of identities and a reduction order $>$
Ensure: A finite convergent rewrite system $R$ equivalent to $E$ if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully
1: $R_0 := \emptyset$; $E_0 := E$; $i := 0$
2: while $E_i \neq \emptyset$ or there is an unmarked rule in $R_i$ do
3: while $E_i \neq \emptyset$ do
4: Choose an identity $(s, t) \in E$
5: Reduce $s$ and $t$ to some $R_i$-normal forms $\tilde{s}$ and $\tilde{t}$
6: if $\tilde{s} = \tilde{t}$ then
7: $R_{i+1} := R_i$; $E_{i+1} := E_i \setminus \{(s, t)\}$; $i := i + 1$
8: else
9: if $\tilde{s} \neq \tilde{t}$ then
10: terminates with output FAIL
11: else
12: let $l$ and $r$ such that $\{l, r\} = \{\tilde{s}, \tilde{t}\}$ and $l > r$
13: $R_{i+1} := \{(g, d) \mid (g, d) \in R_i \land g$ cannot be reduced with $l \rightarrow r \land \tilde{d}$ is a $R_i \cup \{(l, r)\}$-normal form of $d) \cup \{(l, r)\}$
14: $(l, r)$ is not marked and $(g, d)$ is marked in $R_{i+1}$ iff $(g, d)$ is in $R_i$
15: $E_{i+1} := (E_i \setminus \{(s, t)\}) \cup \{(g', d) \mid (g, d) \in R_i \land g$ can be reduced to $g'$ with $l \rightarrow r\}$
16: $i := i + 1$
17: end if
18: end if
19: end while
20: if there is an unmarked rule in $R_i$ then
21: let $(l, r)$ be such a rule
22: $R_{i+1} := R_i$
23: $E_{i+1} := \{(s, t) \mid (s, t)$ is a critical pair of $(l, r)$ with itself or with a marked rule in $R_i\}$
24: $i := i + 1$
25: Mark $(l, r)$
26: end if
27: end while
28: return $R_i$

Exercise 5:
We are now considering Huet’s completion procedure.
1. Do the same study as exercise 4.
2. Prove that the set of identities
\[
\{(\@\text{nil}, x), \\
(\@\text{cons}(x, y), z, \text{cons}(x, \@\text{cons}(y, z))), \\
(\text{rev}(\text{nil}), \text{nil}), (\text{rev}\text{cons}(x, y)), \\
\@\text{rev}(y), \text{cons}(x, \text{nil}))\}
\]
can be oriented to give a convergent TRS. Let \( R \) this TRS.

3. Prove that the associativity \( A \) of \( \@ \), \( \@\text{cons}(x, y), z) = \@\text{cons}(x, \@\text{cons}(y, z)) \) is not a consequence of \( R \).

4. How would you prove associativity of concatenation of lists?

5. Prove that you can complete \((A, R)\). You can use Huet’s completion procedure.

6. Prove that the idempotence \( I \) of \text{rev}, \text{rev}(\text{rev}(x)) = x \) is not a consequence of \( R \).

7. Prove that you can complete \((I, R)\).

8. Prove that Huet’s completion fails to complete \((\{\text{rev}(x) = \@\text{cons}(x, x)\}, R)\).