Techniques de réécriture

TD n°1 : Termination & interpretations

Exercise 1:
Are those rewrite systems terminating?
- \{f(a) \rightarrow f(b) ; g(b) \rightarrow g(a)\}
- \{f(f(x)) \rightarrow f(g(f(x)))\}
- \{f(g(x)) \rightarrow g(f(x))\}
- \{f(g(x)) \rightarrow g(g(f(x)))\}

We recall that a polynomial interpretation on integers is the following data:
- a subset \(A\) of \(\mathbb{N}\);
- for every symbol \(f\) of arity \(n\), a polynomial \(P_f \in \mathbb{Z}[X_1, \ldots, X_n]\)
satisfying that for every symbol \(f\) of arity \(n\):
  - for every \(a_1, \ldots, a_n \in A\), \(P_f(a_1, \ldots, a_n) \in A\)
  - for every \(a_1, \ldots, a_i > a'_i, \ldots, a_n \in A\), \(P_f(a_1, \ldots, a_i, \ldots, a_n) > P_f(a_1, \ldots, a'_i, \ldots, a_n)\).
Then \((A, (P_f)_f, \succ)\) is a well-founded monotone algebra.

Exercise 2:
Prove the termination of the following rewrite systems using the given polynomial interpretation on integers:
1. \{x+0 \rightarrow x ; x+s(y) \rightarrow s(x+y) ; d(0) \rightarrow 0 ; d(s(x)) \rightarrow s(d(x))\} on \(\mathbb{N} \setminus \{0, 1\}\) with \(P_f(X, Y) = X + Y\), \(P_g(X) = X + 1\), \(P_d(X) = 3X\), \(P_c = X^3\) and \(P_0 = 2\).
2. \{f(f(x, y), z) \rightarrow f(x, f(y, z)) ; f(y, f(x, z)) \rightarrow f(x, x)\} on \(\mathbb{N} \setminus \{0, 1, 2\}\) with \(P_f = X^2 + XY\).

Exercise 3:
Prove the termination of the following rewrite system by finding a polynomial interpretation on integers:
\[x \times (y + z) \rightarrow (x \times y) + (x \times z) ; (x + y) + z \rightarrow x + (y + z)\]

Exercise 4:
Let \(R\) be a rewrite system whose termination can be proved using a polynomial interpretation on integers. Let \(A \subseteq \mathbb{N}\) be its domain and for every \(f\) in the alphabet \(F\), \(P_f\) the interpretation of \(f\). Take \(a \in A \setminus \{\emptyset\}\).
1. Define \(\pi_a : T(F, X) \rightarrow A \setminus \{\emptyset\}\) the function which maps every variable \(x\) to \(a\) and every term of the form \(f(t_1, \ldots, t_n)\) to \(P_f(\pi_a(t_1), \ldots, \pi_a(t_n))\). Prove that \(\pi_a(t)\) is greater or equal to the length of every reduction starting from \(t\).
2. Show that there exists \(d\) and \(k\) integers such that \(a \leq d\) and for every \(f \in F\) of arity \(n\) and every \(a_1, \ldots, a_n \in A \setminus \{\emptyset\}\), \(P_f(a_1, \ldots, a_n) \leq d \prod_{i=1}^{n} a_i^k\).
3. Fix \(c \geq k + \log_2(d)\). Prove that \(\pi_a(t) \leq 2^{2^{c|t|}}\).
4. In this question, consider any finite rewrite system and \( f \in F \). Prove that there exists an integer \( k \) such that if \( s \to t \) then \( |t|_f \leq k(|s|_f + 1) \), where \( |.|_f \) is the number of \( f \).

5. Deduce that \( \{ a(0, y) \to s(y) ; a(s(x), 0) \to a(x, s(0)) ; a(s(x), s(y)) \to a(x, a(s(x), y)) \} \)
cannot be proved terminating using a polynomial interpretation on integers.

**hint:** you may use the fact that the Ackermann’s function grows faster than any primitive recursive function.

A polynomial interpretation on real numbers is the following data :
— a subset \( A \) of \( \mathbb{R}_+ \);
— a positive real number \( \delta \);
— for every symbol \( f \) of arity \( n \), a polynomial \( P_f \in \mathbb{R}[X_1, ... , X_n] \)
satisfying that for every symbol \( f \) of arity \( n 
— for every \( a_1, ..., a_n \in A \), \( P_f(a_1, ... , a_n) \in A \)
— for every \( a_1, ..., \alpha > \delta \), \( a_1', ..., a_n \in A \), \( P_f(a_1, ... , a_i, ... , a_n) > \delta \ P_f(a_1, ... , a_i', ... , a_n) \), where \( x > \delta \ y \) iff \( x > y + \delta \).

Then \((A,(P_f)_f, > \delta)\) is a well-founded monotone algebra.

**Exercise 5:**

Define the following two rewrite systems :

\[
R_1 = \{ f(g(x)) \to g(g(f(x))); g(s(x)) \to s(g(x)); g(x) \to h(x, x); s(x) \to h(x, 0); s(x) \to h(x, 0) \}
\]

\[
R_2 = \{ k(k(k(x))) \to h(k(x), k(x)); s(h(k(x), k(x))) \to k(k(k(x))) \}
\]

1. Prove that \( R_1 \cup R_2 \) terminates using the following polynomial interpretation on real numbers : \( \delta = 1 \), \( P_0 = 0 \), \( P_s = X + 4 \), \( P_f = X^2 \), \( P_g = 3X + 5 \), \( P_h = X + Y \) and \( P_k = \sqrt{2X + 1} \).

2. Prove that any polynomial interpretation on integers proving the termination of \( R_1 \) is of the form \( P_s = X + s_0 \), \( P_h = X + Y + h_0 \), \( P_g = g_1X + g_0 \) with \( s_0, h_0, g_0 \geq 1 \), \( g_1 \geq 2 \) and \( P_f \) of degree at least 2.

**hint:** look at the dominant terms of the polynomials computed from the rewrite rules. For example, from the second rule of \( R_1 \), you should be able to prove that \( P_s \) is of degree 1.

3. Deduce that you cannot prove the termination of \( R_1 \cup R_2 \) using a polynomial interpretation on integers.

A matrix interpretation on integers is the following data :
— a positive integer \( d \);
— for every symbol of arity \( n \), \( n \) matrices \( M_{f,1}, ... , M_{f,n} \in \mathbb{N}^{d \times d} \);
— for every symbol of arity \( n \), a vector \( V_f \in \mathbb{N}^d \);
— a non-empty set \( I \subseteq \{ 1, ..., d \} \)
satisfying that for every symbol \( f \) of arity \( n \), the map

\[
L_f : (\mathbb{N}^d)^n \to \mathbb{N}^d \ (X_1, ... , X_n) \mapsto \sum M_{f,i}X_i + V_f
\]
is monotonic with respect to \( >_I \) where \( X >_I Y \) iff for every \( i \in \{ 1, ..., d \} \), \( X[i] \geq Y[i] \) and there is \( j \in I \) such that \( X[j] > Y[j] \).

Then \((\mathbb{N}^d,(L_f)_f, >_I)\) is a well-founded monotone algebra.
**Exercise 6:**
Consider the following rewrite system:

\[ R = \{ f(a) \rightarrow f(g(a)) ; g(b) \rightarrow g(f(b)) \} \]

1. Prove that the termination of \( R \) cannot be proved by a polynomial interpretation on integers.
2. Prove the termination of \( R \) using the following matrix interpretation with \( \succ_{\{1,2\}} \):

\[
L_f(X) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X \\
L_g(X) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X \\
L_a = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
L_b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

3. Why does it fail if we take \( \succ_{\{1\}} \) instead? Is there another matrix interpretation that works with this ordering?

**Exercise 7:**
Prove that the following rewrite system:

\[
\{ f(g(x)) \rightarrow f(a(g(g(f(x)))), g(g(f(x)))) ; h(h(x)) \rightarrow c(h(x)) ; a(x,x) \rightarrow h(x) ;
\]

\[
c(x) \rightarrow x ; f(x) \rightarrow x ; g(x) \rightarrow x
\]

using the following matrix interpretation with \( \succ_{\{1\}} \):

\[
L_a(X,Y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
L_c(X) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
L_f(X) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
L_g(X) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} X + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
L_h(X) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Why does it fail with \( \succ_{\{1,2\}} \)? Is there another matrix interpretation that works with this ordering?

**Exercise 8:**
Prove that termination is undecidable:

**(data)** \( R \) finite rewrite system.

**(question)** Does \( R \) terminate?

*hint*: reduce the Post correspondence problem. From an instance of the PCP, construct a rewrite system that terminates iff the PCP has no solutions.