Basic category theory - Limits

Gouter des doctorants

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Limit I : Final object

A *final object* in a category $\mathcal{C}$ is an object I such that for all object X of $\mathcal{C}$, there exists a unique morphism from X to I.

Ex : In Set, a final object is a singleton

Ex : In Rel, the unique final object is the empty set

Ex : In a preordered set, a final object is a maximal element, which does not necessarily exist

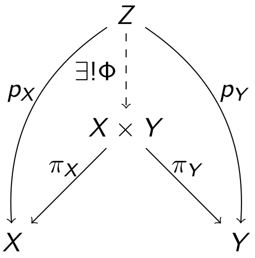
Limit II : Cartesian product

Given two objects X, Y of $\mathcal{C}$, a cartesian product is :

- an object $X \times Y$
- two morphisms $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ (projections)

satisfying that for every object Z of $\mathcal{C}$ and morphisms $p_X : Z \longrightarrow X$ and $p_Y : Z \longrightarrow Y$ there exists a unique morphism $\Phi : Z \longrightarrow X \times Y$ such that :

$$p_X = \pi_X \circ \Phi \qquad p_Y = \pi_Y \circ \Phi$$



Limit II : Cartesian product (examples)

Ex : In Set (and Rel), $X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$ with projections is a cartesian product, but also every set in bijection with it.

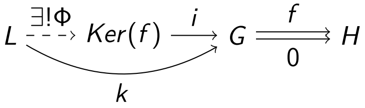
Ex : In Mon, $X \times Y$ with the multiplication component by component is a cartesian product.

Ex : In a preordered set, a cartesian product is an inf.

Limit III : Kernel

Given $f : G \longrightarrow H$ a morphism of groups. The inclusion $i : Ker(f) \longrightarrow G$ satisfies :

- $f \circ i = 0$
- for every morphism of groups $k : L \longrightarrow G$ such that $f \circ k = 0$ there exists a unique morphism of groups $\Phi : L \longrightarrow Ker(f)$ such that $k = i \circ \Phi$



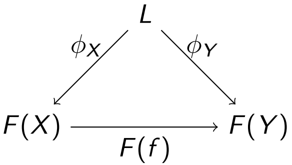
Limit IV : general case, cone

Given a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, a cone of F is :

- an object L of $\mathcal{D}$
- for every object X of $\mathcal{C}$, a morphism $\phi_X : L \longrightarrow F(X)$

verifying that for every morphism $f : X \longrightarrow Y$ of $\mathcal{C}$:

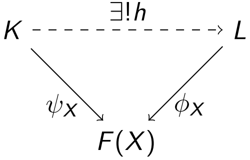
$$F(f) \circ \phi_X = \phi_Y$$



Limit IV : general case, definition

A limit of the functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a cone $(L, (\phi_X)_{X \in Ob(\mathcal{C})})$ of F such that for every other cone $(K, (\psi_X)_{X \in Ob(\mathcal{C})})$ of F there exists a unique morphism $h : K \longrightarrow L$ such that for every object X of $\mathcal{C}$:

$$\psi_X = \phi_X \circ h$$



Remark : a limit does not exist in general and if it exists, is unique up to isomorphism i.e. if L and L' are limits of the same functor F then there exists $f : L \longrightarrow L'$ and $g : L' \longrightarrow L$ such that :

$$g \circ f = id_L \qquad f \circ g = id_{L'}$$

Limit IV : general case in *Set*

A functor $F : \mathcal{C} \longrightarrow Set$ where $\mathcal{C}$ is small has always a limit :

- $L = \{(a_Y)_{Y \in Ob(\mathcal{C})} \mid a_Y \in F(Y) \wedge \forall f : X \longrightarrow Y \; F(f)(a_X) = a_Y\}$
- $\phi_X : L \longrightarrow F(X) \qquad (a_Y)_{Y \in Ob(\mathcal{C})} \longmapsto a_X$

We say that *Set* is complete.