Basic category theory

Gouter des doctorants

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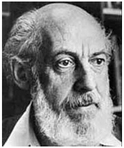
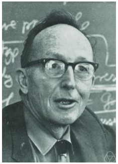
A bit of history

- introduced by Samuel Eilenberg and Saunders Mac Lane in 1942-45 to axiomatize the theory of homology in algebraic topology

- idea of Emmy Noether : if you want to understand a mathematical structure, you have to understand the processes that preserve this structure

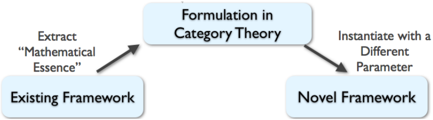


- in France, spread by Alexander Grothendieck



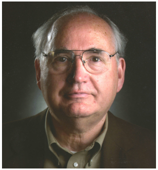
The essence of mathematics

Purpose : extract the mathematical essence of a reasoning, a structure, only keep what is mathematically needed, abstract



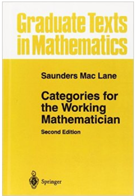
Ex : in computer science, denotational semantics of λ-calculus :

- first non-trivial models in partial orders with some properties in the 70s :
  - P_ω of Plotkin and Scott
  - D_∞ of Scott
- construction of a class of categorical models from the extraction of these properties (reflexive objects in a cartesian closed category) by Scott in the 80s

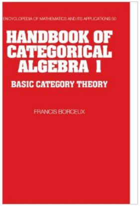


Some basic references

- *Categories for the Working Mathematician*,
  Saunders Mac Lane, Springer, 1998



- *Handbook of Categorical Algebra*,
  Francis Bordeaux, Cambridge University Press, 1994



Definition of a category

General definition of a category $\mathcal{C}$

a class of *objects* $Ob(\mathcal{C})$

for every pair X, Y of objects, a set $\mathcal{C}(X, Y)$ of *morphisms*

for every object X, a particular morphism $id_X : X \longrightarrow X$ called *the identity of X*

for every triple X, Y, Z of objects, a function $\circ : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$ called *composition*

Category *Set*

the class of sets

$Set(X, Y)$ the set of functions from X to Y

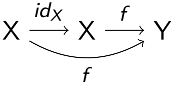
$$\begin{aligned} id_X : X &\longrightarrow X \\ x &\longmapsto x \end{aligned}$$

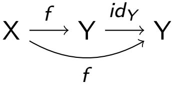
for $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, $\circ(f, g)(x) = g \circ f(x) = g(f(x))$

Axioms of a category

- identities are neutral elements to the left and to the right of the composition

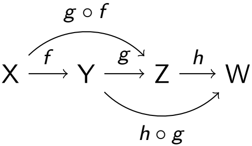
$$\forall f : X \longrightarrow Y,\ f \circ id_X = f \text{ and } id_Y \circ f = f$$





- the composition is associative

$$\forall f : X \longrightarrow Y, g : Y \longrightarrow Z, h : Z \longrightarrow W, h \circ (g \circ f) = (h \circ g) \circ f$$



External example I : *Mon*

- $Ob(Mon)$: monoids i.e. sets X with an associative binary operation . and a left and right neutral element 1 for .

$$x.(y.z) = (x.y).z \qquad 1.x = x = x.1$$

- $Mon(X, Y)$: morphisms of monoids i.e. functions $f : X \longrightarrow Y$ such that

$$f(x.y) = f(x).f(y) \qquad f(1) = 1$$

External example II : *Gra*

- $Ob(Gra)$: oriented multigraphs i.e. quadruples (V, E, s, t) with $s, t : E \longrightarrow V$
- $Gra((V, E, s, t), (V', E', s', t'))$: morphisms of graphs i.e. pairs of functions $(f : V \longrightarrow V', g : E \longrightarrow E')$ such that

$$s' \circ g = f \circ s \qquad t' \circ g = f \circ t$$

Other examples :

- groups/abelian groups with morphisms of groups
- preordered sets with non decreasing functions
- topological spaces with continuous functions
- automata with morphisms of automata

External example III : *Rel*

- $Ob(Rel)$: sets
- $Rel(X, Y)$: relations between X and Y i.e. sets $R \subseteq X \times Y$
- id_X : $\{(x, x) \mid x \in X\}$
- $R' \circ R$: $\{(x, z) \mid \exists y, (x, y) \in R \wedge (y, z) \in R'\}$

Internal examples

What are the following classes of categories ?

- a discrete category i.e. $\mathcal{C}(X, Y)$ is empty if $X \neq Y$, $\{id_X\}$ otherwise
- a category with one object
- a small category such that $\mathcal{C}(X, Y)$ is empty or a singleton

Definition of a functor

A *functor* F from the category $\mathcal{C}$ to the category $\mathcal{D}$ is :

- a function $F : Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$
- for every pair $X, Y \in Ob(\mathcal{C})$, a function $F : \mathcal{C}(X, Y) \longrightarrow \mathcal{D}(F(X), F(Y))$

satisfying :

$$F(g \circ f) = F(g) \circ F(f)$$

$$F(id_X) = id_{F(X)}$$

External example I : Forgetful functor

Given two categories a mathematical structures $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C}$ which has more structures than $\mathcal{D}$.

Ex : monoids/sets, groups/sets, groups/monoids, ...

$\mathcal{U} : \mathcal{C} \longrightarrow \mathcal{D}$ which forgets the supplementary structure.

Ex : $\mathcal{U}$: *Mon* $\longrightarrow$ *Set* which maps a monoid to its underlying set and a morphism of monoids to its underlying function

External example II : Free monoid functor

Given a set X we can construct the free monoids on X, written $X^\star$ as :

- $X^\star$: set of finite words on the alphabet X
- . : concatenation
- 1 : empty word

Given a function $f : X \longrightarrow Y$ we can construct a morphism of monoids $f^\star : X^\star \longrightarrow Y^\star$ as :

$$f^\star(x_1.x_2 \ldots x_k) = f(x_1).f(x_2) \ldots f(x_k)$$

This defines a functor $\_^\star : Set \longrightarrow Mon$.

Internal examples

What is a functor :

- between classes seen as discrete categories ?
- between monoids ?
- between preordered sets ?
- from the category with one object and one morphism to any category ?
- from the category

  $$\bullet \longrightarrow \bullet$$

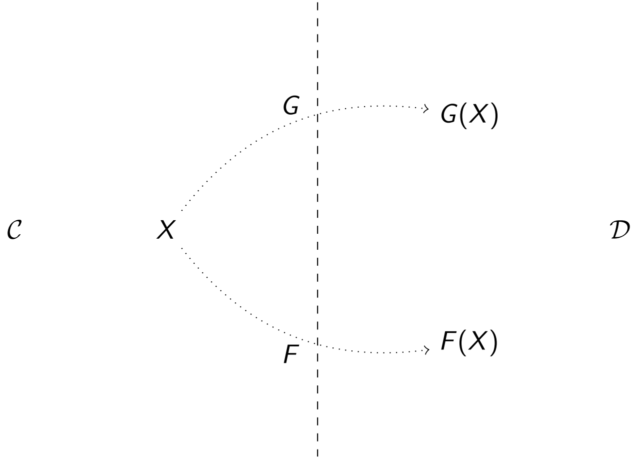
  to any category ?
- from

  $$\bullet \rightrightarrows \bullet$$

  to Set ?

Naturality

Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$. How to 'transform' F to G in preserving the structure of a functor ?



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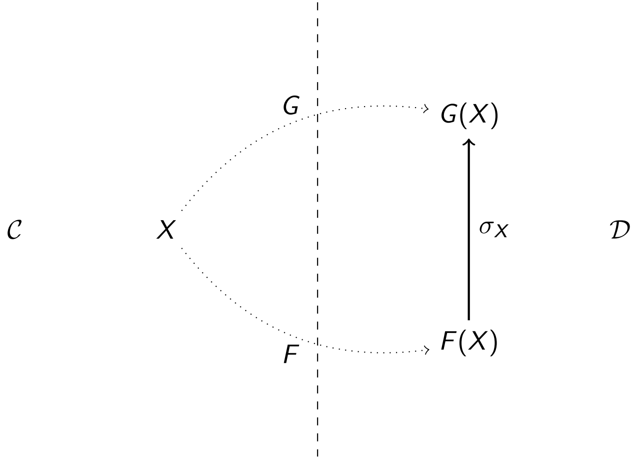

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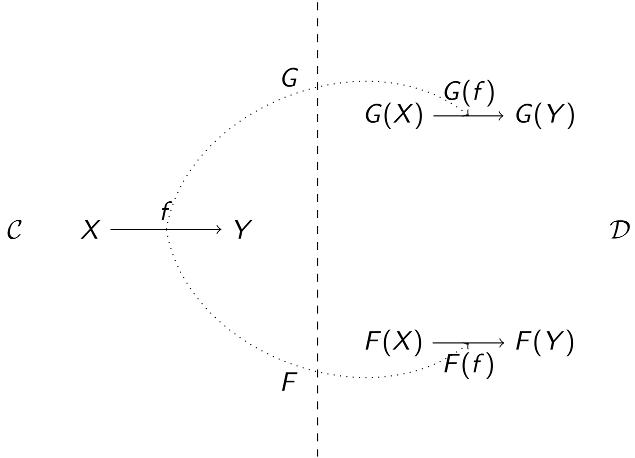

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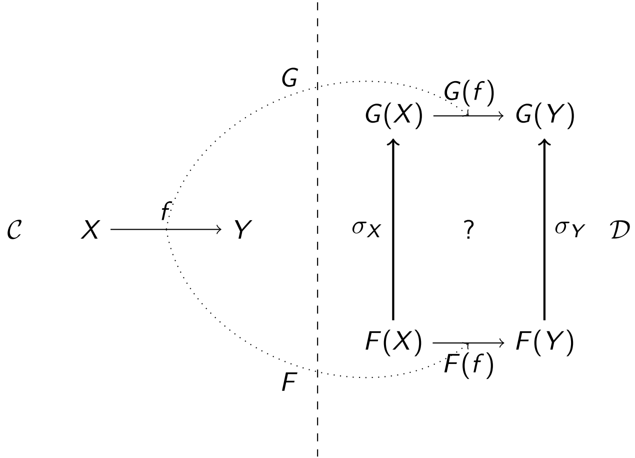

Natural transformation

Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$.
A *natural transformation* $\sigma : F \longrightarrow G$ is the data of a morphism $\sigma_X : F(X) \longrightarrow G(X)$ of $\mathcal{D}$ such that for every morphism $f : X \longrightarrow Y$ of $\mathcal{C}$:

$$\sigma_Y \circ F(f) = G(f) \circ \sigma_X$$

$$\begin{array}{ccc} G(X) & \xrightarrow{G(f)} & G(Y) \\ {\scriptstyle \sigma_X} \uparrow & & \uparrow {\scriptstyle \sigma_Y} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array}$$

Internal examples

- there is a natural transformation from f to g where f, g are non-decreasing functions from X to Y preordered sets iff $\forall x$, $f(x) \leq g(x)$

- a natural transformation between oriented multigraphs is a morphism of *Gra*

Discrete/Chaotic topologies

There are two canonic ways to define a topology on a given set :

- $Dis : Set \longrightarrow Top \qquad X \mapsto (X, \mathcal{P}(X)) \qquad f \mapsto f$
- $Cha : Set \longrightarrow Top \qquad X \mapsto (X, \{X, \varnothing\}) \qquad f \mapsto f$

There is a natural transformation $id : Dis \longrightarrow Cha$
$id_X : (X, \mathcal{P}(X)) \longrightarrow (X, \{X, \varnothing\})$.

There is no natural transformation $Cha \longrightarrow Dis$ because
$id_X : (X, \{X, \varnothing\}) \longrightarrow (X, \mathcal{P}(X))$ is not continuous.