Natural homology
HOMOTOPOY IN CONCURRENCY AND REWRITING

Jérémy Dubut

joint work with
Eric Goubault - LIX, Ecole Polytechnique
Jean Goubault-Larrecq - LSV, ENS Cachan

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General context: verification of concurrent systems
Models of true concurrency

- Petri nets [Petri 62]
- progress graphs [Dijkstra 68]
- trace theories [Mazurkiewicz 70s]
- event structures [Winskel 80s]
- higher dimensional automata (HDA) [Pratt 91]
Plan

I. Geometry of true concurrency

II. Classical homology

III. A candidate of directed homology: natural homology
I. Geometry of true concurrency
A toy language: SU-programs [Afek et al. 90]

- shared global memory
- atomic operations:
  - $S$: scan ALL the memory
  - $U$: update ONLY its OWN part of the memory
- synchronization • (rendez-vous)
- $S$ and $U$ non independent

\[(S|U) \bullet (U.S|U.S)\]
X pospace = space + order

dipath = increasing path = increasing continuous function \( p : [0, 1] \to X \)

= « execution with memory of the time between actions »

dipath space : \( \mathcal{P}(X)(a, b) = \{ p : a \to b \} \)

trace \( \langle p \rangle = \) dipath \( p \) modulo increasing reparametrization

= « execution where only organization of actions is significant »

trace space : \( \mathcal{I}(X)(a, b) = \{ \langle p \rangle \text{ with } p : a \to b \} \)
Dihomotopy

**Dihomotopy**:

- $f$ and $g$ dipaths from $a$ to $b$ in $X$.
- $H : [0, 1] \times [0, 1] \longrightarrow X$ dihomotopy from $f$ to $g$ if:
  - $H$ continuous and increasing in the second coordinate
  - $H(0,.) = f$, $H(1,.) = g$, $H(.,0) = a$ et $H(.,1) = b$
- $f$ and $g$ dihomotopic if there exists a dihomotopy from one to the other

**dihomotopic** = « deforming continuously one to the other while staying a dipath »
Objective

- study those concurrent systems through their geometry (dipaths, traces, dihomotopies)
- homology = essential notion, computable abstraction of homotopy
  \[\Rightarrow\] defining a directed analogue of homology
II. Classical homology
Homology = counting holes

hole of dimension 1

no hole of dimension 1 ... ... but a hole of dimension 2

In first approximation : $H_n(X) \sim \mathbb{Z}^{\text{number of holes of dimension } n}$
In general : $H_n(X) \sim \prod T_i \text{ hole of dimension } n \mathbb{Z}/k_i \mathbb{Z}$
Particular case : $H_0(X) \sim \mathbb{Z}^{\text{number of path-connected components}}$
Properties of homology

(sound) invariant by homotopy, correction
(precis) not too much loss of information (Hurewicz), partial completeness (Whitehead)
(mod) modularity = homology of a space expressible from homology of smaller spaces (Mayer-Vietoris)
(calc) computability in the case of finitely presented spaces (simplicial, pre-cubical sets)

Objective:

- study those concurrent systems through their geometry (dipaths, traces, dihomotopies)
- homology = essential notion, computable abstraction of homotopy
  - defining a directed analogue of homology with the same kind of properties
Existing works

(Goubault 95)  (precis)  (mod)  (calc)

(Goubault 95) -  ×  -  ✓

(Grandis 04) ✓  ×  -  -

(Farhenberg 04) ✓  ×  -  -

(Kahl 13) ✓  ×  -  -

Diagram
Existing works

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<thead>
<tr>
<th>[Goubault 95]</th>
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## Existing works

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![Diagram](image-url)

Jérémy Dubut (LSV, ENS Cachan)
A candidate of directed homology: natural homology
trace spaces vs evolution of trace spaces

\[
\begin{align*}
\alpha & \quad \beta \\
S & \quad U \\
U & \quad S
\end{align*}
\]
(geometric) Natural homology

Natural homology $\overrightarrow{H}_n(X) \ (n \geq 1)$:

- trace $\langle p \rangle$
  - $p$ dipath from $a$ to $b$
  \[ \mapsto \quad H_{n-1}(\overrightarrow{\mathfrak{L}}(X)(a, b)) \]

- extension $((\langle \alpha \rangle, \langle \beta \rangle))$
  - $\alpha$ from $a'$ to $a$, $\beta$ from $b$ to $b'$
  \[ \mapsto \quad H_{n-1}((\langle q \rangle \in \overrightarrow{\mathfrak{L}}(X)(a, b) \mapsto \langle \alpha \ast q \ast \beta \rangle \in \overrightarrow{\mathfrak{L}}(X)(a', b'))) \]

$\overrightarrow{H}_n(X) = \text{functor from the category of factorization of the category of traces to}$
$\text{Ab = natural system on the category of traces}$
Example 1

\[ \overrightarrow{H}_1([0, 1]) \]

\[ [0, 1] \]

\[ [0, y] \quad [x, 1] \]

\[ [0, x] \quad [x, y] \quad [y, 1] \]

\[ 0 \quad x \quad y \quad 1 \]
Example II

\[ \begin{array}{c}
\text{0} \\
\text{x'} \quad \text{b} \quad \text{y'} \\
\text{1} \\
\end{array} \quad \xrightarrow{\text{H}_1(X)} \quad \begin{array}{c}
\text{a} \\
\text{[0, y]} \quad \text{x, 1} \\
\text{[0, x]} \quad \text{x, y} \quad \text{[y, 1]} \\
\text{0} \\
\end{array} \quad \begin{array}{c}
\text{b} \\
\text{[0, y']} \quad \text{x', 1} \\
\text{[0, x']} \quad \text{x', y'} \quad \text{[y', 1]} \\
\text{1} \\
\end{array} \]
Study of \textbf{(mod)}

**Proposition:**

- $\overrightarrow{H}_n$ is a functor from $\text{PoTop}$ to the category of functors $\text{Fun}(\text{Ab})$.
- $\text{Fun}(\text{Ab})$ is not abelian but is homological in the sense of [Grandis 91].

**Proof:**

- morphisms from $F : C \rightarrow \text{Ab}$ to $G : D \rightarrow \text{Ab}$: pairs $(\Phi, \sigma)$ where:
  - $\Phi : C \rightarrow D$
  - $\sigma : F \rightarrow G \circ \Phi$
- null morphisms: $(\Phi, \sigma)$ with $\sigma_c$ are zero
- kernels: $c \mapsto \text{Ker}\sigma_c$
- cokernels: a bit tricky (because colimits in $\text{Fun}(\text{Ab})$ are more complicated)
- + some morphisms are exact (because $\text{Ab}$ is abelian and its morphisms are exact)
Study of \((\text{mod})\)

**Proposition:**
- \(\bar{H}_n\) is a functor from \(\text{PoTop}\) to the category of functors \(\text{Fun}(\text{Ab})\).
- \(\text{Fun}(\text{Ab})\) is not abelian but is homological in the sense of [Grandis 91].

**Theorem (mod) [Grandis 91]:**

Let \(\mathcal{A}\) be a homological category.
For every short exact sequence in \(C_\bullet(\mathcal{A})\):

\[
\begin{array}{c}
\text{U} \\ \downarrow m \end{array} \longrightarrow \begin{array}{c}
\text{V} \\ \downarrow p \end{array} \longrightarrow \begin{array}{c}
\text{W} \\ \downarrow \end{array}
\]

there exists a long sequence of order two in \(\mathcal{A}\):

\[
\cdots \longrightarrow H_n(V) \xrightarrow{H_n(p)} H_n(W) \xrightarrow{\partial_n} H_{n-1}(U) \xrightarrow{H_{n-1}(m)} H_{n-1}(V) \longrightarrow \cdots
\]

natural in the short exact sequence.
Moreover, there are conditions to turn the long sequence to an exact sequence.
Study of (précis)

\[ H_0\left( \tilde{\mathbb{Z}}(X)(\alpha\beta, \gamma\delta) \right) \cong \mathbb{Z}^2 \]

\[ \Rightarrow \text{there exists two dipaths that are not dihomotopic} \]

\[ \Rightarrow \text{we can see it in } \tilde{H}_1(X) ! \]

Theorem (précis) [D.G.G.]:

- if \( X \) 0-diconnected, \( \tilde{H}_1(X) \cong \text{Free} \circ \tilde{\Pi}_1(X) \)
- if \( X \) 1-diconnected, \( \tilde{H}_2(X) \cong \text{Ab} \circ \tilde{\Pi}_2(X) \)
- if \( X \) \((n - 1)\)-diconnected \( (n \geq 3) \), \( \tilde{H}_n(X) \cong \tilde{\Pi}_n(X) \)
Computability

Theorem (**calc** [D.G.G. 15] :)

Given a finite pre-cubical complex, there exists $\overrightarrow{h}_n(X)$ (discrete natural homology):

- computable
- equivalent to $\overrightarrow{H}_n(X)$
Proof (construction)

$X$ a pre-cubical complex

- a discrete trace from $x \in X$ to $y \in X$: sequence $c_0, \ldots, c_n \in X$ such that $c_0 = x$, $c_n = y$ and for all $i$:
  - either $c_{i-1}$ is of the form $\delta_{i_k}^0 \circ \cdots \circ \delta_{i_0}^0(c_i)$
  - either $c_i$ is of the form $\delta_{i_k}^1 \circ \cdots \circ \delta_{i_0}^1(c_{i-1})$

- we map each discrete trace to a geometric one:

  ![Diagram of geometric trace]

- $\vec{h}_n(X)$ is the restriction of $\vec{H}_n(X)$ to those traces
Example I

\[\xymatrix{ & 0, a, 1 \\
0, a & a, 1 \\
0 & a & 1} \quad \xymatrix{ & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \ar@{.>}[urr] \ar@{.>}[ur]}

\[\overrightarrow{h_1([0, 1])}\]
Example II

\[ 0, a, 1 \quad 0, b, 1 \]

\[ 0, a \quad a, 1 \quad 0, b \quad b, 1 \]

\[ \overrightarrow{h_1([0, 1])} \]

\[ \mathbb{Z}^2 \]

\[ \mathbb{Z} \]

\[ \mathbb{Z} \]

\[ \mathbb{Z} \]

\[ \mathbb{Z} \]

\[ \mathbb{Z} \]
Proof (computability)

- enumeration of discrete traces
- construction of a finite representation (prod-simplical complex) of each trace space \([\text{Raussen 09}]\)
- computation of classical homology
Which notion of equivalence?

problem: $\overrightarrow{h}_n(X)$ and $\overrightarrow{H}_n(X)$ non isomorphic by cardinality

solution: existence of a morphism with some lifting properties

equivalence: existence of a span of such morphisms
\( \mathcal{P} \)-open maps [Joyal et al. 94]

\( \mathcal{P} = \) sub-category = category of paths 

paths: \( G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} G_n \)

extensions of paths:

\[
\begin{array}{c}
G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} G_k \\
\downarrow id \quad \downarrow id \quad \downarrow id \\
G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} G_k \xrightarrow{f_k} \cdots \xrightarrow{f_{n-1}} G_n
\end{array}
\]

\( \mathcal{P} \)-open = has the lifting property with respect to those extensions:

\[
\begin{array}{c}
P \xrightarrow{p} X \\
\downarrow g \quad \downarrow r \\
Q \xrightarrow{q} Y
\end{array}
\]

\[
\begin{array}{c}
P \xrightarrow{p} X \\
\downarrow g \quad \downarrow r \quad \downarrow f \\
Q \xrightarrow{q} Y
\end{array}
\]
Bisimilarity - Computer science point of view

Two functors $F : C \to \textbf{Ab}$ and $G : D \to \textbf{Ab}$ are bisimilar if there exists a set

$$R \subseteq \{ (c, f, d) \mid c \in C \land d \in D \land f \in \text{Ab}(F(c), G(d)) \text{ isomorphism} \}$$

such that :

- for all $c \in C$, there exists $d, f$ such that $(c, f, d) \in R$
- for all $d \in D$, there exists $c, f$ such that $(c, f, d) \in R$
- for all $(c, f, d) \in R$ and $i : c \to c'$ there exists $j : d \to d'$ and $g : F(c') \to G(d')$ iso such that $(c', g, d') \in R$ and $f \circ F(i) = G(j) \circ g$
- for all $(c, f, d) \in R$ and $j : d \to d'$ there exists $i : c \to c'$ and $g : F(c') \to G(d')$ iso such that $(c', g, d') \in R$ and $f \circ F(i) = G(j) \circ g$
Bisimilarity

A open map from $H : E \rightarrow Ab$ to $G : D \rightarrow Ab$ is:

- a functor $\Phi : E \rightarrow D$ satisfying:
  - $\Phi$ is surjective on objects
  - $\Phi$ has the lifting property for any morphism of $D$ : for every morphism $j : d \rightarrow d'$ of $D$ and every object $e$ of $E$ such that $\Phi(e) = d$ there exists a morphism $i : e \rightarrow e'$ such that $\Phi(i) = j$

- a natural isomorphism $\sigma : H \rightarrow G \circ \Phi$

We say that $F : C \rightarrow Ab$ and $G : D \rightarrow Ab$ are bisimilar if there exists a span of open maps between them.
Proof (equivalence)

**Proposition:**
There exists an open map $\text{Carrier} : \overrightarrow{H}_n(X) \rightarrow \overrightarrow{h}_n(X)$.

**Proof (construction):**

construction a functor

$\text{Carrier} : \text{geometric traces} \rightarrow \text{discrete traces}$

$\text{Carrier}(\text{trace}) = \text{« the sequence of hypercubes crossed by this trace »}$
Example
Proof (equivalence)

Proposition:

There exists an open map \( \text{Carrier} : \overrightarrow{H}_n(X) \rightarrow \overrightarrow{h}_n(X) \).

Proof (lifting property):

\[
\begin{array}{c}
\text{ext} \quad \text{Carrier} \quad \text{ext} \\
\downarrow \quad \downarrow \\
? \quad \text{Carrier} \\
\end{array}
\]
Proof (equivalence)

Proposition:
Il existe une open map \( \text{Carrier} : \overrightarrow{H}_n(X) \rightarrow \overrightarrow{h}_n(X) \).

Proof (lifting property):

\[ \begin{array}{c}
\text{ext} \\
\downarrow \\
\text{ext}
\end{array} \quad \xrightarrow{\text{Carrier}} \quad 
\begin{array}{c}
\text{ext} \\
\downarrow \\
\text{ext}
\end{array} \]
Proof (equivalence)

**Proposition:**
There exists an open map \( \text{Carrier} : \overrightarrow{H}_n(X) \to \overrightarrow{h}_n(X) \).

**Corollary:**
If \( X' \) is a barycentric subdivision of \( X \), \( \overrightarrow{h}_n(X) \) and \( \overrightarrow{h}_n(X') \) are \( \mathcal{P} \)-bisimilar.
Results and future works

Results :

(sound) invariance by dihomeomorphism, subdivision (action refinement)
(precis) Hurewicz-like theorem
(mod) existence of long sequence in homology by the theory of homological category of [Grandis 91]
(calc) notion of bisimulation of functors, equivalence with a discrete computable natural homology

Future works :

- link with bisimulations [Fahrenberg, Legay 13], observational equivalences [Plotkin, Pratt 90], temporal properties [Baldan, Crafa 10] in true concurrency
- improve the algorithmic
  - better representation of trace spaces
  - decidability of bisimilarity using matrix algorithmic
- link with persistence homology [Carlsson 09]
- applications in higher order rewriting