

# Natural homology

ICALP track B, Kyoto

Jérémy DUBUT (LSV, ENS Cachan, France)

Eric GOUBAULT (LIX, Ecole Polytechnique, France)

Jean GOUBAULT-LARRECQ (LSV, ENS Cachan, France)

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I.

## Homotopy, dihomotopy and directed homology

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# Objective

## Objective :

Compare spaces **with a notion of order** up to continuous deformation **that preserves this order**

Problem coming from :

- geometric semantics of truly concurrent systems
  - ▶ PV-programs [**Dijkstra 68**]
  - ▶ scan/update [**Afek et al. 90**]
  - ▶ higher dimensional automata [**Pratt 91**]
- theory of relativity [**Dodson, Poston 97**]

# Non directed case : algebraic topology

## Non directed case : algebraic topology

Compare spaces ~~with a notion of order~~ up to continuous deformation ~~that preserves this order~~

Homology [**Poincaré 1895**] which is :

- sound (invariant of homotopy)
- partially complete [**Hurewicz 52, Whitehead 49**]
- computable [**Poincaré 1900**]
- modular (homology can be expressed from homology of simpler spaces [**Mayer, Vietoris 30**])

# Dihomotopies

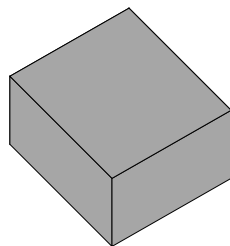
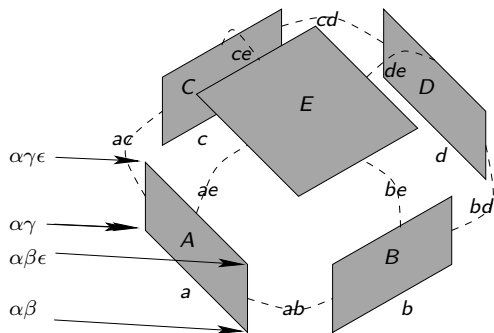
**Dipaths** = **increasing** continuous functions from  $[0, 1]$  to  $X$

2 dipaths are **dihomotopic** = you can deform continuously one into the other **while staying a dipath**

(di)homotopic

non (di)homotopic

# Homotopy vs dihomotopy



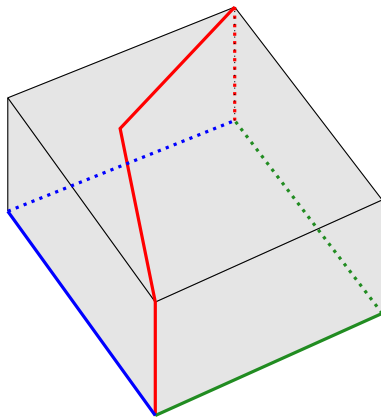
Fahrenberg's matchbox [Fahrenberg 04]

# Homotopy vs dihomotopy

homotopic...



# Homotopy vs dihomotopy



... but not dihomotopic

## Related work contribution

Candidates of directed homology :

- past and future homologies [**Goubault 95**]
- ordered homology groups [**Grandis 04**]
- directed homology via  $\omega$ -categories [**Fahrenberg 04**]
- homology graph [**Kahl 13**]

Not fine enough : do not distinguish Fahrenberg's matchbox from a point

Our contribution (1/4) :

A directed homology fine enough to detect failure of dihomotopy, **natural homology**

## II.

# Natural Homology

# Trace spaces

dipath = continuous increasing map from  $[0, 1]$  to  $X$

trace = dipath modulo reparametrization

Trace space from  $a$  to  $b$  [Raussen 09]

$T(X)(a, b) = \{\text{traces from } a \text{ to } b\}$  with compact open topology

## A first idea

*Not\_so\_good* directed homology :

$Not\_so\_good(X) =$  classical homology of  $T(X)(a, b)$

$$A = (U \parallel S) \bullet (U.S \parallel U.S)$$

$$B = U.U \parallel S.S$$

$$T(A)(a, b) \simeq 6 \text{ point space} \simeq T(B)(a, b)$$

$$Not\_so\_good(A) \simeq \mathbb{Z}^6 \simeq Not\_so\_good(B)$$

# A first (not so) bad idea

make  $a, b$  vary

$T(A)(a, b) \simeq 4 \text{ point space}$

$\text{Not\_so\_good}(A) \simeq \mathbb{Z}^4$

no  $a', b'$  such that

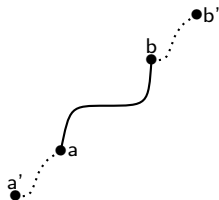
$T(B)(a', b') \simeq 4 \text{ point space}$

$\text{Not\_so\_good}(B) \simeq \mathbb{Z}^4$

# Natural homology

$\mathcal{F}_X$  = category whose :

- objects are traces
- morphisms are extensions



## Natural homology :

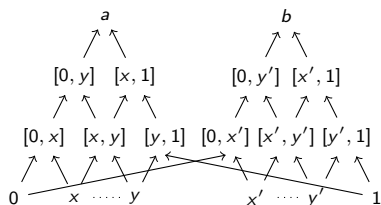
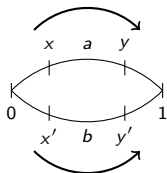
functor  $\vec{H}_n(X) : \mathcal{F}_X \longrightarrow \mathbf{Ab}$

$$(a \xrightarrow{\gamma} b) \longmapsto H_{n-1}(T(X)(a, b))$$

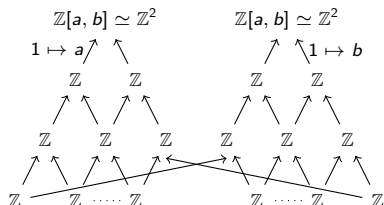
( $H_{n-1}$  = classical homology)

- $\mathcal{F}_X$  = category of factorizations [Mac Lane 71]
- $\vec{H}_n(X)$  = natural system [Leech 73, Baues, Wirsching 85]

# Example : first natural homology of $a + b$



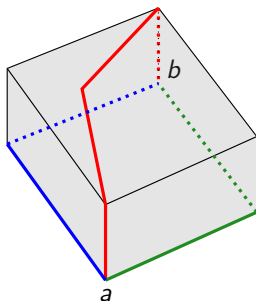
$\vec{H}_1(a+b)$



$\mathcal{F}_{a+b}$



# Natural homology on Fahrenberg's matchbox



2 dipaths non dihomotopic

$\Rightarrow T(X)(a, b) \simeq 2$  point space

$\Rightarrow H_0(T(X)(a, b)) \simeq \mathbb{Z}^2$

$\Rightarrow \vec{H}_1(X)$  not  $\mathbb{Z}$  everywhere

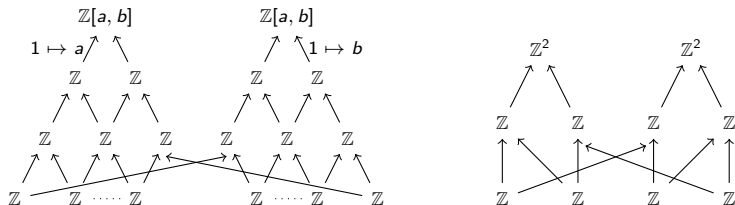
$\Rightarrow$  **natural homology detects failure of dihomotopy in Fahrenberg's matchbox**

### III.

## Comparison of natural systems

# How to compare natural systems ?

$$\vec{H}_n(A) = \vec{H}_n(B) \Rightarrow A = B, \text{ modulo isomorphism}$$



## Our contribution (2/4) :

We define a notion of bisimilarity for natural systems

Similar idea in **[Fiore 00]** for functorial models of concurrent computation

# Bisimulation of functors

Bisimulation between  $F : C \rightarrow \mathbf{Ab}$  and  $G : D \rightarrow \mathbf{Ab}$

= set  $R$  of pairs  $(c, d)$  such that :

- $c$  is an object of  $C$
- $d$  is an object of  $D$

satisfying :

- for every object  $c$  of  $C$ , there exists  $d$  such that  $(c, d) \in R$
- 

$$\begin{array}{ccc} & (c, d) \in R & \\ & & \\ c & & d \\ \downarrow i & & \downarrow j \\ c' & & d' \\ & (c', d') \in R & \end{array}$$

and symmetrically

# Bisimulation of functors

Bisimulation between  $F : C \rightarrow \mathbf{Ab}$  and  $G : D \rightarrow \mathbf{Ab}$

= set  $R$  of triples  $(c, \eta, d)$  such that :

- $c$  is an object of  $C$
- $d$  is an object of  $D$
- $\eta : F(c) \rightarrow G(d)$  is an isomorphism of groups

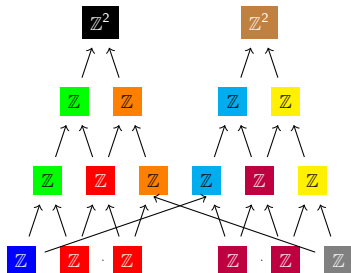
satisfying :

- for every object  $c$  of  $C$ , there exists  $d$  and  $\eta$  such that  $(c, \eta, d) \in R$
- 

$$\begin{array}{ccccc} & & (c, \eta, d) \in R & & \\ & & & & \\ c & Fc & \xrightarrow{\eta} & Gd & d \\ i \downarrow & Fi \downarrow & & \downarrow Gj & \downarrow j \\ c' & Fc' & \xrightarrow{\eta'} & Gd' & d' \\ & & (c', \eta', d') \in R & & \end{array}$$

and symmetrically

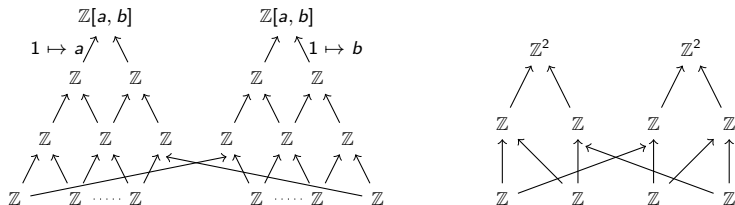
# Example



## IV.

# Computability, invariance by subdivision

# Bisimilarity type of natural homology



Natural homology = uncountable, not computable

Our contribution (3/4) :

When  $X$  is finitely presented, we can compute a finite natural system bisimilar to  $\vec{H}_n(X)$ .



# Discrete natural homology

$X$  presented by a finite cubical complex  
 $\simeq$  glueing of cubes of unit length in  $\mathbb{R}^n$

discrete trace = trace which is a glueing of segments joining center of cubes



$f_X$  = category of discrete traces and extensions by discrete traces

Discrete natural homology  $\vec{h}_n(X)$  :

functor  $\vec{h}_n(X) : \textcircled{f_X} \rightarrow \mathbf{Ab}$

$$(a \xrightarrow{\gamma} b) \mapsto H_{n-1}(T(X)(a, b))$$

# Computability

## Theorem :

Given a finite cubical complex  $X$ ,  $\vec{h}_n(X)$  is :

- computable
- bisimilar to  $\vec{H}_n(X)$

## Proof :

- **Carrier [Fajstrup05]** ( $a \xrightarrow{\gamma} b$ )  $\mapsto$  trace drawn by joining centers of cubes traversed by  $\gamma$



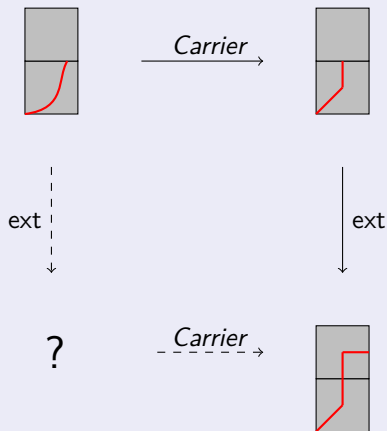
$\xrightarrow{\text{Carrier}}$



- We exhibit a **bisimulation** between  $\gamma$  and  $\text{Carrier}(\gamma)$
- ... compatible with morphisms  $\eta$ ,  $F(i)$ ,  $F(j)$  in **Ab** (not shown here)

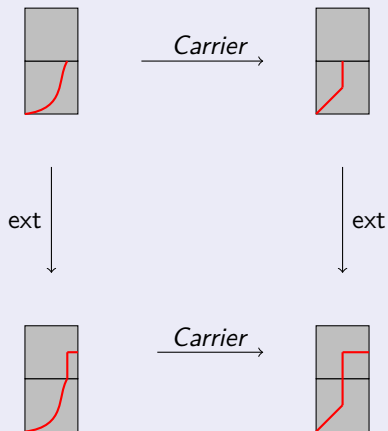
# Proof

## Proof (bisimilarity) :



# Proof

## Proof (bisimilarity) :



# Invariance by subdivision

Our contribution (4/4) — Corollary :

$\vec{H}$  and  $\vec{h}$  are invariant by subdivision.



$\simeq$  invariance by action refinement [**Goltz, van Glabbeek 89**]

# Conclusion

Our contributions :

- definition of the first satisfactory directed homology, **natural homology**
- comparison of natural systems by **bisimilarity**
- definition of a **computable** natural system bisimilar to natural homology, **discrete natural homology**
- invariance by **subdivision**

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