The directed homotopy hypothesis

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I. Directed algebraic topology
Objective

Compare spaces with a notion of direction of time up to continuous deformation that preserves this direction

Problem coming from:

- geometric semantics of truly concurrent systems
  - PV-programs [Dijkstra 68]
  - scan/update [Afek et al. 90]
  - higher dimensional automata [Pratt 91]
- theory of relativity [Dodson, Poston 97]
Non directed case: algebraic topology

Compare spaces with a notion of direction of time up to continuous deformation that preserves this direction.
Dihomotopies

**Directed** space = topological space $X$ with a collection of specified paths (continuous functions from $[0, 1]$ to $X$), called *dipaths*

2 dipaths are *dihomotopic* = you can deform continuously one into the other while staying a dipath
Homotopy vs dihomotopy

Fahrenberg’s matchbox [Fahrenberg 04]
Homotopy vs dihomotopy

homotopic...
Homotopy vs dihomotopy

... but not dihomotopic
Purposes of our paper

- give algebraic representatives of directed spaces up to continuous deformation that preserves direction

- explicit what we mean by continuous deformation that preserves direction (through the notion of directed deformation retract)

- define a algebraic gadget (via a notion of “weak” enriched categories) that reflects directed phenomena

Theorem:
If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.
II.

Grothendieck’s homotopy hypothesis
« Topological spaces are the same as $\infty$-groupoids. »
Topological spaces as $\infty$-groupoids

$\infty$-category = objects + 1-cells (=$\text{morphisms between objects}$) + 2-cells (=$\text{morphisms between 1-cells}$) + ... 

objects = points  
1-cells = paths (= 0-homotopies)  
2-cells = (1-)homotopies  
:  
n-cells = (n-1)-homotopies 

$\infty$-groupoid = $\infty$-category whose n-cells are invertible up-to (n+1)-cells

Here: n-homotopies are invertible up-to (n+1)-homotopies

Ex: a path $\gamma$ has $t \mapsto \gamma(1-t)$ as inverse up-to homotopy
But what are exactly $\infty$-groupoids?

Many ways to « model » $\infty$-groupoids

$\infty$-groupoids $\simeq$ Kan complexes
n-cells $\simeq$ n-simplices
n-cells have inverse up-to (n+1)-cells $\simeq$ n-horns have (n+1)-fillers

Singular simplicial complex $Sing : Top \longrightarrow Kan (\subseteq Simp)$
But what are exactly \( \infty \)-groupoids?

Many ways to « model » \( \infty \)-groupoids

\[
\begin{align*}
\infty \text{-groupoids} & \quad = \quad \text{Kan complexes} \\
n \text{-cells} & \quad = \quad n \text{-simplices} \\
n \text{-cells have inverse up-to (n+1)} \text{-cells} & \quad = \quad n \text{-horns have (n+1)} \text{-fillers}
\end{align*}
\]

Singular simplicial complex \( \text{Sing} : \text{Top} \longrightarrow \text{Kan} (\subseteq \text{Simp}) \)
A formal statement of the homotopy hypothesis

Theorem [Quillen 67]:
The Quillen-Serre model structure on topological spaces is Quillen-equivalent to the Kan-Quillen model structure on simplicial sets.

A few consequences:

- a topological space is weakly homotopy equivalent to the geometric realization of its singular simplicial complex (and so to a CW-complex)
- two topological spaces are weakly homotopy equivalent iff the geometric realization of their singular simplicial complex are weakly homotopy equivalent
A formal statement of the homotopy hypothesis

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A few consequences:

- A topological space is weakly homotopy equivalent to the geometric realization of its singular simplicial complex (and so to a CW-complex).
- Two topological spaces are weakly homotopy equivalent iff the geometric realization of their singular simplicial complex are weakly homotopy equivalent.

« If two topological spaces are equivalent up-continuous deformation then their induced \(\infty\)-groupoids are equivalent (up-to weak equivalence in the suitable model structure) »
III.

A first proposal of directed homotopy hypothesis
Topological spaces as $\infty$-groupoids

$\infty$-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ...

- objects = points
- 1-cells = paths (= 0-homotopies)
- 2-cells = (1-)homotopies
  
  $\vdots$
  
  n-cells = (n-1)-homotopies

$\infty$-groupoid = $\infty$-category whose n-cells are invertible up-to (n+1)-cells

Here: n-homotopies are invertible up-to (n+1)-homotopies

Ex: a path $\gamma$ has $t \mapsto \gamma(1 - t)$ as inverse up-to homotopy
Directed topological spaces as \(\infty\)-groupoids

\(\infty\)-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + \ldots

- objects = points
- 1-cells = dipaths (= 0-dihomotopies)
- 2-cells = (1-)dihomotopies
  \[\vdots\]
- n-cells = (n-1)-dihomotopies

\(\infty\)-groupoid = \(\infty\)-category whose n-cells are invertible up-to (n+1)-cells

Here: n-dihomotopies are invertible up-to (n+1)-dihomotopies
Directed topological spaces as $\infty$-groupoids

$\infty$-category $=$ objects + 1-cells ($=$ morphisms between objects) + 2-cells ($=$ morphisms between 1-cells) + ...  

- objects $=$ points  
- 1-cells $=$ dipaths ($=$ 0-dihomotopies)  
- 2-cells $=$ (1-)dihomotopies  
- $\vdots$  
- n-cells $=$ (n-1)-dihomotopies

$\infty$-groupoid $=$ $\infty$-category whose n-cells are invertible up-to (n+1)-cells

Here: n-dihomotopies are invertible up-to (n+1)-dihomotopies

True for $n \geq 1$, but dipaths are not invertible up-to dihomotopy!
Directed topological spaces as $(\infty,1)$-categories

$\infty$-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + …

- objects = points
- 1-cells = dipaths (= 0-dihomotopies)
- 2-cells = (1-)dihomotopies
  
  …
  
- n-cells = (n-1)-dihomotopies

$(\infty,1)$-category = $\infty$-category whose n-cells are invertible up-to (n+1)-cells for $n \geq 1$

Here: n-dihomotopies are invertible up-to (n+1)-dihomotopies for $n \geq 1$
« Directed topological spaces are the same as \((\infty,1)\)-categories. »
But what are exactly $(\infty, 1)$-categories?

Many ways to « model » $(\infty, 1)$-categories:
- quasi-categories (= weak Kan complexes) [Joyal]
- enriched categories in Kan complexes [Bergner]
- ...

$(\infty, 1)$-categories = enriched categories in Kan complexes
objects = objects
$n$-cells = $(n-1)$-simplices of Hom-objects
$n$-cells have inverse = $(n-1)$-horns of Hom-objects
up-to $(n+1)$-cells for $n \geq 1$ have $n$-fillers for $n \geq 1$
One direction of a directed homotopy hypothesis?

Singular trace category \( \mathbb{T} : dTop \rightarrow KanCat \subseteq SimpCat \) [Porter]
\( \mathbb{T}(X) = \) simplicially enriched category such that:
- objects = points of \( X \)
- Hom-object from \( x \) to \( y \) = singular simplicial complex of \( \mathbb{T}(X)(x, y) \) (space of dipaths from \( x \) to \( y \) up-to increasing reparametrization)

« Can we compare (weak) dihomotopy types of directed spaces by their singular trace categories (up-to weak equivalence) ? »
Yes and no : the case of the directed segment

In many equivalences, $\vec{I}$ is equivalent to a point $*$

$\mathbb{T}(\vec{I})$ and $\mathbb{T}(*)$ are not weakly equivalent.
Two problems

- Specify what we mean by equivalence directed spaces up-to continuous deformations which preserves directedness.
  - the match box not equivalent to a point
  - the directed segment equivalent to a point
  - few algebraic constructions are invariant (directed components \([\text{Goubault, Haucourt 07}],\) natural homology \([\text{DGG 15}]\))

- Fix the directed homotopy hypothesis.
III.

The need for equivalences in directed algebraic topology
Reminder on classical algebraic topology

A (strong) deformation retract of $X$ on a subspace $A$ is a continuous map

$$H : X \rightarrow P(X) = \left[ [0, 1] \rightarrow X \right]$$

such that:

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$.

Theorem:

Two topological spaces are homotopy equivalent iff there is a span of deformation retracts between them.
Definition in directed algebraic topology

A future deformation retract of $X$ on a sub-$d$-space $A$ is a continuous map

$$H : X \longrightarrow \vec{P}(X)$$

such that:

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0, 1]$, the map $H_t : x \mapsto H(x)(t)$ is a $d$-map;
- for every $\delta$ of $A$ from $z$ to $H_1(x)$ there is a dipath $\gamma$ of $X$ from $y$ to $x$ with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to $\delta$.

Definition:

Two $d$-spaces are dihomotopy equivalent iff there is a zigzag of future and past deformation retracts between them.
Something’s wrong, isn’t it?

There is a future deformation retract from the matchbox to its upper face (and so to its upper corner)!
Something’s wrong, isn’t it?

There is a future deformation retract from the matchbox to its upper face (and so to its upper corner)!

Problem: the dipaths along which we deform do not preserve the fact that dipaths are not dihomotopic.
Inessential dipaths

Idea from [Fajstrup, Goubault, Haucourt, Raussen] for category of components.

The set $\mathcal{I}(X)$ of inessential dipaths of $X$ is the largest set of dipaths such that:

- it is closed under concatenation and dihomotopy;
- for every $\gamma \in \mathcal{I}(X)$ from $x$ to $y$, for every $z \in X$ such that $\overrightarrow{P}(X)(z, x)$, the map $\gamma \ast \_ : \overrightarrow{P}(X)(z, x) \rightarrow \overrightarrow{P}(X)(z, y)$ $\delta \mapsto \gamma \ast \delta$ is a homotopy equivalence;
- symmetrically for $\_ \ast \gamma$;
- $\mathcal{I}(X)$ has the right and left Ore condition modulo dihomotopy:

Ex: $\epsilon$ is not inessential in the matchbox
Better definition in directed algebraic topology

A future deformation retract of $X$ on a sub-dspace $A$ is a continuous map

$$H : X \rightarrow \mathcal{J}(X)$$

such that:

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0, 1]$, the map $H_t : x \mapsto H(x)(t)$ is a dmap;
- for every $\delta$ of $A$ from $z$ to $H_1(x)$ there is a dipath $\gamma$ of $X$ from $y$ to $x$ with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to $\delta$.

Definition:

Two dspaces are dihomotopy equivalent iff there is a zigzag of future and past deformation retracts between them.
First results

- the directed segment is dihomotopy equivalent to a point
- the matchbox is not dihomotopy equivalent to a point
- if two dspaces are dihomotopy equivalent then they have the same directed components and their natural homology are bisimilar
IV.

A new proposal of directed homotopy hypothesis
Fixation of the directed homotopy hypothesis

- replacing enriched categories by partially enriched categories (which encode accessibility)
- changing weak equivalences
- proving the following:

**Theorem:**
If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.
Fixation of the directed homotopy hypothesis

- replacing enriched categories by partially enriched categories (which encode accessibility)
- changing weak equivalences
- proving the following:

**Theorem:**
If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.

« One can compare directed spaces by comparing their partially enriched category (up-to weak equivalence). »
Conclusion

Summary:
- We have defined a dihomotopy equivalence, which behaves well on examples and for which natural homology is an invariant.
- We have defined a new structure, closed to $(\infty, 1)$-categories, and designed its weak equivalence, for which it is an invariant of dihomotopy equivalence.

Many open questions:
- Are there two weakly equivalent dspaces that are not dihomotopy equivalent?
- Are there model structures on dspaces (or partially enriched categories) for which the weak equivalence is dihomotopy equivalence (or weak equivalence)?
- Do we have a kind of geometric realization from partially enriched categories to dspaces in order to formulate a complete directed homotopy equivalence?
- Are the partially enriched categories (in Top or Simp) a nice model of $(\infty, 1)$-categories?
IV.

A new proposal of directed homotopy hypothesis
The symptomatic case of the directed segment

\[ 0 \rightarrow 1 \]

In any reasonable equivalence, \( \vec{I} \) is equivalent to a point \(*\)

\[ \mathbb{T}(\vec{I}) \text{ and } \mathbb{T}(\ast) \text{ are not weakly equivalent:} \]

- for \( x < y \), \( \mathbb{T}(\vec{I})(y, x) \) is empty while \( \mathbb{T}(\ast)(\ast, \ast) \) is not
- their category of components are not equivalent (one has empty Hom-sets while the other has not)
The symptomatic case of the directed segment

In any reasonable equivalence, $\mathbb{I}$ is equivalent to a point $\ast$

$\mathbb{T}(\mathbb{I})$ and $\mathbb{T}(\ast)$ are not weakly equivalent:

- for $x < y$, $\mathbb{T}(\mathbb{I})(y, x)$ is empty while $\mathbb{T}(\ast)(\ast, \ast)$ is not
- their category of components are not equivalent (one has empty Hom-sets while the other has not)

Empty path spaces have a particular behavior that must be studied with care
Reminder on enriched categories and functors

Let \((V, U, \otimes)\) be a monoidal category.

A (small) enriched category \(C\) on \(V\) consists in the following data:

- a set of objects \(\text{Ob}(C)\)
- for every pair of objects \(A, B\), an object \(C(A, B)\) of \(V\)
- for every triple of objects \(A, B, C\), a morphism in \(V\)

\[
\circ_{A,B,C} : C(A, B) \otimes C(B, C) \rightarrow C(A, C)
\]

- for every object \(A\), a morphism in \(V\)

\[
u_A : U \rightarrow C(A, A)
\]

satisfying some coherence diagrams (associativity, unity).

An enriched functor \(F : C \rightarrow D\) on \(V\) consists in the following data:

- a function \(F : \text{Ob}(C) \rightarrow \text{Ob}(D)\);
- for every pair of objects \(A, B\) of \(C\), a morphism in \(V\)

\[
F_{A,B} : C(A, B) \rightarrow D(F(A), F(B))
\]

satisfying some coherence diagrams (composition, unity).
A better definition to handle emptiness

Let \((V, U, \otimes)\) be a monoidal category.

A (small) partially enriched category \(\mathcal{C}\) on \(V\) consists in the following data:

- a preordered set of objects \(\text{Ob}(\mathcal{C}), \leq\)
- for every pair of objects \(A \leq B\), an object \(\mathcal{C}(A, B)\) of \(V\)
- for every triple of objects \(A \leq B \leq C\), a morphism in \(V\)

\[
\circ_{A,B,C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)
\]

- for every object \(A\), a morphism in \(V\)

\[
u_A : U \to \mathcal{C}(A, A)
\]

satisfying some coherence diagrams (associativity, unity), compatible with \(\leq\).

An enriched functor \(F : \mathcal{C} \to \mathcal{D}\) on \(V\) consists in the following data:

- a monotonic function \(F : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})\);
- for every pair of objects \(A \leq B\) of \(\mathcal{C}\), a morphism in \(V\)

\[
F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))
\]

satisfying some coherence diagrams (composition, unity), compatible with \(\leq\).
From dTop to PeCat(HoTop): the dipath category

\( \mathbb{P}(X) = \) partially enriched category on \( \text{HoTop} \):
- objects = points of \( X \);
- \( x \leq y \) iff \( \overrightarrow{\mathbb{P}}(X)(x, y) \neq \emptyset \);
- for \( x \leq y \), \( \mathbb{P}(X)(x, y) = \overrightarrow{\mathbb{P}}(X)(x, y) \);
- composition = concatenation up-to homotopy;
- unit = constant path.

We can have defined it with value in \( \text{HoSimp} \) or \( \text{Ab} \) by composing with singular simplicial complex or homology.

We recover the fundamental category \( \pi_1(X) \) by composing with the connected components functor.
What about the category of components?

For [Bergner], it is just $\pi_1(X)$. 
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No good for directed segment.
Already known since [Fajstrup, Goubault, Haucourt, Raussen].
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We have to define a category of « directed » components.
Yoneda morphisms, category of directed components

A slight modification of [Fajstrup, Goubault, Haucourt, Raussen]

The set \( \mathcal{Y}(C) \) of Yoneda morphisms of a category \( C \) is the largest set of morphisms such that:

- it is closed under concatenation;
- for every \( f : c \to c' \in \mathcal{Y}(C) \), for every object \( c'' \) of \( C \) such that \( C(c', c'') \neq \emptyset \), the function \( \_ \circ f : C(c', c'') \to C(c, c'') \) \( g \mapsto g \circ f \) is a bijection;
- symmetrically for \( f \circ \_ \);
- it has right and left Ore conditions

\[
\pi_0(C) = C[\mathcal{Y}(C)^{-1}] = C \text{ in which we inverse the morphisms in } \mathcal{Y}(C)
\]

\[
\pi_0(X) = \pi_0(\pi_1(X))
\]
Example: the directed segment

\[ \begin{array}{c}
\text{Example: the directed segment} \\
0 \quad \rightarrow \quad 1 \\
\end{array} \]

\[ \mathbb{P}(\vec{I}) \text{ is such that:} \]
- \[ x \leq y \text{ is the usual ordering on } I; \]
- \[ \text{for every } x \leq y, \mathbb{P}(\vec{I})(x, y) \text{ is contractible.} \]

The fundamental category \( \pi_1(\vec{I}) \) is the poset \((I, \leq)\).

The category of components \( \pi_0(\vec{I}) \) is the preordered set \((I, I \times I)\), which is equivalent to the category with one object and one morphism.
Weak dihomotopy equivalence

We say that a dmap \( f : X \rightarrow Y \) is a weak dihomotopy equivalence iff

- it induces an equivalence between the categories of directed components
- it induces a fully-faithful enriched functor between dipath categories i.e. for \( x \leq x' \), the map

\[
P(f)_{x,x'} : P(X)(x,x') \rightarrow P(Y)(f(x), f(x'))
\]

which maps \( \gamma \) to \( f \circ \gamma \) is a homotopy equivalence.

We say that two dspaces are weakly dihomotopy equivalent iff there is zigzag of weak dihomotopy equivalence between them.
Examples

→ $I$ is weakly equivalent to a point.

$\mathbb{P}(s, t)$ is homotopy equivalent to a two point space, so the match box cannot be weakly equivalent to a point.
Invariance

**Theorem:**

If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.
Invariance

Theorem:
If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.

« One can compare dspaces by comparing their dipath category (up-to weak equivalence). »
Invariance

**Theorem:**
If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.

« One can compare dspaces by comparing their dipath category (up-to weak equivalence). »

« Are dspaces the same as partially enriched categories in HoTop (or HoSimp)? »