Bisimulations and unfolding in $\mathcal{P}$-accessible categorical models

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Jérémy Dubut (LSV, ENS Cachan, France)

Eric Goubault (LIX, Ecole Polytechnique, France)

Jean Goubault-Larrecq (LSV, ENS Cachan, France)

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Computing systems in the language of category theory

Mainly, two types :

- (co)algebraic approach **[Jacobs, ...]**
- lifting approach **[Winskel, Joyal, Nielsen, ...]**

approach	class type	system type	bisimulations
coalgebraic	category + functor (monad)	coalgebra	span of morphisms
lifting	category + sub-category	object	span of morphisms with lifting property w.r.t. the sub-category

Example : TS I - category of TS

Fix an alphabet Σ.

Transition system :

A **TS** $T = (Q, i, \Delta)$ on Σ is the following data :

- a set Q (of states) ;
- a initial state $i \in Q$;
- a set of transitions $\Delta \subseteq Q \times \Sigma \times Q$.

Morphism of TS :

A **morphism of TS** $f : T_1 = (Q_1, i_1, \Delta_1) \longrightarrow T_2 = (Q_2, i_2, \Delta_2)$ is a function $f : Q_1 \longrightarrow Q_2$ such that $f(i_1) = i_2$ and for every $(p, a, q) \in \Delta_1$, $(f(p), a, f(q)) \in \Delta_2$.

$\mathbf{TS(\Sigma)}$ = category of TS on Σ and morphisms of TS

Example : TS II - relational bisimulations

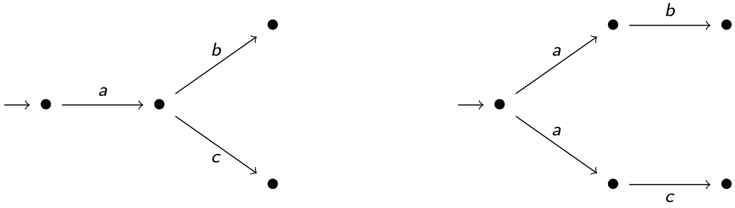
Bisimulations **[Park81]** :

A **bisimulation** between $T_1 = (Q_1, i_1, \Delta_1)$ and $T_2 = (Q_2, i_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that :

(i) $(i_1, i_2) \in R$;

(ii) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$;

(iii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.



Example : TS III - runs

Branch :

A n-**branch on** Σ is a transition system $\langle a_0, \ldots, a_{n-1} \rangle = ([n], 0, \Delta)$ where :

- $[n]$ is the set $\{0, \ldots, n\}$;
- Δ is of the form $\{(i, a_i, i+1) \mid i \in [n-1]\}$ for some a_0, ..., a_{n-1} in Σ.

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \quad \cdots \quad n-1 \xrightarrow{a_{n-1}} n$$

Branch of a TS :

A n-**branch of a TS** T is a morphism of TS from any $\langle a_0, \ldots, a_{n-1} \rangle$ to T.

$\mathbf{Br}(\Sigma)$ = full sub-category of $\mathbf{TS}(\Sigma)$ of branches

Example : TS IV - from states to runs

A bisimulation R between T_1 and T_2 induces a relation R_n between n-branches of T_1 and n-branches of T_2 by :

$$R_n = \{(f_1 : B \longrightarrow T_1, f_2 : B \longrightarrow T_2) \mid \forall i \in [n], (f_1(i), f_2(i)) \in R\}$$

Properties :

- $(\iota_{T_1}, \iota_{T_2}) \in R_0$ by (i) ;
- by (ii), if $(f_1, f_2) \in R_n$ and if $(f_1(n), a, q_1) \in \Delta_1$ then there is $q_2 \in Q_2$ such that $(f_2(n), a, q_2) \in \Delta_2$ and $(f_1', f_2') \in R_{n+1}$ where $f_i'(j) = f_i(j)$ if $j \leq n$, q_i otherwise ;
- symmetrically with (iii) ;
- if $(f_1, f_2) \in R_{n+1}$ then $(f_1', f_2') \in R_n$ where f_i' is the restriction of f_i to $[n]$.

Fact : bisimilarity is equivalent to the existence of such a relation between branches

Categorical models

Categorical models :

A **categorical model** is a category $\mathcal{M}$ with a small subcategory $\mathcal{P}$ which have a common initial object I.

- $\mathcal{M}$ = category of systems (Ex : $\mathbf{TS(\Sigma)}$) ;
- $\mathcal{P}$ = sub-category of execution shapes (Ex : $\mathbf{Br(\Sigma)}$) ;
- unique morphism $I \longrightarrow X$ = initial state of X (Ex : $I = [0]$).

Other examples : 1-safe Petri nets with event structures **[Winskel]**, HDA with HDA paths **[van Glabbeek]**, ...

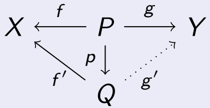
Relational bisimilarity in categorical models

Strong path bisimulation **[Joyal, Nielsen, Winskel]**

A **strong path-bisimulation** R between X and Y, objects of $\mathcal{M}$ is a set of elements of the form $X \xleftarrow{f} P \xrightarrow{g} Y$ with P object of $\mathcal{P}$ such that :

(a) $X \xleftarrow{\iota_X} I \xrightarrow{\iota_Y} Y$ belongs to R;

(b) if $X \xleftarrow{f} P \xrightarrow{g} Y$ belongs to R then for every morphism $p : P \longrightarrow Q$ in $\mathcal{P}$ and every $f' : Q \longrightarrow X$ such that $f' \circ p = f$ then there exists $g' : Q \longrightarrow Y$ such that $g' \circ p = g$ and $X \xleftarrow{f'} Q \xrightarrow{g'} Y$ belongs to R;



(c) symmetrically;

(d) if $X \xleftarrow{f} P \xrightarrow{g} Y$ belongs to R and if we have a morphism $p : Q \longrightarrow P \in \mathcal{P}$ then $X \xleftarrow{f \circ p} Q \xrightarrow{g \circ p} Y$ belongs to R.

Few remarks

- strong path bisimilarity coincides with classical bisimilarity in some cases (TS, Petri nets, ...)
- a Hennessy-Milner-like theorem holds for strong path bisimulation

Bisimilarity as span

$\mathcal{P}$-bisimilarity

We say that a morphism $f : X \longrightarrow Y$ of $\mathcal{M}$ is **($\mathcal{P}$-)open** iff for all commutative diagrams :

$$\begin{array}{ccc} P & \xrightarrow{x} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[y]{} & Y \end{array}$$

with $p : P \longrightarrow Q \in \mathcal{P}$, there exists a morphism $\theta : Q \longrightarrow X$ such that the following diagram commutes :

$$\begin{array}{ccc} P & \xrightarrow{x} & X \\ {\scriptstyle p}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[y]{} & Y \end{array}$$

We then say that two objects X and Y of $\mathcal{M}$ are $\mathcal{P}$-**bisimilar** iff there exists a span $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ where f and g are $\mathcal{P}$-open.

Link between those two bisimilarities

- $\mathcal{P}$-bisimilarity always implies strong path bisimilarity
- in many concrete cases, $\mathcal{P}$-bisimilarity is equivalent to classical bisimilarity, and so to strong path bisimilarity
- there is no general theorem of equivalence between those two bisimilarities

How to prove strong path bisimilarity $\Rightarrow$ $\mathcal{P}$-bisimilarity ?

Given a relation $R \subseteq \{X \xleftarrow{f} P \xrightarrow{g} Y\}$, how can we construct a span :

$$\begin{array}{ccc} & W_R & \\ {}^{\Phi_R}\swarrow & & \searrow^{\Psi_R} \\ X & & Y \end{array}$$

where Φ_R and Ψ_R are open ?

Accessibility

Idea : R = set of formal paths ; W_R = glueing of those paths

Trees

We call **tree** any colimit in $\mathcal{M}$ of a small diagram with values in $\mathcal{P}$.
We say that a categorical model **has trees** if those colimits exists.
We note $\mathbf{Tree(}\mathcal{M}, \mathcal{P}\mathbf{)}$ the full sub-category of $\mathcal{M}$ of trees.

Ex : $\mathbf{Tree(TS(\Sigma), Br(\Sigma))}$ = synchronization trees

By universal property, W_R, Φ_R and Ψ_R exist but Φ_R and Ψ_R not open
True if the trees does not have more branches than the ones used to construct it :

Accessibility

We say that a categorical model is accessible if it has trees and every path in a tree $p : P \in \mathcal{P} \longrightarrow \mathit{colim}\ F$ where F is non-empty can be factorized as $\kappa_c \circ q$ where :

- c is an object of the domain of F ;
- κ_c is the universal morphism from $F(c)$ to $\mathit{colim}\ F$;
- q is a morphism of $\mathcal{P}$.

Summary and remarks

Theorem [Dubut, Goubault*2] :

If $(\mathcal{M}, \mathcal{P})$ is an accessible categorical model then $\mathcal{P}$-bisimilarity is equivalent to strong path bisimilarity.

Few remarks :

- this implies that $\mathcal{P}$-bisimilarity is an equivalence relation ;
- TS, word automata, timed transition systems, pre-sheaf models are accessible ;
- accessibility is preserved by coreflection.

Example : TS VI - unfolding

Unfolding of a TS = synchronization tree obtained by delooping

Given a TS $T = (Q, i, \Delta)$, its unfolding is the TS whose :

- its states are branches of T ;
- its initial states is the 0-branch ι_T ;
- its transition are $(f : \langle a_0, \ldots, a_{n-1} \rangle \longrightarrow T, a_n, g : \langle a_0, \ldots, a_n \rangle \longrightarrow T)$, where the restriction of g to $\langle a_0, \ldots, a_{n-1} \rangle$ is f.

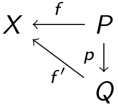
Its a synchronization tree.

Unfolding in a accessible categorical model

Idea : $\mathrm{Unfold}(X)$ = glueing of all paths of X

Form the following diagram $F_X : \mathcal{P} \downarrow X \longrightarrow \mathcal{P}$:

- objects of $\mathcal{P} \downarrow X$ = paths of X = morphisms from any $P \in \mathcal{P}$ to X
- morphisms are morphisms p of $\mathcal{P}$ such that :



- $F_R(p : P \longrightarrow X) = P$

$\mathrm{Unfold}(X) = \mathit{colim}\ F_X$

Properties of the unfolding

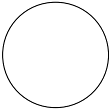
Theorem [Dubut, Goubault*2] :

- When $(\mathcal{M}, \mathcal{P})$ has trees, $\mathrm{Unfold}(X)$ always exists and Unfold is a functor from $\mathcal{M}$ to **Tree($\mathcal{M}, \mathcal{P}$)**.
- When $(\mathcal{M}, \mathcal{P})$ is accessible, the canonical map from $\mathrm{Unfold}(X)$ to X is open.
- When $(\mathcal{M}, \mathcal{P})$ is accessible, Unfold is the right adjoint of the injection of **Tree($\mathcal{M}, \mathcal{P}$)** in $\mathcal{M}$.

Universal covering of a topological space = unfolding ?

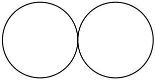
Universal covering = complete unlooping

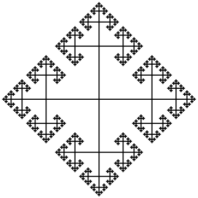
S^1



$\mathbb{R}$

$S^1 \vee S^1$





the definition is too technical and it does not always exist

Example : SPCG I - the accessible categorical model

SPCG :

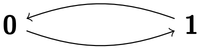
A **small pointed connected groupoid** is :

- a small category $\mathcal{C}$ such that :
  - every morphism is invertible ;
  - between two objects, there is always at least one morphism.
- an object c of $\mathcal{C}$.

A **morphism of SPCG** is a functor that preserves the points. We note **SPCG** this category.

Let I be the full sub-category of **SPCG** whose objects are :

- 0 = the SPCG with one object an one morphism ;
- 1 =



Proposition :

$(\mathbf{SPCG}, I)$ is accessible.

Example : SPCG II - (universal) covering

Covering :

A **covering of a SPCG** $(\mathcal{C}, c)$ is I-open map $F : (\mathcal{D}, d) \longrightarrow (\mathcal{C}, c)$ whose lifts are unique i.e.

$$\begin{array}{ccc} \mathbf{0} & \longrightarrow & (\mathcal{D}, d) \\ \downarrow & \exists ! \nearrow & \downarrow F \\ \mathbf{1} & \longrightarrow & (\mathcal{C}, c) \end{array}$$

We say that it is **universal** if $\mathcal{D}(d, d)$ is a singleton.

We can prove that the universal covering always exists and is unique up-to isomorphism.

Proposition :

Let $F : \mathcal{D} \longrightarrow \mathcal{C}$ be a universal covering and $G : \mathcal{E} \longrightarrow \mathcal{C}$ be a covering. Then, there exists a unique morphism $H : \mathcal{D} \longrightarrow \mathcal{E}$ (which is a covering) such that $F = G \circ H$. In particular, the universal covering is initial among coverings.

Universality of the unfolding

Fix an accessible categorical model $(\mathcal{M}, \mathcal{P})$.

$\mathcal{P}$-covering :

A morphism $f : X \longrightarrow Y$ is a $\mathcal{P}$-**covering** if it is a $\mathcal{P}$-open map whose lifts are unique.

Note $unf_X : \mathrm{Unfold}(X) \longrightarrow X$ the canonical morphism.

Theorem [Dubut, Goubault*2] :

i) unf_X is a $\mathcal{P}$-covering.

ii) For every $\mathcal{P}$-covering $f : Y \longrightarrow X$, there exists a unique morphism $g : \mathrm{Unfold}(X) \longrightarrow Y$ (which is a $\mathcal{P}$-covering) such that $unf_X = f \circ g$.

In particular, the unfolding is initial among $\mathcal{P}$-coverings.

Corollary :

The universal covering and the I-unfolding coincide.

Conclusion and future works

Summary : we have designed a general framework **accessible categorical models** in which :

- strong path bisimilarity and $\mathcal{P}$-bisimilarity coincide ;
- a nice notion of unfolding exists ;
- classical phenomena are captured (TS, timed TS, automata, sheaf models, ...) ;
- the classical notion of universal covering coincides with the unfolding.

And now ?

- truly concurrent systems (Petri nets, HDA, ...) ;
- natural accessible structure on the category of topological spaces for which the unfolding extends the universal covering.