Natural homology
Computability and Eilenberg-Steenrod axioms
Applied and Computational Algebraic Topology
HIM, Bonn

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True concurrency

- Petri nets [Petri 62]
- progress graphs [Dijkstra 68]
- trace theories [Mazurkiewicz 70s]
- event structures [Winskel 80s]
- higher dimensional automata (HDA) [Pratt 91]
Interleaving vs continuity

\[ X := 0 \parallel Y := 1 \]

Interleaving behaviors: \( A \) then \( B \) or \( B \) then \( A \)
Interleaving vs continuity

\[ X := 0 \parallel Y := 1 \]

Continuous behaviors: any scheduling of \( A \) and \( B \)
True concurrency, geometrically

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True concurrency, geometrically

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Problem: executions are directed, paths are not
True concurrency, geometrically

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D-spaces and dipaths [Grandis 01]

A d-space is a topological $X$ with a subset $\overrightarrow{P}(X)$ of paths, called dipaths, such that:

- constant paths are dipaths,
- dipaths are closed under concatenation,
- dipaths are closed under non-decreasing reparametrization, $\gamma \circ r$ with $r : [0, 1] \longrightarrow [0, 1]$ continuous monotonic.

\[
\begin{align*}
\gamma_1 \ast \gamma_2(t) &= \gamma_1(2t) & \text{if } t \leq \frac{1}{2} \\
&= \gamma_2(2t - 1) & \text{if } t \geq \frac{1}{2}
\end{align*}
\]

The set of paths can be equipped with the compact-open topology $\overrightarrow{P}(X)$ and $\overrightarrow{P}(X)(a, b)$ can be equipped with the subspace topology.

A dimap is a continuous function $f : X \longrightarrow Y$ such that for every $\gamma \in \overrightarrow{P}(X)$, $f \circ \gamma \in \overrightarrow{P}(Y)$. 
The different d-space structures of the segment

$\overrightarrow{[0, 1]}$: dipaths are monotonic paths,

$\overline{[0, 1]}$: dipaths are constant paths,

$\overleftarrow{[0, 1]}$: dipaths are all the paths.

Ex: dipaths of $X = \text{dimaps from } \overrightarrow{[0, 1]} \text{ to } X$
Homotopy

A **homotopy** from $\gamma$ to $\tau$, paths from $a$ to $b$, is a continuous function

$$H : [0, 1] \times [0, 1] \to X$$

such that:

- $H(0, \_ ) = a$ and $H(1, \_ ) = b,$
- $H(\_, 0) = \gamma$ and $H(\_, 1) = \tau.$

Equivalently, it is a path in the space of paths $P(X)(a, b)!$

Two paths are **homotopic** if there is a homotopy between them, or equivalently, if they are in the same path-connected components of $P(X)(a, b).$
Dihomotopy

A **dihomotopy** from $\gamma$ to $\tau$, dipaths from $a$ to $b$, is a dimap

$$H : [0,1] \times [0,1] \rightarrow X$$

such that:

- $H(0,\_)=a$ and $H(1,\_)=b$,
- $H(\_,0)=\gamma$ and $H(\_,1)=\tau$.

Equivalently, it is a path in the space of dipaths $\hat{\mathcal{P}}(X)(a,b)$!

Two paths are **dihomotopic** if there is a dihomotopy between them, or equivalently, if they are in the same path-connected components of $\hat{\mathcal{P}}(X)(a,b)$. 
Example 1

dihomotopic

non-dihomotopic
Example II : Fahrenberg’s matchbox
A category of dipaths?

Can we form the following category?

- objects are points,
- morphisms are dipaths,
- identities are constant paths,
- composition is concatenation.
Can we form the following category?
- objects are points,
- morphisms are dipaths,
- identities are constant paths,
- composition is concatenation.

Answer: No, the concatenation is not associative...
A category of dipaths?

...but we can form the fundamental category $\pi_1^{\rightarrow}$:

- objects are points,
- morphisms are dipaths modulo dihomotopy,
- identities are dihomotopy classes of constant paths,
- composition is concatenation modulo dihomotopy.

because concatenation is associative modulo dihomotopy.
A category of traces

Actually, concatenation is associative modulo reparametrization
\( \gamma \) reparametrizes to \( \rho \) if there is a surjective, monotonic and continuous function \( r : [0, 1] \to [0, 1] \) such that \( \rho = \gamma \circ r \). We call \textbf{trace} of a dipath \( \gamma \) and note \( \langle \gamma \rangle \), the equivalence class of \( \gamma \) modulo reparametrization.

We can form the \textbf{category of traces} \( \overrightarrow{T}(X) \):
- objects are points,
- morphisms are traces,
- identities are traces of constant paths,
- composition is concatenation modulo reparametrization.

We can also define the \textbf{trace space} \( \overrightarrow{T}(X)(a, b) \) as the quotient space of \( \overrightarrow{P}(X)(a, b) \) modulo reparametrization.
Objective

- study those concurrent systems through their geometry (dipaths, traces, dihomotopies)

- homology = essential notion, computable abstraction of homotopy
  - defining a directed homology
  - proving classical properties of this homology
Directed Homology
Related work

Candidates of directed homology:
- past and future homologies [Goubault 95]
- ordered homology groups [Grandis 04]
- directed homology via $\omega$-categories [Fahrenberg 04]
- homology graph [Kahl 13]

Not fine enough: do not distinguish Fahrenberg’s matchbox from a point
A first idea

**Not so good** directed homology:

\[ \text{Not so good} (X) = \text{classical homology of } T(X)(a, b) \]

\[
T(A)(a, b) \simeq 6 \text{ point space} \simeq T(B)(a, b)
\]

\[ \text{Not so good}(A) \simeq \mathcal{R}^6 \simeq \text{Not so good}(B) \]
A first (not so) bad idea

- make $a, b$ vary
  - $T(A)(a, b') \simeq 4$ point space
  - $\text{Not}_\text{so}_\text{good}(A) \simeq \mathcal{R}^4$

- no $a', b'$ such that
  - $T(B)(a', b') \simeq 4$ point space
  - $\text{Not}_\text{so}_\text{good}(B) \simeq \mathcal{R}^4$
Natural homology

$\mathcal{F}_X = \text{category whose}:
\begin{itemize}
  \item objects are traces
  \item morphisms are extensions
\end{itemize}

\textbf{Natural homology :}

functor $\overrightarrow{H}_n(X) : \mathcal{F}_X \longrightarrow \text{Mod}(\mathcal{R})$

$\begin{align*}
(a \xrightarrow{\gamma} b) & \longmapsto H_{n-1}(T(X)(a, b))
\end{align*}$

($H_{n-1} = \text{classical singular homology}$)

- $\mathcal{F}_X = \text{category of factorizations} \ [\text{Mac Lane 71}]
- \overrightarrow{H}_n(X) = \text{natural system} \ [\text{Leech 73, Baues, Wirsching 85}]$
Natural homotopy

\[ \mathcal{F}_X = \text{category whose}: \]

- objects are traces
- morphisms are extensions

\[ (a \xrightarrow{\gamma} b) \mapsto \pi_{n-1}(T(X)(a, b), \gamma) \quad (\pi_{n-1} = \text{classical homotopy}) \]

- \( \mathcal{F}_X = \text{category of factorizations} \) [Mac Lane 71]
- \( \overrightarrow{\pi}_n(X) = \text{natural system} \) [Leech 73, Baues, Wirsching 85]
**Bimodule homology**

\( \mathcal{E}_X \) = category whose:
- pairs of points \((a, b)\), s.t. \( \exists \) a dipaths from \( a \) to \( b \)
- morphisms are extensions

\[ \mathcal{E}_X \to \operatorname{Mod} \mathcal{R} \]

\((a, b) \mapsto H_{n-1}(T(X)(a, b)) \quad (H_{n-1} = \text{classical singular homology}) \]

- \( \mathcal{E}_X \) = enveloping category
- \( \overrightarrow{H}_n(X) \) = bimodule [Mitchell 72]
Example: first natural homology of \( a + b \)
Example: first bimodule homology of $a + b$
2 dipaths non dihomotopic
\[ \Rightarrow T(X)(a, b) \simeq 2 \text{ point space} \]
\[ \Rightarrow H_0(T(X)(a, b)) \simeq \mathbb{R}^2 \]
\[ \Rightarrow \overrightarrow{H}_1(X) \text{ not trivial} \]
\[ \Rightarrow \text{natural homology detects failure of dihomotopy} \]
\[ \text{in Fahrenberg's matchbox} \]
Comparison of diagrams
Category of diagrams \( \text{Diag}(\mathcal{M}) \)

Fix a category \( \mathcal{M} \), typically \( \text{Mod}(\mathcal{R}), \text{Set}, \ldots \)

A diagram in \( \mathcal{M} \) is a functor from any small category to \( \mathcal{M} \).

Ex : \( \overrightarrow{H}_n(X), \overrightarrow{H}'_n(X), \overrightarrow{\pi}_n(X) \) are diagrams

A morphism of diagrams from \( F : \mathcal{C} \to \mathcal{M} \) to \( G : \mathcal{D} \to \mathcal{M} \) is a pair \( (\Phi, \sigma) \) where:

- \( \Phi : \mathcal{C} \to \mathcal{D} \) is a functor,
- \( \sigma : F \to G \circ \Phi \) is a natural transformation.

Isos : \( \Phi \) isofunctor, \( \sigma \) natural iso

\( \overrightarrow{H}_n, \overrightarrow{H}'_n, \overrightarrow{\pi}_n \) are functors from \( \text{dTop} \) to \( \text{Diag}(\mathcal{M}) \)
How to compare natural homologies?

\[ \overrightarrow{H}_n(A) = \overrightarrow{H}_n(B) \Rightarrow A = B, \text{ modulo isomorphism} \]

\[ \mathcal{R}[a, b] \]

Crucial idea:

Compare natural homologies up-to evolutions of homology of trace spaces with time.

Idea similar to bisimulations in concurrent systems.
Bisimulations of diagrams

Based on the general theory of bisimulations of [Joyal et al. 94]

An **open map** is a morphism of diagrams \((\Phi, \sigma)\) from \(F : C \rightarrow M\) to \(G : D \rightarrow M\) such that:

- \(\sigma\) is a natural iso,
- \(\Phi\) is surjective on objects,
- \(\Phi\) is fibrational in the following sense: for every morphism of the form \(j : F(c) \rightarrow d'\) in \(D\), there is a morphism \(i : c \rightarrow c'\) in \(C\) with \(F(i) = j\).

We say that two diagrams are **bisimilar** if there is a zigzag of open maps between them.
\( \overrightarrow{H}_n(X) \) is bisimilar to \( \overrightarrow{H'}_n(X) \)

the first natural homology of the matchbox is not bisimilar to the one of a point space
Computability
Cubical complex and discrete traces

Euclidian cubical complex: any subspace of $\mathbb{R}^n$ which is a finite union of cubes of the form

$$[a_1, a_1 + \alpha_1] \times \ldots \times [a_n, a_n + \alpha_n]$$

with $a_i \in \mathbb{Z}$ and $\alpha_i \in \{0, 1\}$.

discrete trace = trace which is a glueing of segments joining center of cubes

$f_X = \text{category of discrete traces and extensions by discrete traces}$

Discrete natural homology $\overrightarrow{h}_n(X)$:

functor $\overrightarrow{h}_n(X) : (X) \rightarrow \text{Ab}$

$$(a \gamma \rightarrow b) \mapsto H_{n-1}(T(X)(a, b))$$
Computability

Theorem:

Given an Euclidian cubical complex $X$, $\vec{h}_n(X)$ is:
- computable
- bisimilar to $\vec{H}_n(X)$

Proof of computability:
- compute a finite representation of trace spaces [Raussen, Ziemianski],
- compute its homology.

Corollary:

Given two Euclidian cubical complexes, it is decidable whether they have the same natural homology (when computed in real numbers).

Proof:
Bisimilarity is decidable by reducing the existential theory of the reals.
Exactness Axiom
Exactness axiom in classical homology

If \((A, X)\) is a topological pair, the relative homology \(H_n(X, A)\) is the homology of the chain complex \(C_n(X)/C_n(A)\)

**Exactness axiom:**

There is a long exact sequence:

\[
\cdots \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(p)} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots
\]
Exactness axiom in classical homology, general form

the sequence:

\[ C(A) \xrightarrow{i} C(X) \xrightarrow{p} C(X)/C(A) \]

is short exact

Long exact sequence in homology:

If

\[ A \xrightarrow{i} B \xrightarrow{p} C \]

is a short exact sequence of chain complexes, then there is a long exact sequence in modules of the form:

\[ \cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(p)} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \to \cdots \]

nice for computations, more general in Abelian categories
Exactness in diagrams?

\[ \text{Diag(Mod}(\mathcal{R})) \text{ is not Abelian} \]

Which ingredients?
- zero objects,
- kernels,
- images/cokernels,
- subquotients (exactness of some morphisms),
Exactness in diagrams?

\[ \text{Diag}(\text{Mod}(R)) \] is not Abelian

Which ingredients?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels,
- images/cokernels,
- subquotients (exactness of some morphisms),
Exactness in diagrams?

\[ \text{Diag}(\text{Mod}(\mathcal{R})) \] is not Abelian

Which ingredients?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels, OK defined levelwise,
- images/cokernels,
- subquotients (exactness of some morphisms),
Exactness in diagrams?

\[ \text{Diag}(\text{Mod}(\mathcal{R})) \text{ is not Abelian} \]

Which ingredients?

- zero objects, \textit{no} zero objects, but null objects (diagrams with values 0),
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Exactness in diagrams?

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Exactness in diagrams?

\[
\text{Diag} (\text{Mod}(\mathcal{R})) \text{ is not Abelian}
\]

Which ingredients?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels, OK defined levelwise,
- images/cokernels, OK, but more complicated,
- subquotients (exactness of some morphisms), OK.

Theorem:

\[
\text{Diag} (\text{Mod}(\mathcal{R})) \text{ is a homological category in the sense of [Grandis 91].}
\]
Almost exactness in diagrams

**Theorem [Grandis 91]**:

Let \( \mathcal{A} \) be a homological category.
For every short exact sequence in \( C_\bullet(\mathcal{A}) \):

\[
\begin{array}{c}
U & \xrightarrow{m} & V & \xrightarrow{p} & W
\end{array}
\]

there exists a long sequence of order two in \( \mathcal{A} \):

\[
\cdots \longrightarrow H_n(V) \xrightarrow{H_n(p)} H_n(W) \xrightarrow{\partial_n} H_{n-1}(U) \xrightarrow{H_{n-1}(m)} H_{n-1}(V) \longrightarrow \cdots
\]

natural in the short exact sequence.
Moreover, there are conditions to turn the long sequence to an exact sequence. In particular, \( \mathcal{A} \) is modular iff this sequence is always exact.

Bad news: \textbf{Diag(Mod(\mathcal{R}))} is not modular
Homotopy Axioms
Homotopy axioms for classical homology

Homotopy axiom, original form
If $f, g : X \to Y$ are homotopic, then $H_n(f) = H_n(g)$.

Homotopy axiom, v.1
If $f : X \to Y$ is a homotopy equivalence, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism.

Homotopy axiom, v.2
If $f : X \to Y$ is a weak homotopy equivalence, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism.
Dihomotopy axioms for natural homology, v.2

For free:

### Dihomotopy axiom, v.2.0

Let $f : X \to Y$ be a dimap. If for every $n$,

$$
\overrightarrow{\pi}_n(f) : \overrightarrow{\pi}_n(X) \to \overrightarrow{\pi}_n(Y)
$$

is a isomorphism of diagrams, then for every $n$,

$$
\overrightarrow{H}_n(f) : \overrightarrow{H}_n(X) \to \overrightarrow{H}_n(Y)
$$

is a isomorphism of diagrams.

Proof:
Apply the homotopy axiom, form v.2 on trace spaces.
Dihomotopy axioms for natural homology, form 2

Better, (almost) as free as v.2.0:

### Dihomotopy axiom, v.2.1

Let $f : X \to Y$ be a dimap. If for every $n$,

$$\overrightarrow{\pi}_n(f) : \overrightarrow{\pi}_n(X) \to \overrightarrow{\pi}_n(Y)$$

is a open map, then for every $n$,

$$\overrightarrow{H}_n(f) : \overrightarrow{H}_n(X) \to \overrightarrow{H}_n(Y)$$

and

$$\overrightarrow{H'}_n(f) : \overrightarrow{H'}_n(X) \to \overrightarrow{H'}_n(Y)$$

are open maps.

Proof:

Apply the homotopy axiom, form 2 on trace spaces + reasoning on open maps.
Which dihomotopy equivalences for v.1?

A future deformation retract of $X$ on a sub-dspace $A$ is a continuous map

$$H : X \rightarrow \mathcal{I}(X) \subseteq \mathcal{P}(X)$$

such that:

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0, 1]$, the map $H_t : x \mapsto H(x)(t)$ is a dmap;
- for every $\delta$ of $A$ from $z$ to $H_1(x)$ there is a dipath $\gamma$ of $X$ from $y$ to $x$ with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to $\delta$.

Inessential equivalence:

Two dspaces are inessentially equivalent iff there is a zigzag of future and past deformation retracts between them.
Why not $\overrightarrow{P}(X)$?

In classical algebraic topology:
if $f : X \longrightarrow Y$ is a homotopy equivalence, $P(x, y)$ and $P(f(x), f(y))$ are homotopically equivalent because paths induces homotopy equivalence by concatenation:

$$\gamma \star _{\_} : P(X)(z, x) \longrightarrow P(X)(z, y) \quad \delta \mapsto \gamma \star \delta$$

is a homotopy equivalence.

In directed algebraic topology: dipaths do not have this property.
Inessential dipaths

Idea from [Fajstrup, Goubault, Haucourt, Raussen] for category of components.

The set $\mathcal{I}(X)$ of inessential dipaths of $X$ is the largest set of dipaths such that:

- for every $\gamma \in \mathcal{I}(X)$ from $x$ to $y$, for every $z \in X$ such that $\overrightarrow{P}(X)(z, x) \neq \emptyset$, the map $\gamma \star \_ : \overrightarrow{P}(X)(z, x) \rightarrow \overrightarrow{P}(X)(z, y)$ $\delta \mapsto \gamma \star \delta$ is a homotopy equivalence;
- symmetrically for $\_ \star \gamma$;
- $\mathcal{I}(X)$ has the right and left Ore condition modulo dihomotopy:

\[
\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow \hspace{2cm} \Downarrow \hspace{2cm} \\text{mod. dihomot.} & & \\downarrow \hspace{2cm} \\text{mod. dihomot.} \\
\hspace{2cm} f' \in \mathcal{I}(X) & & \hspace{2cm} \downarrow \\
\hspace{2cm} z & \xrightarrow{g} & y \\
X & \xrightarrow{g'} & W \\
\end{array}
\]
Why inessential equivalence?

- classify as we expect many dspaces (for example, distinguish the matchbox and the point),

- in-between Grandis’ reversible and dihomotopy equivalences,

- its action on the fundamental category corresponds to the category of components,

- because of the preservation of the homotopy type of the space of dipaths, it has a deep relation with $(\infty, 1)$-categories (directed homotopy hypothesis).
Dihomotopy axiom, v.1

If two Euclidian cubical complexes $X$ and $Y$ are inesentially equivalent then $\overrightarrow{H}_n(X)$, $\overrightarrow{H}_n'(X)$, $\overrightarrow{H}_n(Y)$ and $\overrightarrow{H}'_n(Y)$ are bisimilar.

Conclusion : natural and bimodule homologies are invariant of inessential equivalence, at least on Euclidian cubical complexes, where we can do computations.