Partial-Observation Stochastic Games: How to Win when Belief Fails

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&

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IST Austria

GT Jeux
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Outline

• Game model: example
• Challenges & Results: examples
• Solution insights: examples
Examples

- Poker
  - partial-observation
  - stochastic
Examples

- Poker
  - partial-observation
  - stochastic

- Bonneteau
2 black card, 1 red card

Initially, all are face down

Goal: find the red card
Bonneteau

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Rules:

1. Player 1 points a card

2. Player 2 flips one remaining black card

3. Player 1 may change his mind, wins if pointed card is red
Bonneteau

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Bonneteau: Game Model

\[ \begin{array}{c}
R \quad B \quad B \\
B \quad R \quad B \\
B \quad B \quad R \\
\end{array} \]
Bonneteau: Game Model
Game Model

[Diagram showing a game model with nodes labeled R B B and B R B, and arrows indicating transitions between states.]
Game Model
Game Model
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Game Model
Observations (for player 1)
Observations (for player 1)
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Observations (for player 1)
This strategy is observation-based, e.g. after 1, 2, 3 it plays 3.
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e.g. after , , it plays 3
This strategy is winning with probability $\frac{2}{3}$.
This game is:
- turn-based
- (almost) non-stochastic
- player 2 has perfect observation
General case: concurrent & stochastic

\[ \delta : Q \times A_1 \times A_2 \rightarrow D(Q) \]

Player 1’s move

Player 2’s move

Players choose their moves \textit{simultaneously and independently}
General case: **concurrent & stochastic**

\[ \delta : Q \times A_1 \times A_2 \to \mathcal{D}(Q) \]

\[ \mathcal{D}(Q) = \{ f : Q \to [0, 1] \mid \sum_q f(q) = 1 \} \]

Probability distribution on successor state

Diagram:

- Player 1’s move
- Player 2’s move

\[ \delta(q, a, b)(r) = \frac{1}{3} \]

\[ \delta(q, a, b)(s) = \frac{2}{3} \]
Special cases:

- **player-1 state**

  ![Diagram]

- **player-2 state**

  ![Diagram]

Note: $q \ast r$ means $q \rightarrow r$
Observations: partitions induced by coloring

General case: 2-sided partial observation

Two partitions $\text{Obs}_1 \subseteq 2^Q$ and $\text{Obs}_2 \subseteq 2^Q$
Partial-observation

Observations: partitions induced by coloring

General case: 2-sided partial observation

Two partitions $\text{Obs}_1 \subseteq 2^Q$ and $\text{Obs}_2 \subseteq 2^Q$
Partial-observation

Observations: partitions induced by coloring

Special case: **1-sided** partial observation

\[ \text{Obs}_1 = \{ \{q\} \mid q \in Q \} \quad \text{or} \quad \text{Obs}_2 = \{ \{q\} \mid q \in Q \} \]

View of **perfect-observation** player
A strategy for Player \( i \) is a function \( \sigma_i : \text{Obs}_i^+ \rightarrow \mathcal{D}(A_i) \) that maps histories (sequences of observations) to probability distribution over actions.
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Reachability objective: \( T \subseteq Q \)

Winning probability of \( \sigma_1 : \inf_{\sigma_2} Pr_{q_0,\sigma_1,\sigma_2} (\exists i \geq 0 : q_i \in T) \)
Qualitative analysis

The following problem is undecidable:
(already for probabilistic automata [Paz71])

Decide if there exists a strategy for player 1 that is winning with probability at least $\frac{1}{2}$.

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(already for probabilistic automata [Paz71])

Decide if there exists a strategy for player 1 that is winning with probability at least $\frac{1}{2}$.

Qualitative analysis:

- **Almost-sure**: ... winning with probability 1
- **Positive**: ... winning with probability $> 0$

\[
\exists \sigma_1 \cdot \forall \sigma_2 : Pr_{q_0}^{\sigma_1,\sigma_2}(\exists i \geq 0 : q_i \in T) \begin{cases} = 1 \\ > 0 \end{cases}
\]
Applications in verification

- Control with inaccurate digital sensors
- multi-process control with private variables
- multi-agent protocols
- planning with uncertainty/unknown

```c
void main () { 
    int got_lock = 0; 
    do { 
        if (*) { 
            lock (); 
            got_lock++; 
        }
        if (got_lock != 0) { 
            unlock (); 
        }
        got_lock--;
    } while (*);
}

void lock () { 
    assert(L == 0);
    L = 1;
    }

void unlock () { 
    assert(L == 1);
    L = 0;
    }
```
Outline

• Game Model: example

• Challenges & Results: examples

• Solution insights: examples
Randomization is necessary

Player 1 partial, player 2 perfect

$\text{Obs}_1 = \{\square, \blacksquare\}$

$\sigma_i : \text{Obs}_i^+ \rightarrow \mathcal{D}(A_i)$
Player 1 partial, player 2 perfect

$$\text{Obs}_1 = \{ \square, \blacksquare \}$$

Randomization is necessary

$$\sigma_i : \text{Obs}_i^+ \to \mathcal{D}(A_i)$$

No pure strategy of Player 1 is winning with probability 1 (example from [CDHR06]).
Randomization is necessary

Player 1 partial, player 2 perfect

\[ \text{Obs}_1 = \{ \bullet, \square \} \]

\[ \sigma_i : \text{Obs}_i^+ \rightarrow \mathcal{D}(A_i) \]

No pure strategy of Player 1 is winning with probability 1 (example from [CDHR06]).
Memory and Randomization

Player 1 partial, player 2 perfect

Player 1 wins with probability 1, and needs randomization

\( \sigma_i : \text{Obs}_i^+ \rightarrow D(A_i) \)

Belief-based randomized strategies are sufficient
Example 2

Player 1 partial, player 2 perfect

\[ \sigma_i : \text{Obs}_i^+ \rightarrow \mathcal{D}(A_i) \]
Example 2

Player 1 partial, player 2 perfect

To win with probability 1, player 1 needs to observe his own actions. (example from [CDH10]).

Randomized action-visible strategies: $\sigma_i : (\text{Obs}_i A_i)^* \text{Obs}_i \rightarrow D(A_i)$
Classes of strategies

Classification according to the power of strategies

rand. action-visible

rand. action-invisible

pure
Classes of strategies

Classification according to the power of strategies

Poly-time reduction from decision problem of rand. act.-vis. to rand. act.-inv.

The model of rand. act.-inv. is more general
Classes of strategies

Classification according to the power of strategies

- rand. action-visible
  - rand. action-invisible
    - pure

Computational complexity (algorithms)

Strategy complexity (memory)
## Known results

Reachability - Memory requirement (for player 1)

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## Known results

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### Positive

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About beliefs

Three prevalent beliefs:

• Belief is sufficient.

• Randomized action invisible or visible almost same.

• The general case memory is similar (or in some cases exponential blow up) as compared to the one-sided case.
Pure Strategies

Belief
  • Belief is sufficient.

Proofs
  • Doubts.
Pure Strategies

Belief
- Belief is sufficient.

Proofs
- Doubts.

Lesson:
Doubt your belief and believe in your doubts!! See the unexpected.
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning.
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning winning

There are two belief-based-only pure strategies:

1. When belief is \( \{q_1, q_2\} \), play \( a \)

2. When belief is \( \{q_1, q_2\} \), play \( b \)
When belief fails (1/2)

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There are two belief-based-only pure strategies:

1. When belief is $\{q_1\}$, play $a$.
2. When belief is $\{q_2\}$, play $b$.

Neither is winning!
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning

When belief is \( \{q_1, q_2\} \), alternate \( a \) and \( b \)
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning.

When belief is \( \{q_1, q_2\} \), alternate \( a \) and \( b \)

This strategy is almost-sure winning!
When belief fails (2/2)

Using the trick of “repeated actions” we construct an example where belief-only randomized action-invisible strategies are not sufficient (for almost-sure winning)
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player 1 partial
player 2 perfect
When belief fails (2/2)

Using the trick of “repeated actions” we construct an example where belief-only randomized action-invisible strategies are not sufficient (for almost-sure winning)

Almost-sure winning requires to play pure strategy, with more-than-belief memory!
# New results

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## Positive

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Pure Strategies: Player 1 Perfect, Player 2 Partial

- **Pl1 Perfect, Pl 2 Partial:** Stochastic, Randomized. Memoryless
- **Pl1 Perfect, Pl 2 Partial:** Non-stochastic, Pure. Memoryless
- **Pl1 Perfect, Pl 2 Partial:** Stochastic, Pure.
- **Pl1 Partial, Pl 2 Perfect:** Stochastic, Pure. Exponential
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Pure Strategies: Player 1 Perfect, Player 2 Partial

- **Pl1 Perfect, Pl 2 Partial**: Stochastic, Randomized. Memoryless
  - Restrict to pure

- **Pl1 Perfect, Pl 2 Partial**: Stochastic, Pure. Non-elementary complete
  - Add probability

- **Pl1 Partial, Pl 2 Perfect**: Stochastic, Pure. Exponential
  - Pi 1 more informed, Pi 2 less informed

- **Pl1 Perfect, Pl 2 Perfect**: Stochastic, Pure. Memoryless
  - Pi 2 less informed
## New results

Reachability - Memory requirement (for player 1)

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Player 1 wins from more states, but needs more memory!
Player 1 perfect, player 2 partial

Memory of non-elementary size for pure strategies

- lower bound: simulation of counter systems with increment and division by 2

- upper bound:
  - positive: non-elementary counters simulate randomized strategies
  - almost-sure: reduction to iterated positive

Counter systems with \{+1, ÷2\} require non-elementary counter value for reachability

\[
2 \cdot \left( \frac{2}{2} \right)^{\text{height } n}
\]
Player 1 perfect, player 2 partial

More information:

- Win from more places.
- Winning strategy is very hard to implement.

Information is useful, but ignorance is bliss 😊 !
# New results

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Pure $\equiv$ randomized-invisible

Equivalence of the decision problems for almost-sure reach with pure strategies and rand. act.-inv. strategies

- Reduction of rand. act.-inv. to pure choice of a subset of actions (support of prob. dist.)
- Reduction of pure to rand. act.-inv. repeated-action trick (holds for almost-sure only)

It follows that the memory requirements for pure hold for rand. act.-inv. as well!
## New results

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Beliefs

Three prevalent beliefs:

• Belief is sufficient.

• Randomized action invisible or visible almost same.

• The general case memory is similar (or in some cases exponential blow up) as compared to the one-sided case.

Belief fails!
Summary of our results

Pure strategies (for almost-sure and positive):

- player 1 partial: exponential memory, belief not sufficient
- player 1 perfect: non-elementary memory (complete)
- 2-sided: finite, at least non-elementary memory
  (as compared to previously claimed exponential upper bound)

Randomized action-invisible strategies (for almost-sure):

- player 1 partial: exponential memory, belief not sufficient
- 2-sided: finite, at least non-elementary memory
More results & open questions

Computational complexity for 1-sided:

- Player 1 partial: reduction to Büchi game, EXPTIME-complete
- Player 2 partial: non-elementary complexity
  (note: almost-sure Büchi is poly-time equivalent to almost-sure reachability, positive Büchi is undecidable [BBG08])

Open questions:

- Whether non-elementary size memory is sufficient in 2-sided
- Exact computational complexity
Details can be found in:


Extended abstract @ LICS’12:

Details can be found in:


Other references:

Outline

- Game Model: example
- Challenges & Results: examples
- Solution insights: examples
Some proof ideas
## New results

### Reachability - Memory requirement (for player 1)

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When belief fails

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning

player 1 partial
player 2 perfect
When belief fails

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning

When belief is \( \{q_1, q_2\} \), then ensure that the target is reached from both \( q_1 \) and \( q_2 \).
When belief fails

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At this point, obligation of $q_2$ is satisfied

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Here, obligation of $q_1$ is satisfied

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Empty obligation set $\iff$ All obligations fulfilled
When belief fails

Positive reachability: ensure empty obligation once

Reachability condition

Almost-sure reachability: ensure empty obligation infinitely often (and recharge when empty)

Büchi condition

Empty obligation set ⇔ All obligations fulfilled
When belief fails

Positive reachability: ensure empty obligation once

Reachability condition

Almost-sure reachability: ensure empty obligation infinitely often (and recharge when empty)

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Empty obligation set $\iff$ All obligations fulfilled
# New results

Reachability - Memory requirement (for player 1)

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Positive

| rand. act.-vis. | memoryless | memoryless | memoryless |
| rand. act.-inv. | memoryless | memoryless | memoryless |
| pure | exponential (belief not sufficient) | non-elementary complete | finite (at least non-elementary) |
Player 1 perfect, player 2 partial
1. Play 3 times $a$ to generate $2^4$ indistinguishable paths 
(with observation $LLLL$)
2. Play $b$, then in $q_4$ play $a$ over half of the paths $LLLL$ and the rest $LLLL$
3. In $q_i$, play analogously, and ensure $2^{i-1}$ paths $LLLL$
4. Reach $q_0$ with positive probability
\[
\begin{align*}
&LLLL \\
&LLLL \\
&LLLL \\
&LLLL \\
\end{align*}
\]
1. Play 3 times \( a \) to generate \( 2^4 \) indistinguishable paths
2. Play \( b \), then in \( q_4 \) play \( a \) over half of the paths, \( b \) over the rest
3. In \( q_i \), play analogously, and ensure \( 2^{i-1} \) paths \( LLLLLq_4 \)
4. Reach \( q_0 \) with positive probability

\[ \text{...} \]
Player 1 perfect, player 2 partial

1. Play 3 times $a$ to generate $2^4$ indistinguishable paths
2. Play $b$, then in $q_4$ play $a$ over half of the paths, $b$ over the rest
3. In $q_i$, play analogously, and ensure $2^{i-1}$ paths in $q_{i-1}$
4. Reach $q_0$ with positive probability
Player 1 perfect, player 2 partial

\[
\begin{align*}
L & L L L L \rightarrow q_4 \\
L & L L R \rightarrow q_4 \\
L & L R L L \rightarrow q_4 \\
L & L R R \rightarrow q_4 \\
L & R R L \rightarrow q_4 \\
L & R R R \rightarrow q_4 \\
R & L L L L \rightarrow q_4 \\
R & L L R \rightarrow q_4 \\
R & L R L L \rightarrow q_4 \\
R & L R R \rightarrow q_4 \\
R & R L L \rightarrow q_4 \\
R & R L R \rightarrow q_4 \\
R & R R L \rightarrow q_4 \\
R & R R R \rightarrow q_4 \\
R & R R R R \rightarrow q_4
\end{align*}
\]
Player 1 perfect, player 2 partial

$$L^5 q_4$$
$$L^4 R q_4$$
$$L^3 R L q_4$$
$$L^3 R R q_4$$

$$\{ a, a \}$$

$$R^5 L q_4$$
$$R^4 L R q_4$$
$$R^3 L L L q_4$$
$$R^3 L L R q_4$$

$$\{ b, a \}$$

$q_4^2 q_3 q_2 q_1 q_0$
Player 1 perfect, player 2 partial
Player 1 perfect, player 2 partial

$\{LLL L q_4, \text{ } L L L R q_4, \text{ } L L R L q_4, \text{ } L L R R q_4, \text{ } L R L L q_4, \text{ } L R L R q_4, \text{ } L R R L q_4, \text{ } L R R R q_4, \text{ } R L L L q_4, \text{ } R L L R q_4, \text{ } R L R L q_4, \text{ } R L R R q_4, \text{ } R R L L q_4, \text{ } R R L R q_4, \text{ } R R R L q_4, \text{ } R R R R q_4\}$

$\{a, a \}$

$\{b, a\}$

$\{a, a \}$

$\{b, a\}$

$\{a, a \}$

$\{b, a\}$

$\{a, a \}$

$\{b, a\}$

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$\{b, a\}$

$\{a, a \}$

$\{b, a\}$

1 out of 16 paths reaches $q_0$

Winning probability = $1/16$!
Player 1 perfect, player 2 partial

1 out of 16 paths reaches $q_0$
Winning probability = $1/16$!
1. Play 3 times $a$ to generate $2^4$ indistinguishable paths
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4. Reach $q_0$ with positive probability …
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3. In $q_i$, play analogously, and ensure $2^{i-1}$ paths in $q_{i-1}$
4. Reach $q_0$ with positive probability …
   … using exponential memory
Flavor of a counter system with:

- increment,
- division by 2 (size of alphabet)
Player 1 perfect, player 2 partial

Show that:

1. games can simulate increment and division by 2

2. Such counter systems require non-elementary counter value for reachability

\[
2 \cdot 2^n \}
height n
Counter system with \{+1, \div 2\}
Counter system with \( \{+1, \div 2\} \)

Reachability?
Counter system with \( \{+1, \div 2\} \)
Counter system with \{+1, ÷2\}

\[
\begin{align*}
(\cdot, \cdot, +1) & \quad (\cdot, +1, \div 2) & \quad (\cdot, +1, \div 2, \div 2) & \quad (+1, \div 2, \div 2, \div 2) \\
q_4 \rightarrow q_3 \rightarrow q_2 \rightarrow q_1 \rightarrow q_0 \\
\text{[2, 2, 2, 2]} & \quad [1, 1, 1, 1]
\end{align*}
\]
Counter system with \{+1, ÷2\}
Counter system with \{+1, ÷2\}

\((\cdot, \cdot, +1)\)  \((\cdot, +1, \div 2)\)  \((+1, \div 2, \div 2)\)  \((+1, \div 2, \div 2, \div 2)\)
Counter system with \{+1, \div 2\}
Counter system with \{+1, ÷2\}

\[
\begin{align*}
2^{11} \cdot 2^{2^{11}} &\quad (\cdot, \cdot, 1, +1) \\
2^{3} \cdot 2^{2^{3}} &\quad (\cdot, \cdot, +1, ÷2) \\
2 \cdot 2^{2} &\quad (\cdot, +1, ÷2, ÷2) \\
2 &\quad (+1, ÷2, ÷2, ÷2)
\end{align*}
\]

\[
\begin{align*}
q_4 &\quad [0, 0, 0, 2^{2059}] \\
q_3 &\quad [0, 0, 2^{11}, 2^{11}] \\
q_2 &\quad [0, 2^{3}, 2^{3}, 2^{3}] \\
q_1 &\quad [2, 2, 2, 2] \\
.q_0 &\quad [1, 1, 1, 1]
\end{align*}
\]
Counter system with \( \{+1, \div 2\} \)

\[
\begin{align*}
2^{11} \cdot 2^{11} & \quad (\cdot, \cdot, +1) \\
2^3 \cdot 2^3 & \quad (\cdot, +1, \div 2) \\
2 \cdot 2^2 & \quad (+1, \div 2, \div 2) \\
2 & \quad (\div 2, \div 2, \div 2)
\end{align*}
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\[
\begin{array}{c}
q_4 \quad [0, 0, 0, 2^{2059}] \\
q_3 \quad [0, 0, 2^{11}, 2^{11}] \\
q_2 \quad [0, 2^3, 2^3, 2^3] \\
q_1 \quad [2, 2, 2, 2] \\
q_0 \quad [1, 1, 1, 1]
\end{array}
\]

non-elementary growth!
Show that:

1. games can simulate **increment** and **division by 2**

2. Such counter systems require non-elementary counter value for reachability

\[ 2 \cdot 2 \left\{ \frac{2}{2} \right\} \text{height } n \]
Game gadgets for \( \{\text{idle}, +1, \div 2\} \)
Game gadgets for \{idle, +1, ÷2\}

counter 1          counter 2          counter 3

idle  · · · · ·   idle  · · · · ·   inc  \(\cdot, \cdot, +1\)

\#1   \#1         \#1

idle  · · · · ·   inc  · · · · ·   div_2 \(\cdot, +1, ÷2\)

\#2   \#2         \#2

inc  · · · · ·   div_2  · · · · ·   div_2 \(+1, ÷2, ÷2\)

\#3   \#3         \#3

div_2  · · · · ·   div_2  · · · · ·   div_2 \(÷2, ÷2, ÷2\)
Game that simulates 3 counters...
Pure Strategies: Player 1 Perfect, Player 2 Partial

PI1 Perfect, PI 2 Partial:
Stochastic, Pure.
Non-elementary lower bound
Player 1 perfect, player 2 partial

Memory of **non-elementary** size for pure strategies

- lower bound: simulation of counter systems with increment and division by 2

- upper bound:
  - **positive**: non-elementary counters simulate randomized strategies
  - **almost-sure**: reduction to iterated positive

Counter systems with \{+1, ÷2\} require non-elementary counter value for reachability

\[
2^{2 \cdot 2^{\text{height } n}}
\]
## New results

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Thank you!

Questions?