How Much Memory is Needed to Win in Partial-Observation Games

Laurent Doyen
LSV, ENS Cachan & CNRS

&

Krishnendu Chatterjee
IST Austria

GAMES’11
How Much Memory is Needed to Win in Partial-Observation Games

Laurent Doyen
LSV, ENS Cachan & CNRS

&

Krishnendu Chatterjee
IST Austria

stochastic

GAMES’11
Examples

- Poker
  - partial-observation
  - stochastic
Examples

• Poker
  - partial-observation
  - stochastic

• Bonneteau
2 black card, 1 red card

Initially, all are face down

Goal: find the red card
Bonneteau

2 black card, 1 red card

Initially, all are face down
Goal: find the red card

Rules:
1. Player 1 points a card
2. Player 2 flips one remaining black card
3. Player 1 may change his mind, wins if pointed card is red
Bonneteau

2 black card, 1 red card

Initially, all are face down

Goal: find the red card

Rules:

1. Player 1 points a card
2. Player 2 flips one remaining black card
3. Player 1 may change his mind, wins if pointed card is red
2 black card, 1 red card

Initially, all are face down

Goal: find the red card

Rules:
1. Player 1 points a card
2. Player 2 flips one remaining black card
3. Player 1 may change his mind, wins if pointed card is red
Bonneteau: Game Model
Bonneteau: Game Model
Game Model
Game Model
Game Model
Observations (for player 1)
Observations (for player 1)
Observations (for player 1)
Observations (for player 1)
This strategy is observation-based, e.g. after $\Box$, $\Box$, $\Box$ it plays 3
This strategy is observation-based, e.g. after \[\Box, \Box, \Box\] it plays \[3\]
Optimal observation-based strategy

This strategy is winning with probability $\frac{2}{3}$
This game is:
- turn-based
- (almost) non-stochastic
- player 2 has perfect observation
General case: concurrent & stochastic

\[ \delta : Q \times A_1 \times A_2 \rightarrow \mathcal{D}(Q) \]

Player 1’s move

Player 2’s move

Players choose their moves simultaneously and independently
General case: concurrent & stochastic

\[ \delta : Q \times A_1 \times A_2 \rightarrow \mathcal{D}(Q) \]

- Player 1’s move
- Player 2’s move
- Probability distribution on successor state

2\(\frac{1}{2}\)-player games
Special cases:

- Turn-based games

  - player-1 state

  - player-2 state
Partial-observation

Observations: partitions induced by coloring

General case: 2-sided partial observation

Two partitions $\mathcal{O}_1 \subseteq 2^Q$ and $\mathcal{O}_2 \subseteq 2^Q$
Partial-observation

Observations: partitions induced by coloring

General case: 2-sided partial observation

Two partitions $\mathcal{O}_1 \subseteq 2^Q$ and $\mathcal{O}_2 \subseteq 2^Q$

Player 1’s view

Player 2’s view
Partial-observation

Observations: partitions induced by coloring

Special case: 1-sided partial observation

\[ \mathcal{O}_1 = \{\{q\} \mid q \in Q\} \quad \text{or} \quad \mathcal{O}_2 = \{\{q\} \mid q \in Q\} \]
A strategy for Player $i$ is a function $\sigma_i : \mathcal{O}_i^+ \rightarrow \mathcal{D}(A_i)$ that maps histories (sequences of observations) to probability distribution over actions.
A strategy for Player $i$ is a function $\sigma_i : \mathcal{O}_i^+ \rightarrow \mathcal{D}(A_i)$ that maps histories (sequences of observations) to probability distribution over actions.
A strategy for Player $i$ is a function $\sigma_i : \mathcal{O}_i^+ \to \mathcal{D}(A_i)$ that maps histories (sequences of observations) to probability distribution over actions.

Reachability objective: $\mathcal{T} \subseteq \mathcal{Q}$

Winning probability: $\inf_{\sigma_2} Pr_{q_0}^{\sigma_1,\sigma_2}(\exists i \geq 0 : q_i \in \mathcal{T})$
Qualitative analysis

The following problem is undecidable:
(already for probabilistic automata [Paz71])

Decide if there exists a strategy for player 1 that is winning with probability at least 1/2
Qualitative analysis

The following problem is undecidable:
(already for probabilistic automata [Paz71])

Decide if there exists a strategy for player 1 that is winning with probability at least 1/2

Qualitative analysis:

• **Almost-sure**: ... winning with probability 1

• **Positive**: ... winning with probability > 0

\[
\exists \sigma_1 \cdot \forall \sigma_2 : \Pr_{q_0, \sigma_1, \sigma_2} (\exists i \geq 0 : q_i \in T) \begin{cases} = 1 \\ > 0 \end{cases}
\]
Example 1

Player 1 partial, player 2 perfect

\[ \sigma_i : \mathcal{O}_i^+ \rightarrow \mathcal{D}(A_i) \]
Example 1

Player 1 partial, player 2 perfect

No pure strategy of Player 1 is winning with probability 1

\[ \sigma_i : \mathcal{O}_i^+ \rightarrow \mathcal{D}(A_i) \]
Example 1

Player 1 partial, player 2 perfect

\[ \sigma_i : \mathcal{O}_i^+ \to \mathcal{D}(A_i) \]

No pure strategy of Player 1 is winning with probability 1
Example 1

Player 1 partial, player 2 perfect

\[ \sigma_i : \mathcal{O}_i^+ \rightarrow \mathcal{D}(A_i) \]

Player 1 wins with probability 1, and needs randomization

Belief-based-only randomized strategies are sufficient
Example 2

Player 1 partial, player 2 perfect

\[ \sigma_i : \mathcal{O}_i^+ \rightarrow D(A_i) \]
Example 2

Player 1 partial, player 2 perfect

To win with probability 1, player 1 needs to observe his own actions.

Randomized action-visible strategies: $\sigma_i : (O_iA_i)^*O_i \rightarrow D(A_i)$
Classes of strategies

rand. action-visible

rand. action-invisible

pure

Classification according to the power of strategies
Classes of strategies

Poly-time reduction from decision problem of rand. act.-vis. to rand. act.-inv.

The model of rand. act.-inv. is more general
Classes of strategies

Classification according to the power of strategies

- rand. action-visible
- rand. action-invisible
- pure

Computational complexity (algorithms)

Strategy complexity (memory)
### Known results

**Reachability - Memory requirement (for player 1)**

<table>
<thead>
<tr>
<th>Almost-sure</th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td></td>
<td></td>
<td>Dark grey</td>
</tr>
<tr>
<td>pure</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Known results

### Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th></th>
<th>Almost-sure</th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR’06]</td>
<td>memoryless [BGG’09]</td>
<td>exponential (belief) [BGG’09]</td>
<td></td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (belief) [CDHR’06(remark), GS’09]</td>
<td></td>
<td>exponential (belief) [GS’09]</td>
<td></td>
</tr>
<tr>
<td>pure</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>


[CDHR06] Chatterjee, Doyen, Henzinger, Raskin. *Algorithms for $\omega$-Regular games with Incomplete Information*. CSL’06.

## Known results

### Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th>Almost-sure</th>
<th>player 1 partial</th>
<th>player 1 perfect</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR’06]</td>
<td>memoryless [BGG’09]</td>
<td>exponential (belief) [BGG’09]</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (belief) [CDHR’06(remark), GS’09]</td>
<td>memoryless</td>
<td>exponential (belief) [GS’09]</td>
</tr>
<tr>
<td>pure</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Positive</th>
<th>player 1 partial</th>
<th>player 1 perfect</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>pure</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
Belief-based-only pure strategies are **not sufficient**, both for positive and for almost-sure winning.

(player 1 partial)

(player 2 perfect)
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning winning

There are two belief-based-only pure strategies:

1. When belief is \( \{q_1, q_2\} \), play \( a \)

2. When belief is \( \{q_1, q_2\} \), play \( b \)
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning.

There are two belief-based-only pure strategies:

1. When belief is \( \{q_1, q_2\} \), play \( a \) not winning

2. When belief is \( \{q_1, q_2\} \), play \( b \)
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning.

There are two belief-based-only pure strategies:

1. When belief is \(\{q_1, q_2\}\), play \(a\)

2. When belief is \(\{q_1, q_2\}\), play \(b\) not winning
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning.

There are two belief-based-only pure strategies:

1. When belief is \( q_0 \), play \( a \).
2. When belief is \( \{q_1, q_2\} \), play \( b \).

Neither is winning!
When belief fails (1/2)

Belief-based-only pure strategies are not sufficient, both for positive and for almost-sure winning.

When belief is \( \{q_1, q_2\} \), alternate \( a \) and \( b \).

player 1 partial
player 2 perfect
When belief fails (1/2)

Belief-based-only pure strategies are **not sufficient**, both for positive and for almost-sure winning

When belief is \(\{q_1, q_2\}\), alternate \(a\) and \(b\)

This strategy is almost-sure winning!
When belief fails (2/2)

Using the trick of “repeated actions” we construct an example where belief-only randomized action-invisible strategies are not sufficient (for almost-sure winning)

player 1 partial
player 2 perfect
When belief fails (2/2)

Using the trick of “repeated actions” we construct an example where belief-only randomized action-invisible strategies are not sufficient (for almost-sure winning)
Using the trick of “repeated actions” we construct an example where belief-only randomized action-invisible strategies are not sufficient (for almost-sure winning). When belief fails (2/2)

Almost-sure winning requires to play pure strategy, with more-than-belief memory!
# New results

## Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th>Almost-sure</th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR’06]</td>
<td>memoryless [BGG’09]</td>
<td>exponential (belief) [BGG’09]</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (belief) <a href="remark">CDHR’06</a>, GS’09</td>
<td>memoryless</td>
<td>exponential (belief) [GS’09]</td>
</tr>
<tr>
<td>pure</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

## Positive

<table>
<thead>
<tr>
<th></th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>pure</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
## New results

### Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th></th>
<th>player 1 partial</th>
<th>player 1 perfect</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Almost-sure</strong></td>
<td>player 1 partial</td>
<td>player 1 perfect</td>
<td>2-sided both partial</td>
</tr>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief)</td>
<td>memoryless</td>
<td>exponential (belief)</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (more than belief)</td>
<td>memoryless</td>
<td>exponential (belief)</td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>memoryless</td>
<td>exponential (belief)</td>
</tr>
</tbody>
</table>

|                | player 1 partial | player 1 perfect | 2-sided both partial |
| Positive       | player 1 partial | player 1 perfect | 2-sided both partial |
| rand. act.-vis.| memoryless       | memoryless       | memoryless           |
| rand. act.-inv.| memoryless       | memoryless       | memoryless           |
| pure           | exponential (more than belief) | memoryless       | memoryless           |
## New results

### Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th>Almost-sure</th>
<th>player 1 partial</th>
<th>player 1 perfect</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>player 2 perfect</td>
<td>player 2 partial</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>memoryless</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>memoryless [BGG'09]</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>exponential (belief) [CDHR'06]</td>
<td></td>
</tr>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR'06]</td>
<td>memoryless [BGG'09]</td>
<td>exponential (belief) [BGG'09]</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (more than belief)</td>
<td>memoryless</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>exponential (belief) [GS'09]</td>
<td></td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Positive</td>
<td>player 1 partial</td>
<td>player 1 perfect</td>
<td>2-sided both partial</td>
</tr>
<tr>
<td></td>
<td>player 2 perfect</td>
<td>player 2 partial</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>memoryless</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>memoryless</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>rand. act.-vis.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>memoryless</td>
<td></td>
<td>memoryless</td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

### Positive

- player 1 partial
- player 2 perfect
- player 1 perfect
- player 2 partial
- 2-sided both partial
- memoryless
- memoryless
- memoryless
- memoryless
- exponential (more than belief)
- ?
- ?
- ?
## New results

Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th>Almost-sure</th>
<th>player 1 partial</th>
<th>player 1 perfect</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR'06]</td>
<td>memoryless [BGG'09]</td>
<td>exponential (belief) [BGG'09]</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (more than belief)</td>
<td>memoryless complete</td>
<td>exponential (belief) [GS'09]</td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>non-elementary complete</td>
<td>?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Positive</th>
<th>player 1 partial</th>
<th>player 1 perfect</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>non-elementary complete</td>
<td>?</td>
</tr>
</tbody>
</table>
**New results**

Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th></th>
<th>Almost-sure</th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR’06]</td>
<td>memoryless [BGG’09]</td>
<td>exponential (belief) [BGG’09]</td>
<td></td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (more than belief)</td>
<td>memoryless complete</td>
<td>exponential (belief) [GS’09]</td>
<td></td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>non-elementary complete</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Positive</td>
<td>player 1 partial player 2 perfect</td>
<td>memoryless memoryless</td>
<td>memoryless memoryless</td>
<td></td>
</tr>
<tr>
<td>rand. act.-vis.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
<td></td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
<td></td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>non-elementary complete</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

**Player 1 wins from more states, but needs more memory!**
Player 1 perfect, player 2 partial

Memory of **non-elementary** size for pure strategies

- lower bound: simulation of counter systems with increment and division by 2

- upper bound:
  - **positive**: non-elementary counters simulate randomized strategies
  - **almost-sure**: reduction to iterated positive

Counter systems with \(\{+1, \div 2\}\) require non-elementary counter value for reachability

\[
2^{2 \cdot 2^2} \text{ height } n
\]
## New results

Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th>Almost-sure</th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR'06]</td>
<td>memoryless [BGG'09]</td>
<td>exponential (belief) [BGG'09]</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (more than belief)</td>
<td>memoryless complete</td>
<td>exponential (belief) [GS'09]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pure</th>
<th>exponential (more than belief)</th>
<th>non-elementary complete</th>
<th>finite (at least non-elementary)</th>
</tr>
</thead>
</table>

## Positive

<table>
<thead>
<tr>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1 partial player 2 perfect</td>
</tr>
<tr>
<td>player 1 perfect player 2 partial</td>
</tr>
</tbody>
</table>

### Memory Requirement

- **memoryless**
- **exponential (belief)**
- **exponential (more than belief)**
- **finite (at least non-elementary)**
Equivalence of the decision problems for almost-sure reach with **pure** strategies and **rand. act.-inv.** strategies

- Reduction of rand. act.-inv. to pure choice of a subset of actions (support of prob. dist.)

- Reduction of pure to rand. act.-inv. repeated-action trick (holds for almost-sure only)

It follows that the memory requirements for pure hold for rand. act.-inv. as well!
# New results

## Reachability - Memory requirement (for player 1)

<table>
<thead>
<tr>
<th>Almost-sure</th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>exponential (belief) [CDHR’06]</td>
<td>memoryless [BGG’09]</td>
<td>exponential (belief) [BGG’09]</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>exponential (more than belief)</td>
<td>memoryless memoryless complete</td>
<td>finite (at least non-elementary)</td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>non-elementary complete</td>
<td>finite (at least non-elementary)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Positive</th>
<th>player 1 partial player 2 perfect</th>
<th>player 1 perfect player 2 partial</th>
<th>2-sided both partial</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand. act.-vis.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>rand. act.-inv.</td>
<td>memoryless</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>pure</td>
<td>exponential (more than belief)</td>
<td>non-elementary complete</td>
<td>finite (at least non-elementary)</td>
</tr>
</tbody>
</table>
Summary of our results

Pure strategies (for almost-sure and positive):

• player 1 partial: exponential memory, more than belief
• player 1 perfect: non-elementary memory (complete)
• 2-sided: finite, at least non-elementary memory

Randomized action-invisible strategies (for almost-sure):

• player 1 partial: exponential memory, more than belief
• 2-sided: finite, at least non-elementary memory
More results & open questions

Computational complexity for 1-sided:

- Player 1 partial: reduction to Büchi game, \textsc{EXPTIME}-complete
- Player 2 partial: non-elementary complexity

Open questions:

- Whether non-elementary size memory is sufficient in 2-sided
- Exact computational complexity
Details

Details can be found in:

THANK YOU
Details can be found in:


Other references: