Staying alive in the dark

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joint work with

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Synthesis problem

Specifcation

avoid failure, ensure progress, etc.

Correctness relation
Synthesis problem

System - Model

Specification

avoid failure, ensure progress, etc.

Correctness relation

Solved as a game – system vs. environment

solution = winning strategy

This talk: quantitative games (resource-constrained systems)
Energy games
(staying alive)
Energy games (CdAHS03,BFLM08)

Maximizer
Minimizer

positive weight = reward

play: (1,4) (4,1) (1,4) (4,1) ...

weights: -1 +2 -1 +2 ...

energy level: 1 0 2 1 3 2 4 3 ...
Energy games \((\text{CdAHS03,BFL+08})\)

Play:

- \((1,4)\) \((4,1)\) \((1,4)\) \((4,1)\) ...

Weights:

- \(-1\) \(+2\) \(-1\) \(+2\) ...

Energy level:

- \(1\) \(0\) \(2\) \(1\) \(3\) \(2\) \(4\) \(3\) ...

Initial credit
Energy games

Strategies:

Maximizer $\sigma : Q^* \cdot Q_0 \rightarrow Q$

Minimizer $\pi : Q^* \cdot Q_\Box \rightarrow Q$

**play:** outcome$(q, \sigma, \pi)$

Infinite sequence of edges consistent with strategies $\sigma$ and $\pi$

**outcome is winning if:**

$$c_0 + \sum_{i=0}^{n-1} w_i \geq 0 \text{ for all } n \geq 0$$
Energy games

Decision problem:

Decide if there exist an initial credit $c_0$ and a strategy of the maximizer to maintain the energy level always nonnegative.
Energy games

Decision problem:

Decide if there exist an initial credit $c_0$ and a strategy of the maximizer to maintain the energy level always nonnegative.

For energy games, memoryless strategies suffice.

$$\sigma : Q_{\square} \rightarrow Q$$

$$\pi : Q_{\circ} \rightarrow Q$$
Energy games

Decision problem:

Decide if there exist an initial credit $c_0$ and a strategy of the maximizer to maintain the energy level always nonnegative.

For energy games, memoryless strategies suffice.

A memoryless strategy $\sigma$ is winning if all cycles are nonnegative when $\sigma$ is fixed.
Energy games

Decision problem:
Decide if there exists an initial credit $c_0$ and a strategy of the maximizer to maintain the energy level always nonnegative.

For energy games, memoryless strategies suffice.

A memoryless strategy $\sigma$ is winning if all cycles are nonnegative when $\sigma$ is fixed.
Algorithm
Algorithm for energy games

Initial credit is useful to survive before a cycle is formed

Length(AcyclicPath) ≤ Q

Q: #states
E: #edges
W: maximal weight
Algorithm for energy games

Initial credit is useful to survive before a cycle is formed

Q: #states
E: #edges
W: maximal weight

Minimum initial credit is at most $Q \cdot W$
The minimum initial credit $v(\cdot)$ is such that:

in Maximizer state $q$:

$$v(q) + w(q, q') \geq v(q')$$ for some $(q, q') \in E$$

in Minimizer state $q$:

$$v(q) + w(q, q') \geq v(q')$$ for all $(q, q') \in E$$

Compute successive under-approximations of the minimum initial credit.
Algorithm for energy games

Fixpoint algorithm:
- start with $v(q) = 0$
Algorithm for energy games

Fixpoint algorithm:

- start with \( v(q) = 0 \)

- iterate

  at Maximizer states:

  \[
  v(q) \leftarrow \min \{ v(q') - w(q, q') \mid (q, q') \in E \}
  \]

  at Minimizer states:

  \[
  v(q) \leftarrow \max \{ v(q') - w(q, q') \mid (q, q') \in E \}
  \]
Algorithm for energy games

Fixpoint algorithm:

- start with $v(q) = 0$

- iterate
  
  at Maximizer states:
  
  $$v(q) \leftarrow \min\{v(q') - w(q, q') \mid (q, q') \in E\}$$

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Algorithm for energy games

Fixpoint algorithm:
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  at Minimizer states:
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Termination argument: monotonic operators, and finite codomain $v(q) \in \{0, 1, \ldots, Q \cdot W\}$

Complexity: $O(E \cdot Q \cdot W)$
Mean-payoff games
Mean-payoff games (EM79)

play: (1,4) (4,1) (1,4) (4,1) ...

weights: -1 +2 -1 +2 ...

mean-payoff value: $\frac{1}{2}$
(limit of weight average)
Mean-payoff games (EM79)

Mean-payoff value:
either $\liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w_i$ or $\limsup_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w_i$

Decision problem:
Given a rational threshold $\nu$, decide if there exists a strategy of the maximizer to ensure mean-payoff value at least $\nu$.

Note: we can assume $\nu = 0$
e.g. by shifting all weights by $\nu$. 
Mean-payoff games

Decision problem:
Given a rational threshold $\nu$, decide if there exists a strategy of the maximizer to ensure mean-payoff value at least $\nu$.

Mean-payoff value:
either $\liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w_i$ or $\limsup_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w_i$

Assuming $\nu = 0$

A memoryless strategy $\sigma$ is winning if all cycles are nonnegative when $\sigma$ is fixed.
Mean-payoff games

Mean-payoff value:
either $\liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w_i$ or $\limsup_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w_i$

Decision problem:
Given a rational threshold $\nu$, decide if there exists a strategy of the maximizer to ensure mean-payoff value at least $\nu$.

log-space equivalent to energy games [BFL+08]

Assuming $\nu = 0$

A memoryless strategy $\sigma$ is winning if all cycles are nonnegative when $\sigma$ is fixed.
## Complexity

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Deterministic Pseudo-polynomial algorithms
Outline

- Perfect information
  - Mean-payoff games
  - Energy games
  - Algorithms

- Imperfect information
  - Energy with fixed initial credit
  - Energy with unknown initial credit
  - Mean-payoff
Imperfect information
(staying alive in the dark)
Imperfect information – Why?

- Private variables/internal state
- Noisy sensors

Strategies should not rely on hidden information
Imperfect information – How?

- Coloring of the state space

  observations = set of states with the same color
Imperfect information – How?

Maximizer chooses an action (a or b)

1. Maximizer chooses an action (a or b)
2. Minimizer chooses successor state (compatible with Maximizer’s action)
3. The color of the next state is visible to Maximizer
Imperfect information – How?

Actions \( \Sigma = \{a, b\} \)

Observations \( \text{Obs} = \left\{ \{q_1, q_2, q_3\}, \{q_4, q_5\} \right\} \)
Imperfect information – How?

Observation-based strategies

\[ \sigma : \text{Obs}^+ \rightarrow \Sigma \]

Goal: all outcomes have
- nonnegative energy level,
- or nonnegative mean-payoff value

Actions \( \Sigma = \{ a, b \} \)

Observations \( \text{Obs} = \{ \{ q_1, q_2, q_3 \}, \{ q_4, q_5 \} \} \)
## Complexity

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Two variants for Energy games:
- fixed initial credit
- unknown initial credit

Observation-based strategies
\[ \sigma : \text{Obs}^+ \rightarrow \Sigma \]

Goal: all outcomes have
- nonnegative energy level,
- or nonnegative mean-payoff value
Fixed initial credit

Can you win with initial credit = 3?

Actions \( \Sigma = \{a, b\} \)
Observations \( \text{Obs} = \{\{1\}, \{2, 3\}\} \)
Fixed initial credit

Can you win with initial credit = 3?

Keep track of
- which can be the current state, and
- what is the worst-case energy level

Initially: (3,⊥,⊥)
Example

\[ v(q) \leftarrow \min \{v(q') + w(q', q) \mid v(q') \neq \bot \} \]
Example

\[ v(q) \leftarrow \min \{ v(q') + w(q', q) \mid v(q') \neq \bot \} \]
Example

Stop search whenever
- negative value, or
- comparable ancestor

\[v(q) \leftarrow \min\{v(q') + w(q', q) \mid v(q') \neq \bot\}\]
Example

Stop search whenever:
- negative value, or
- comparable ancestor
Example

Initial credit = 3 is not sufficient!
Search will terminate because \( \mathbb{N}^d \) is well-quasi ordered.
Example

Upper bound: non-primitive recursive

Lower bound: EXPSPACE-hard

Proof (not shown in this talk): reduction from the infinite execution problem of Petri Nets.

Search will terminate because $\mathbb{N}^d$ is well-quasi ordered.
# Complexity

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With imperfect information:

Corollary: Finite-memory strategies suffice in energy games
Memory requirement

With imperfect information:

Corollary: Finite-memory strategies suffice in energy games

In mean-payoff games:

- **infinite** memory may be required
- limsup vs. liminf definition do **not** coincide
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<td>memoryless</td>
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<tr>
<td><strong>Imperfect information</strong></td>
<td>finite memory</td>
<td>infinite memory</td>
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Theorem

The unknown initial credit problem for energy games is undecidable.
(even for blind games)

Proof:

Using a reduction from the halting problem of 2-counter machines.
2-counter machines

- 2 counters $c_1$, $c_2$
- increment, decrement, zero test

```
q1: inc c₁ goto q2
q2: inc c₁ goto q3
q3: if c₁ == 0 goto q6
    else dec c₁ goto q4
q4: inc c₂ goto q5
q5: inc c₂ goto q3
q6: halt
```
2-counter machines

- 2 counters $c_1, c_2$
- increment, decrement, zero test

$q_1$: inc $c$ goto $q_2$
$q_2$: inc $c_1$ goto $q_3$
$q_3$: if $c_1 == 0$ goto $q_6$
else dec $c_1$ goto $q_4$
$q_4$: inc $c_2$ goto $q_5$
$q_5$: inc $c_2$ goto $q_3$
$q_6$: halt

$q$: inc $c$ goto $q'$

$q$: if $c = 0$ then goto $q'$ else dec $c$ goto $q''$. 

$(q, \text{inc}, c, q')$

$(q, 0?, c, q')$ and $(q, \text{dec}, c, q'')$

\[
\begin{align*}
(q_1) & \xrightarrow{(q_1, \text{inc}, c_1, q_2)} (q_2) \\
(0) & \xrightarrow{(q_2, \text{inc}, c_1, q_3)} (q_3) \\
(0) & \xrightarrow{(q_3, \text{dec}, c_1, q_4)} (q_4) \\
(0) & \xrightarrow{(q_4, \text{inc}, c_2, q_5)} (q_5) \\
(1) & \xrightarrow{(q_5, 1)} \ldots
\end{align*}
\]
Reduction

Halting problem:
Given $M$ and state $q_{\text{halt}}$, decide if $q_{\text{halt}}$ is reachable (i.e., $M$ halts).

\[
\begin{align*}
q_1 &: \text{inc } c_1 \text{ goto } q_2 \\
q_2 &: \text{inc } c_1 \text{ goto } q_3 \\
q_3 &: \text{if } c_1 = 0 \text{ goto } q_6 \\
&\quad \text{else dec } c_1 \text{ goto } q_4 \\
q_4 &: \text{inc } c_2 \text{ goto } q_5 \\
q_5 &: \text{inc } c_2 \text{ goto } q_3 \\
q_6 &: \text{halt}
\end{align*}
\]

Reduction:
Given $M$, construct $G_M$ such that $M$ halts iff there exists a winning strategy in $G_M$ (with some initial credit).

- Deterministic machine
- Nonnegative counters
Reduction

- Blind game (unique observation)
- Initial nondeterministic jump to several gadgets
- Winning strategy = \((\#\text{AcceptingRun})^\omega\)
Gadgets

Gadget 1: « First symbol is # »

Reminder: Winning strategy = #AcceptingRun#AcceptingRun#...
Gadgets

Reminder: Winning strategy = #AcceptingRun#AcceptingRun#…

Gadget 2:
« Every $\sigma_1$ is followed by $\sigma_2$ »

E.g., $\sigma_1 = (q, \cdot, \cdot, q')$ and $\sigma_2 = (q', \cdot, \cdot, q'')$
Gadgets

Gadget 3: « Infinitely many # »
(and a bit more...)

Guess: this is the last #

Reminder: Winning strategy = #AcceptingRun#AcceptingRun#...
Gadgets

Check zero tests on c

Gadget 4:
« Counter correctness »
Gadgets

Check zero tests on c

Gadget 4:
« Counter correctness »
Gadgets

$\sigma \neq \#$

$\sigma, \begin{cases} -1 & \text{if } \sigma = (\cdot, inc, c, \cdot) \\ 1 & \text{if } \sigma = (\cdot, dec, c, \cdot) \\ 0 & \text{otherwise} \end{cases}$

Check zero tests on c

Gadget 4:
« Counter correctness »

Check non-zero test on c
If M halts, then $(\#\text{AcceptingRun})^\omega$ is a winning strategy with initial credit $\text{Length}(\text{AcceptingRun})$.

If there exists a winning strategy with finite initial credit, then $\#$ occurs infinitely often, and finitely many cheats occur. Hence, M has an accepting run.
Mean-payoff games

Theorem
Mean-payoff games are undecidable (not co-r.e.).
(even blind games)

Proof:
Using a reduction from the halting problem of 2-counter machines.

Nota: the proof works for both limsup and liminf, but only for strict mean-payoff objective (i.e., $MP > \nu$)
Mean-payoff games

Theorem

Mean-payoff games are undecidable (not co-r.e.).

(even blind games)

Proof:

Using a reduction from the halting problem of 2-counter machines.

Reduction:

Given M, construct $G_M$ such that M halts iff there exists a strategy to ensure strictly positive mean-payoff value.
Gadgets

Reminder: Winning strategy = #AcceptingRun#AcceptingRun#...

Gadget 1:
« First symbol is # »
Gadgets

Gadget 2:
« Every $\sigma_1$ is followed by $\sigma_2$ »

E.g., $\sigma_1 = (q, \ldots, q')$ and $\sigma_2 = (q', \ldots, q''')$

Reminder: Winning strategy = #AcceptingRun#AcceptingRun#...
Gadgets

Gadget 3: « Infinitely many # »

Guess: this is the last #

Reminder: Winning strategy = #AcceptingRun#AcceptingRun#...
Gadgets

$(\sigma \neq \#)$

-1 if $\sigma = (\cdot, \text{inc}, c, \cdot)$
1 if $\sigma = (\cdot, \text{dec}, c, \cdot)$
0 otherwise

Check zero tests on c

Check non-zero test on c
## Complexity

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Mean-payoff games

Theorem

Mean-payoff games are undecidable (not r.e.).
(for games with at least 2 observations)

Proof:

Using a reduction from the non-halting problem of 2-counter machines.

Nota: the proof works only for limsup and non-strict mean-payoff objective (i.e., $\text{MP} \geq \nu$)
Mean-payoff games

Theorem

Mean-payoff games are undecidable (not r.e.).
(for games with at least 2 observations)

Proof:

Using a reduction from the non-halting problem of 2-counter machines.

Reduction:

Given $M$, construct $G_M$ such that $M$ does \textbf{not halt} iff there exists a strategy to ensure strictly nonnegative mean-payoff value.
Reduction

- 2-observation game
- Initial nondeterministic jump to several gadgets (+ back-edges)
- Winning strategy = Non-terminatingRun
Gadgets

Gadget 3:
« avoid halting state »

Reminder: Winning strategy = Non-terminatingRun
Gadgets

\[
\begin{align*}
\sigma, & \begin{cases} 
1 & \text{if } \sigma = (\cdot, \text{inc}, c, \cdot) \\
-1 & \text{if } \sigma = (\cdot, \text{dec}, c, \cdot) \\
0 & \text{otherwise}
\end{cases} \\
(\cdot, \text{dec}, c, \cdot), & -1 \\
q_0
\end{align*}
\]

Check non-zero test on \(c\)

Gadget 5 and 6:
« Counter correctness »
Gadgets

Gadget 5 and 6:
« Counter correctness »

Check zero tests on c
• If M does not halt, then Non-terminatingRun is a winning strategy.

• If M halts, then Maximizer has to cheat within L steps where $L = \text{Size}(\text{AcceptingRun})$, or reaches halting state, thus he ensures mean-payoff at most $-1/L$. 
### Complexity

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Nota: whether there exists a finite-memory winning strategy in mean-payoff games is also undecidable.
Energy and mean-payoff games with visible weights are decidable (EXPTIME-complete).

Weights are visible if

\[ q_1 \xrightarrow{a, w} q_2, \quad q'_1 \xrightarrow{a, w'} q'_2 \]

implies \( w = w' \)

Weighted subset construction is finite
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<td>EXPTIME-complete</td>
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Conclusion

- Quantitative games with imperfect information
  - Undecidable in general
  - Energy with fixed initial credit **decidable**
  - Visible weights **decidable**

- Open questions
  - Strict vs. non-strict mean-payoff
  - Liminf vs. Limsup
  - Blind mean-payoff games

- Related work
  - Incorporate liveness conditions (e.g. parity)
Thank you!

Questions?
References

