Automatic Rectangular Refinement of Affine Hybrid Automata

Laurent Doyen

ULB

Jean-François Raskin

ULB

Tom Henzinger

EPFL

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Overview

- Automatic analysis of affine hybrid systems

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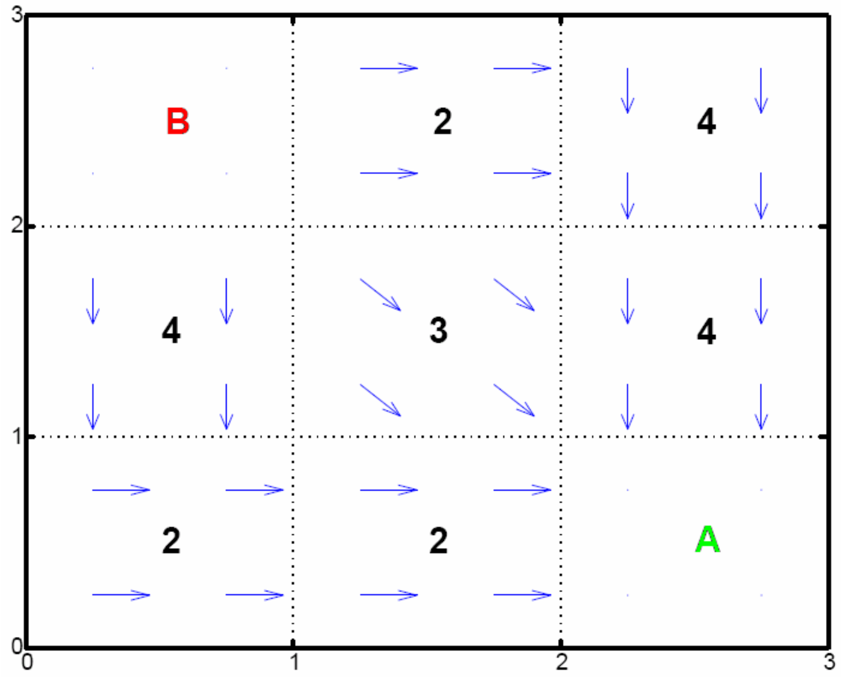
- Example:



Navigation Benchmark

$$\begin{cases} \dot{x} = v \\ \dot{v} = A(v - v_d) \end{cases}$$

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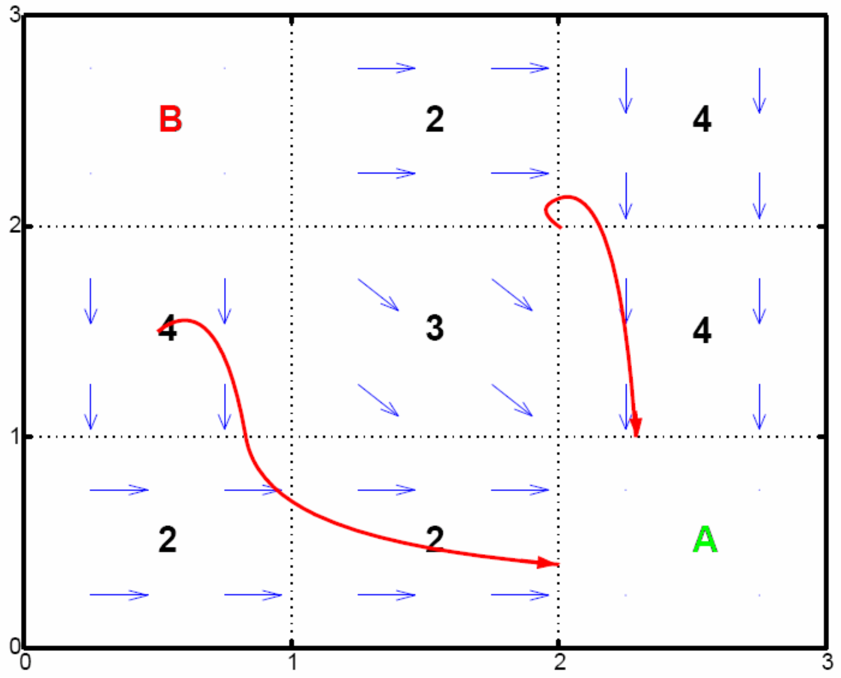

Two trajectories

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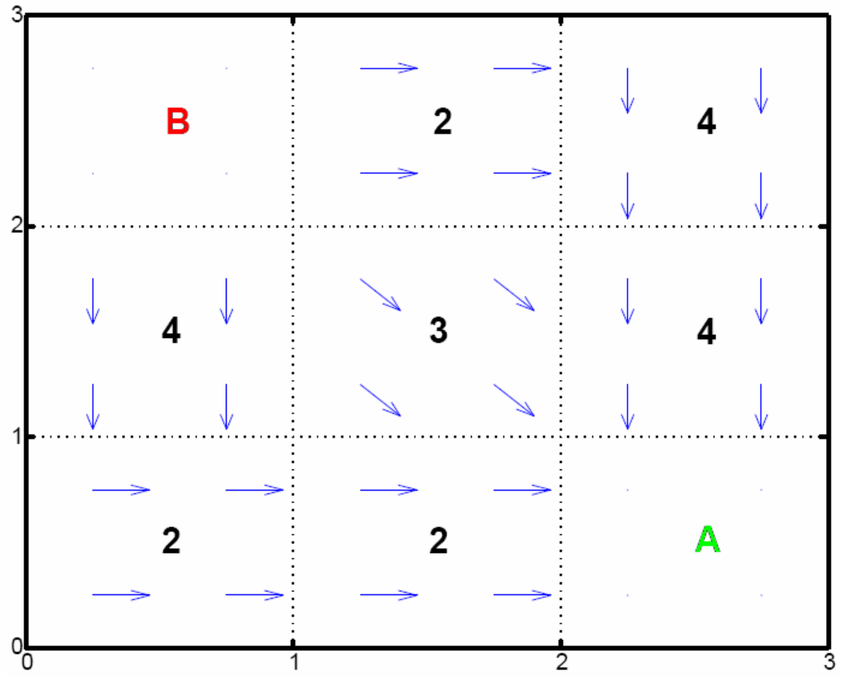

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Affine dynamics

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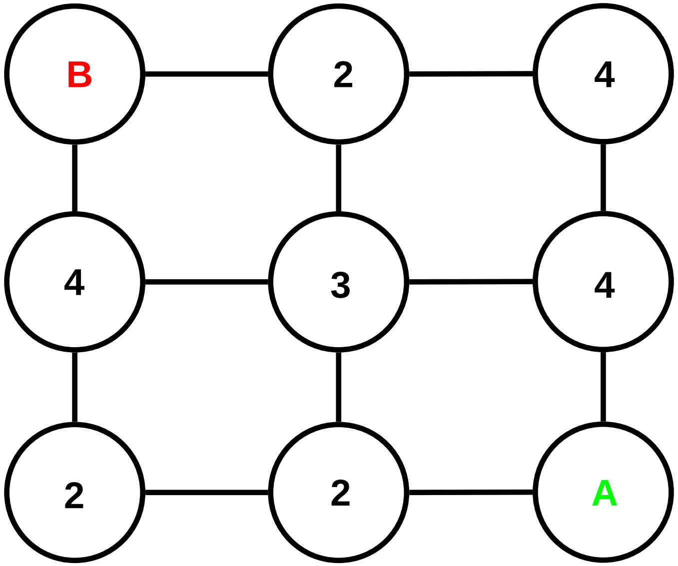

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Discrete states + **Affine dynamics**

Reminder

- Some classes of hybrid automata:
  - Timed automata ($\dot{x} = 1$)
  - Rectangular automata ($\dot{x} \in [a, b]$)
  - Linear automata ($\sum a_i \dot{x}_i \sim b$)

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  - etc.

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Limit for symbolic computation of Post with HyTech

Limit for decidability of Language Emptiness

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➡ Abstract it !

- Abstraction is too coarse ?

➡ Refine it !

HOW ?

Methodology

- 1. Abstraction: over-approximation

Affine dynamics → Rectangular dynamics

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$$0 \le x \le 3$$

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Affine dynamics → Rectangular dynamics

$$\dot{x} = 2 - x \qquad 0 \le x \le 3$$

→

$$\dot{x} \in [-1,2] \qquad 0 \le x \le 3$$

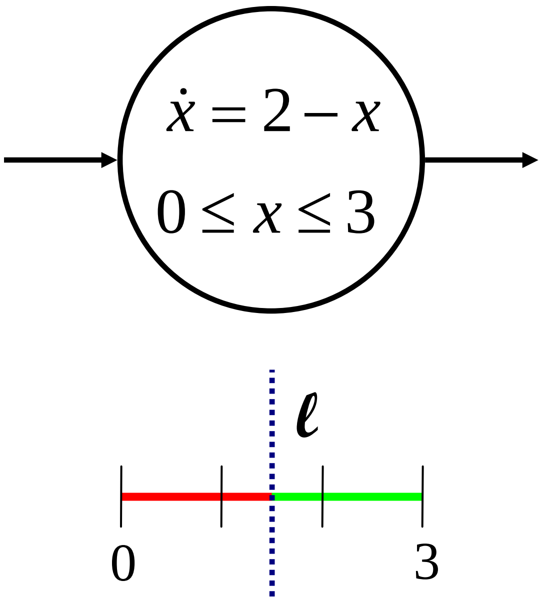
Let $\begin{cases} f(x) = 2\text{-}x \\ \mathrm{Inv} = \{0 \le x \le 3\} \end{cases}$

Then $[-1,2] = [\min_{x \in \mathrm{Inv}} f(x), \max_{x \in \mathrm{Inv}} f(x)]$

Methodology

- 2. Refinement: split locations by a line cut

Line $\boldsymbol{\ell} \equiv x = \frac{3}{2}$



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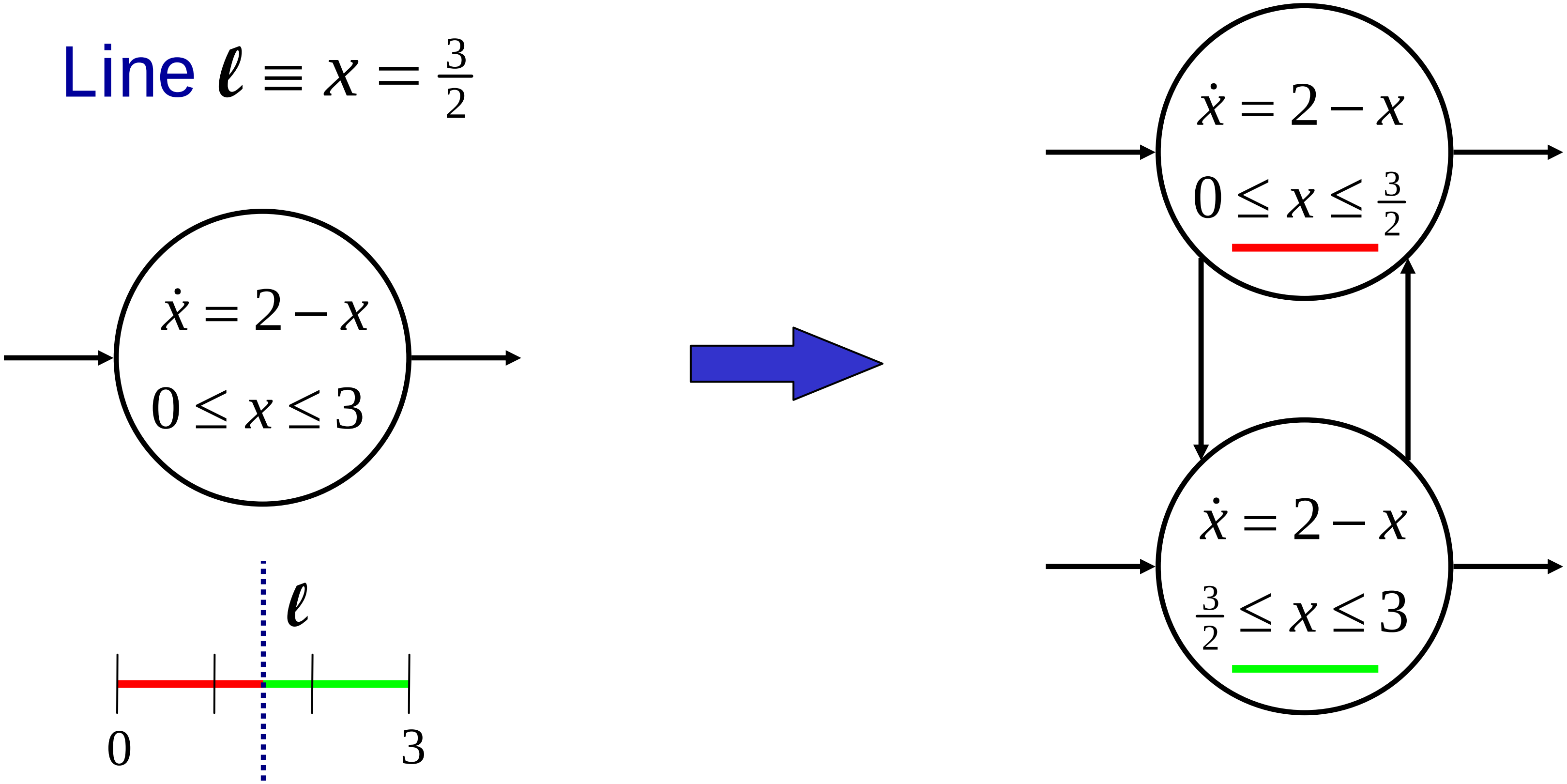

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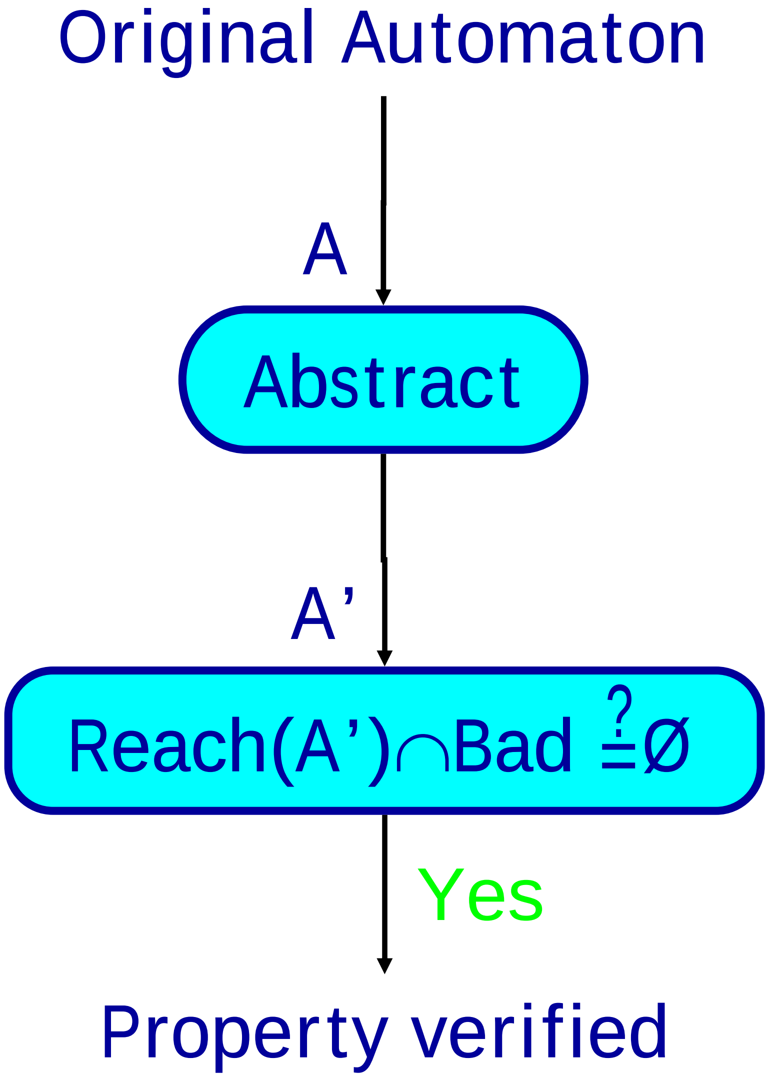

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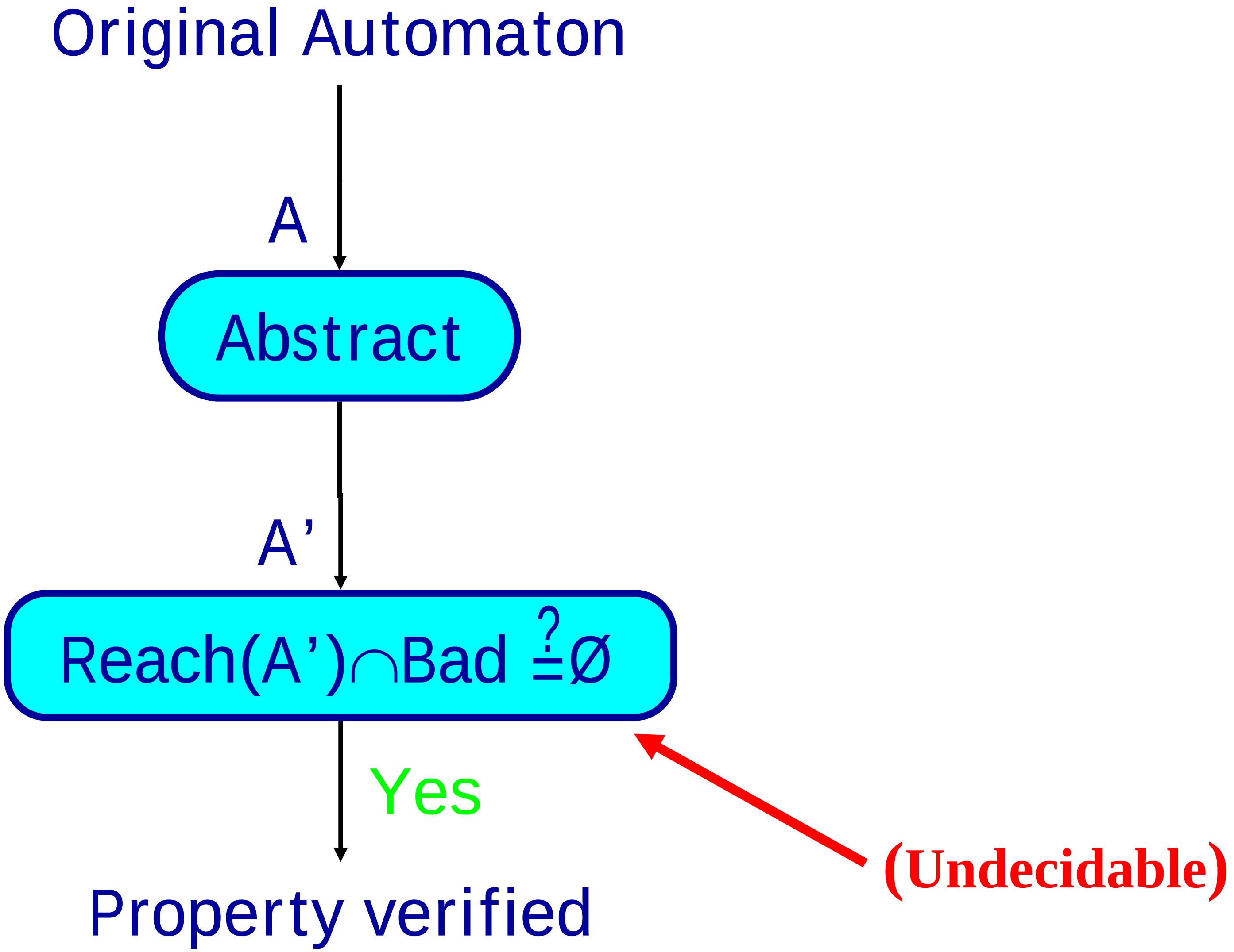

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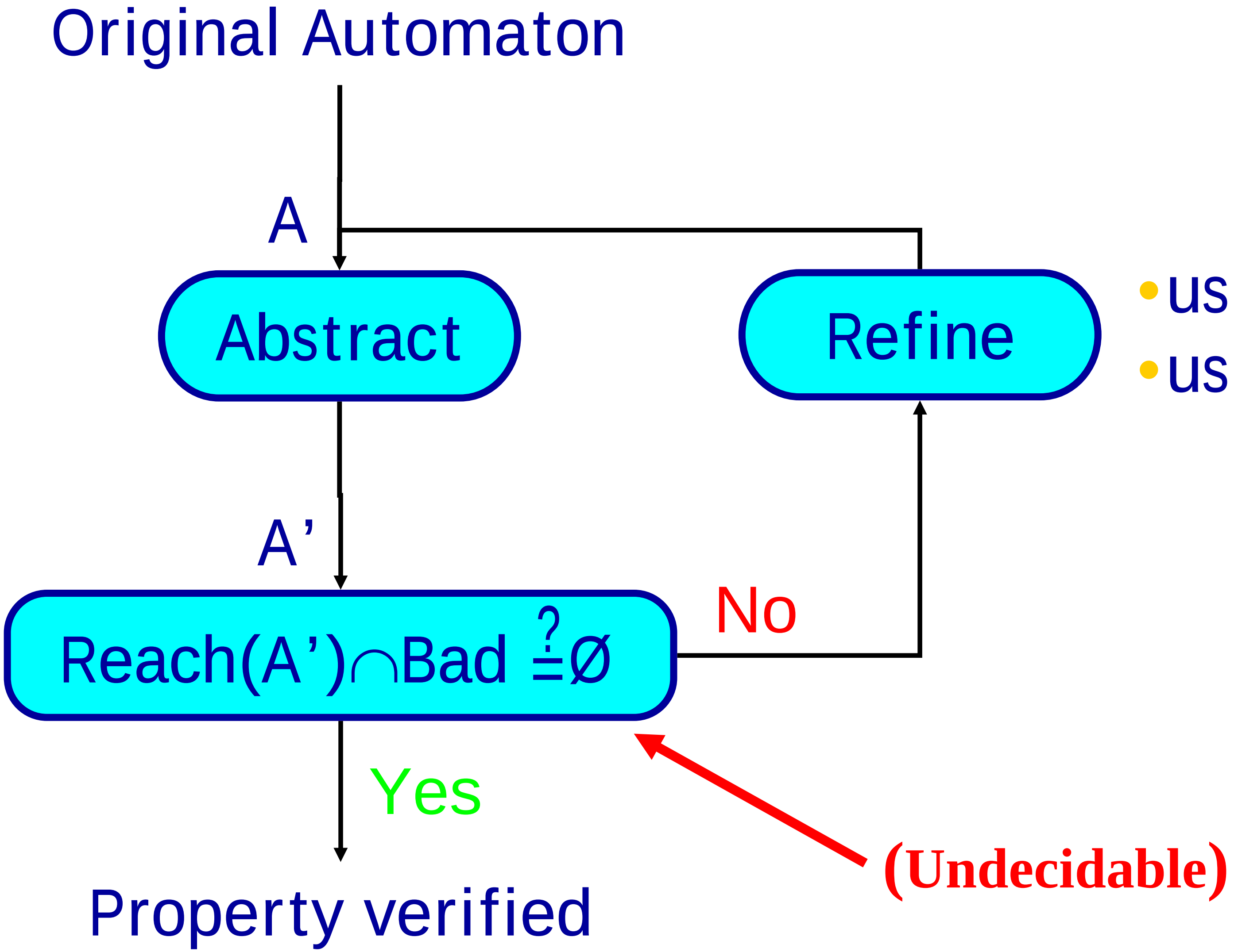

- using Reach(A')
- using Pre*(Bad)

Refinement

- 2. Refinement: split locations by a line cut
- Which location(s) ?
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- Which location(s) ?
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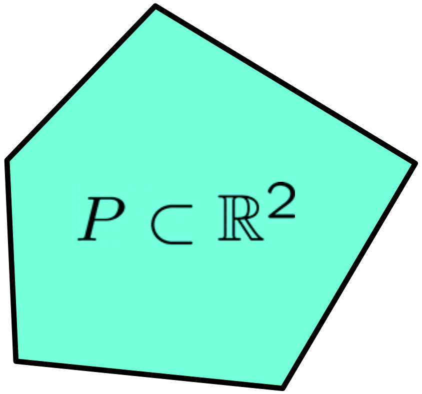
  - Better: replace the state space by Loc_2
- Which line cut ?
  - The best cut for some *criterion* characterizing the *goodness* of the resulting approximation.

Notations

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

$$(x, y) \in P$$



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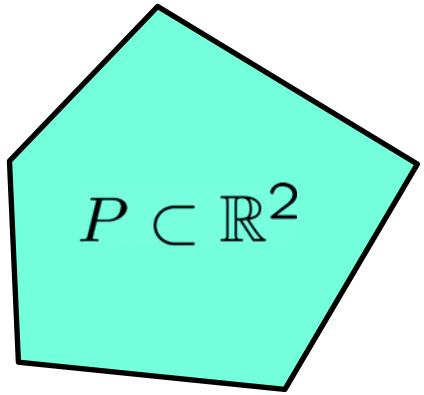

$$[\dot{x}_{\text{min}}, \dot{x}_{\text{max}}] = f(P) \qquad r_x = \dot{x}_{\text{max}} - \dot{x}_{\text{min}}$$

$$[\dot{y}_{\text{min}}, \dot{y}_{\text{max}}] = g(P) \qquad r_y = \dot{y}_{\text{max}} - \dot{y}_{\text{min}}$$

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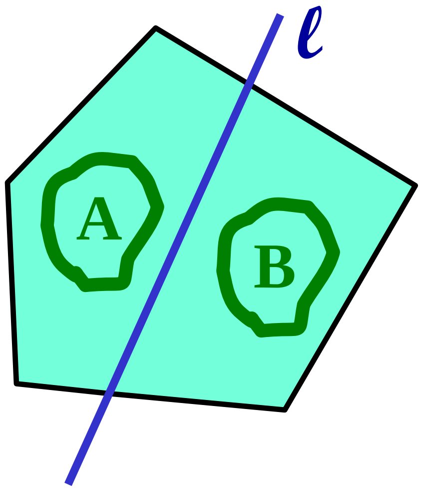
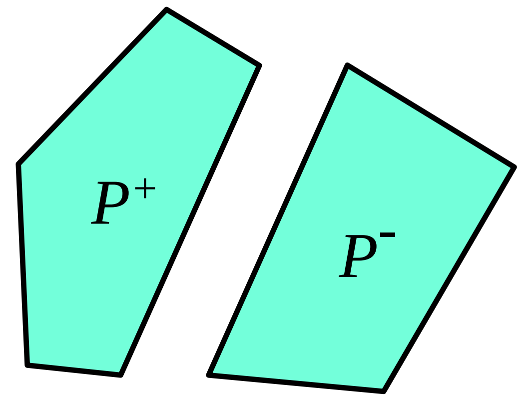


$$P/\ell = \langle P^+, P^- \rangle$$



Definition Let $A \subseteq P$ and $B \subseteq P$.
We say that ℓ *separates* A and B
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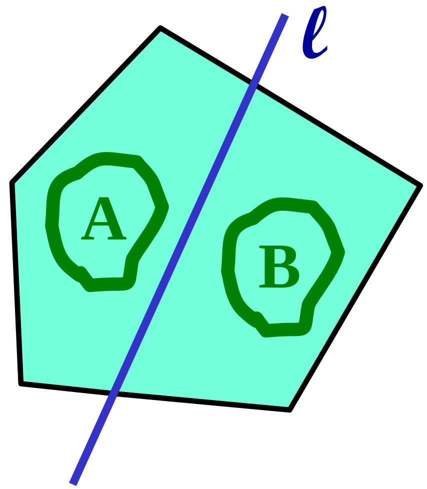

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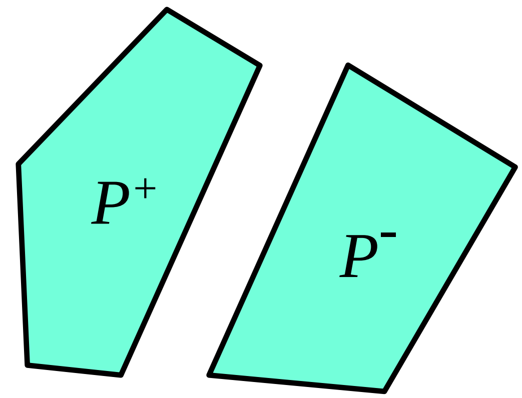

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$[a^+, b^+] = f(P^+)$ $\quad$ $[a^-, b^-] = f(P^-)$

$[c^+, d^+] = g(P^+)$ $\quad$ $[c^-, d^-] = g(P^-)$

$$\text{sizeRange}_x^\sim(P/\ell) = b^\sim - a^\sim$$

$$\text{sizeRange}_y^\sim(P/\ell) = d^\sim - c^\sim$$

$$\sim \in \{+, -\}$$

Goodness of a cut

- A good cut should minimize
  - $\max_{x \in \mathsf{Var}, \sim \in \{+,-\}} \mathsf{sizeRange}_x^{\sim}(P/\ell)$?

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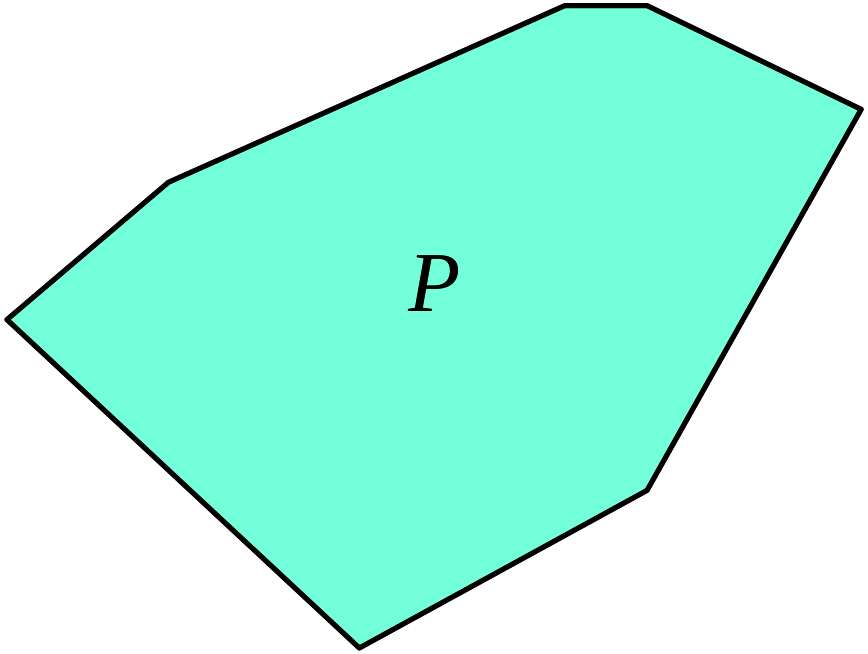
Our choice

Finding the optimal cut



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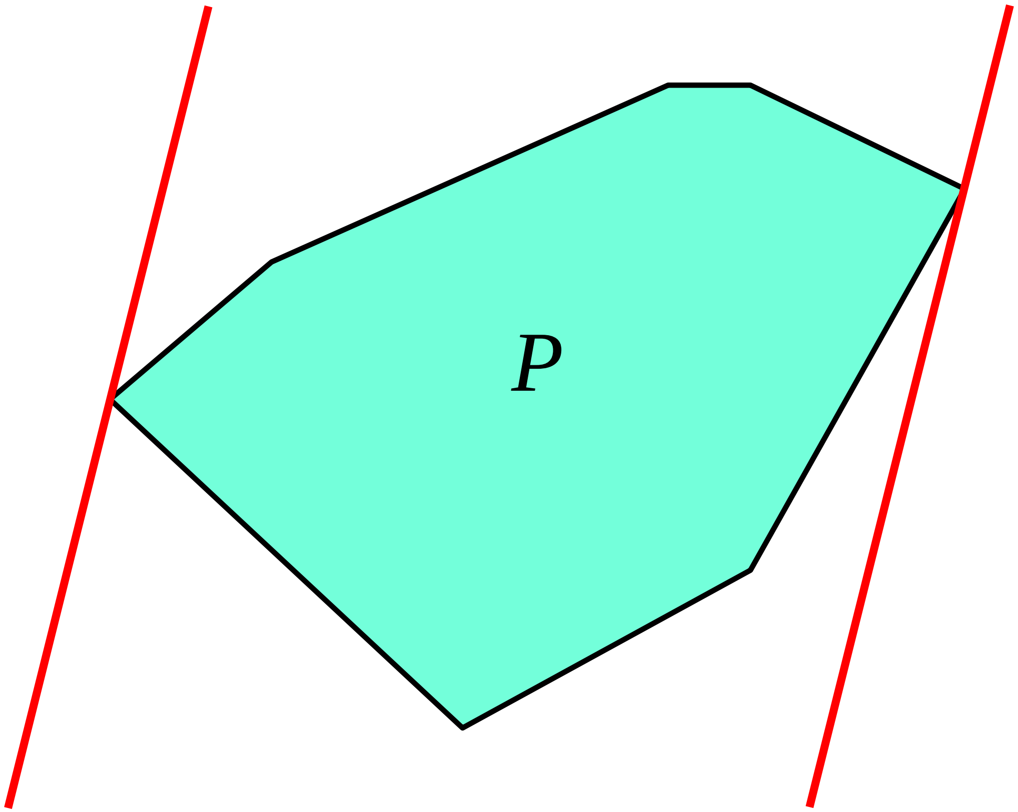
Extremal level sets of f(x,y)



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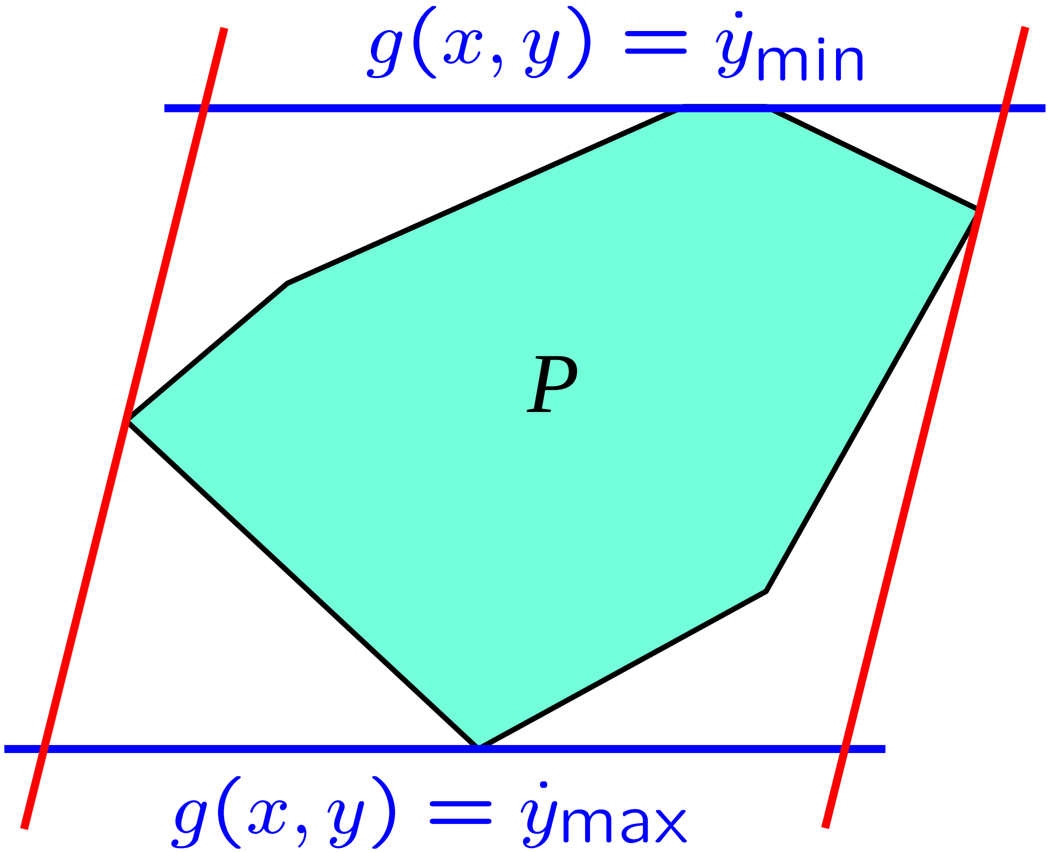
$$f(x, y) = \dot{x}_{\mathsf{min}} \qquad f(x, y) = \dot{x}_{\mathsf{max}}$$

Extremal level sets of g(x,y)



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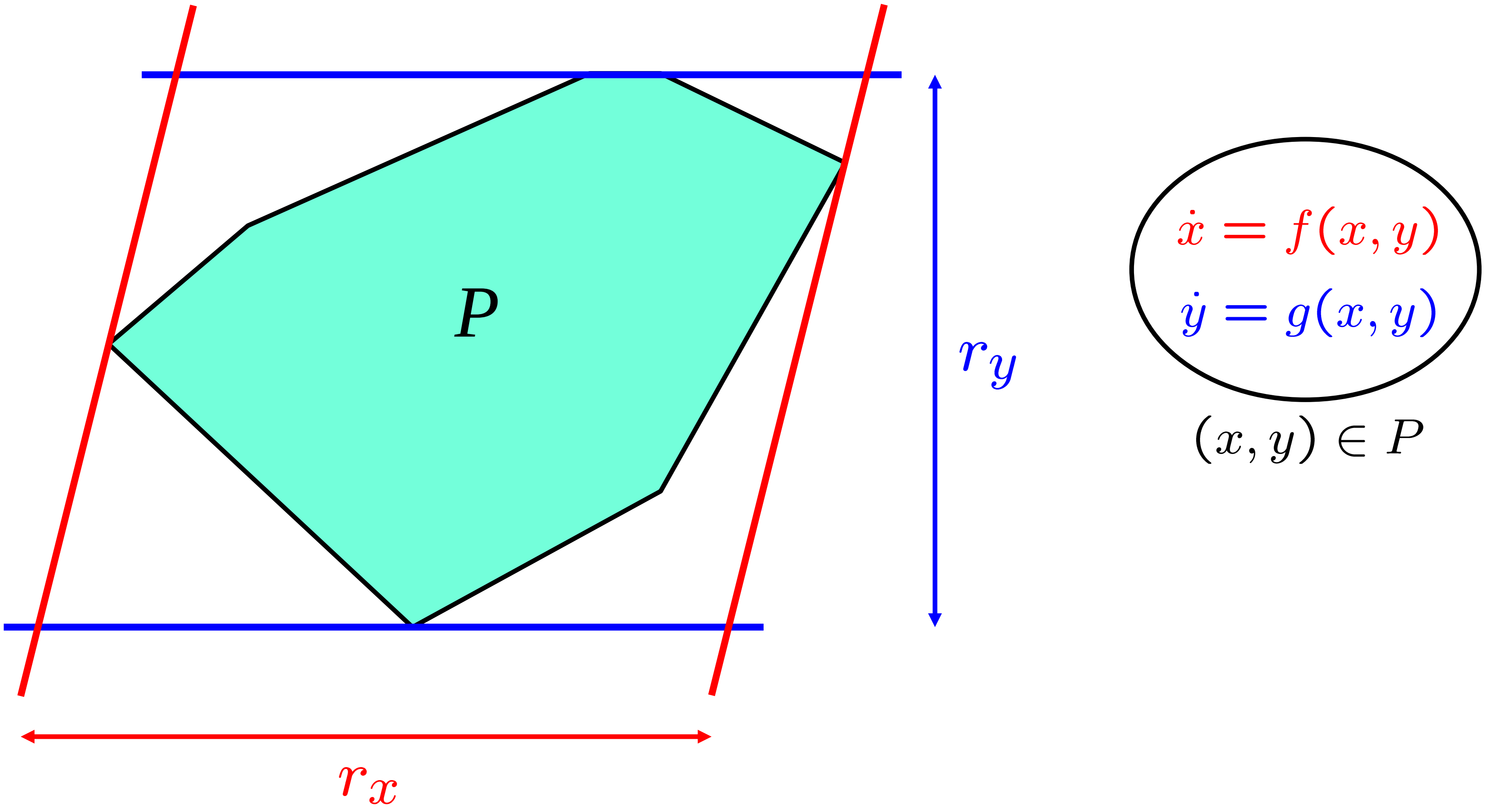
Example



Assume $r_x > r_y$

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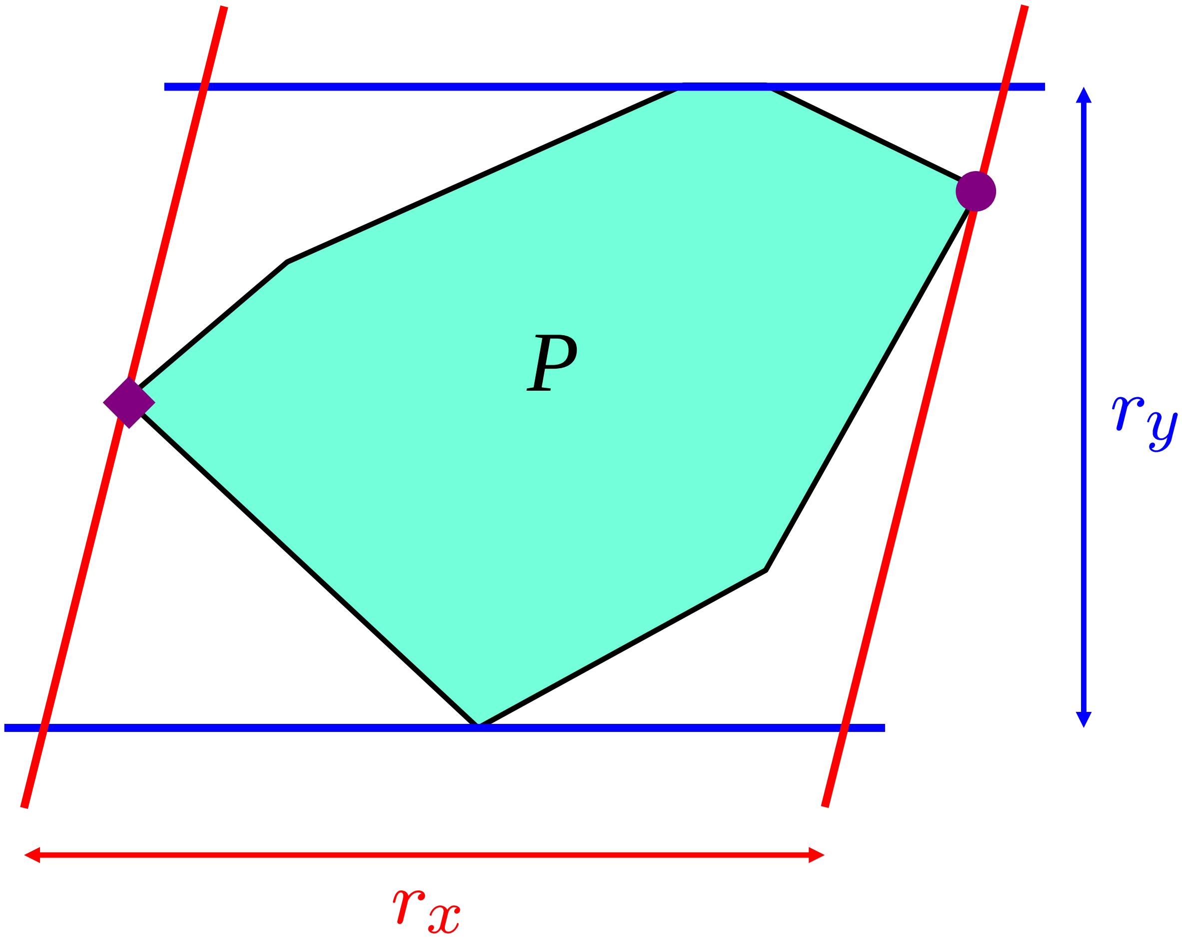

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Then any line separating {●} and {◆}

is <u>better</u> than any other line.

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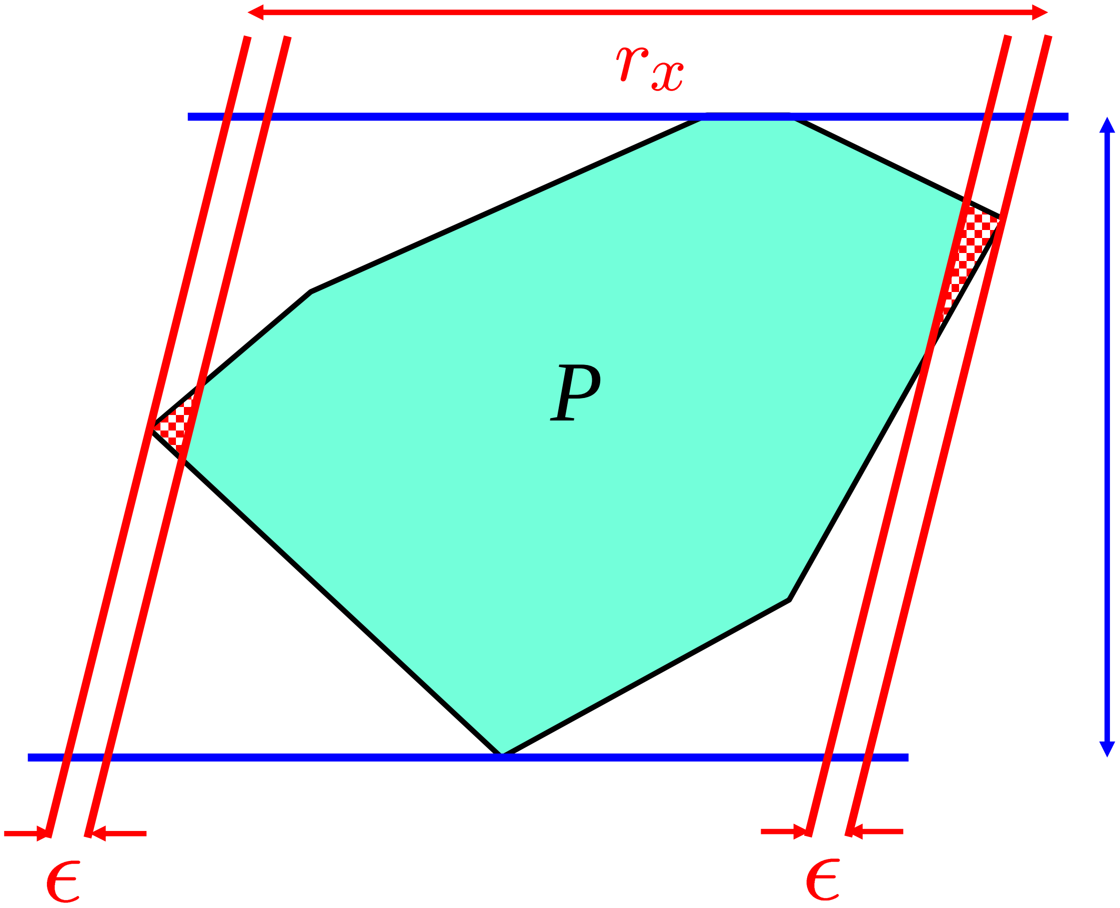

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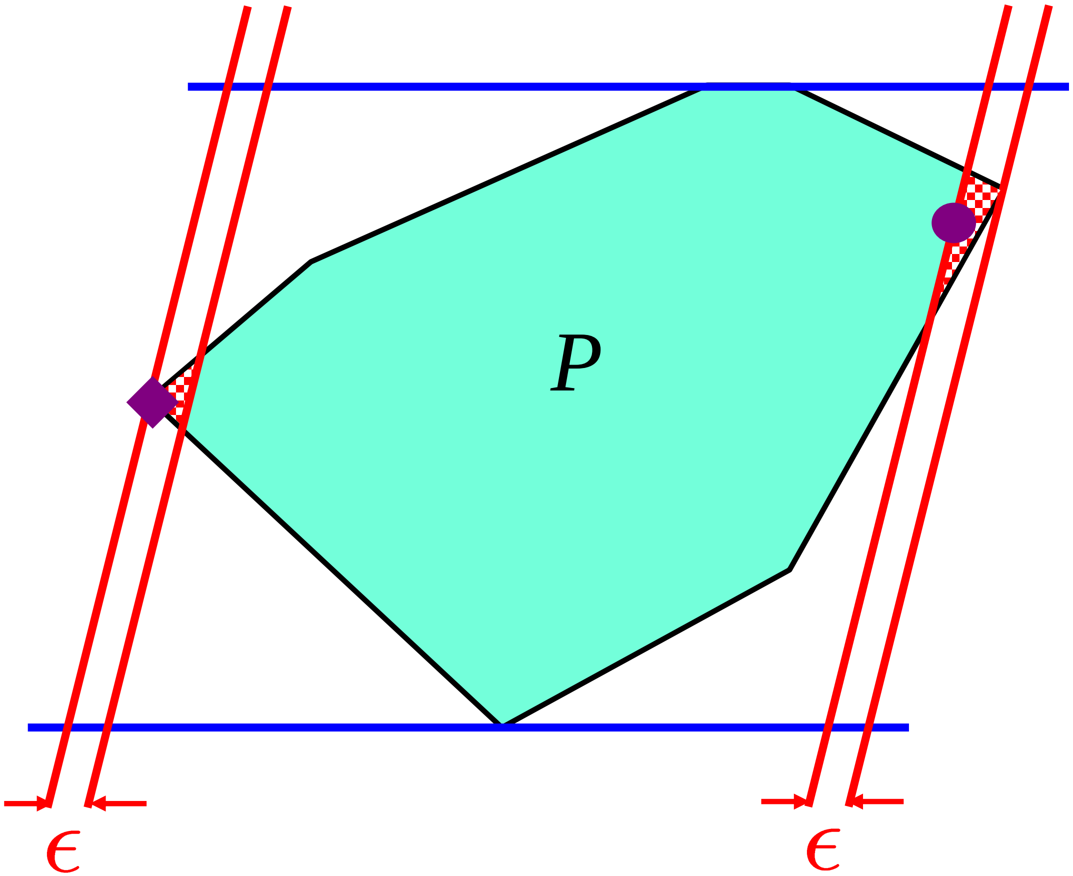

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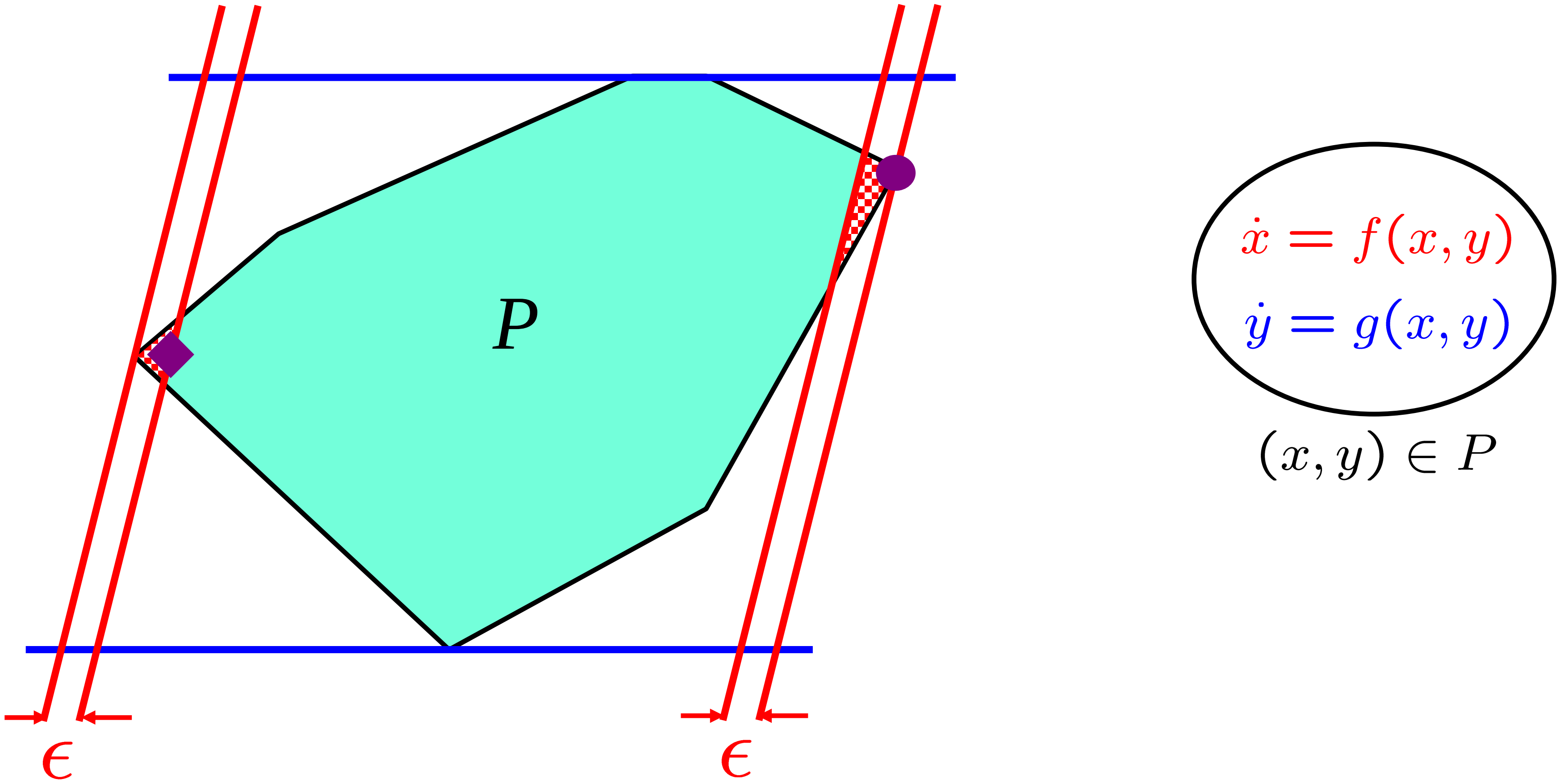

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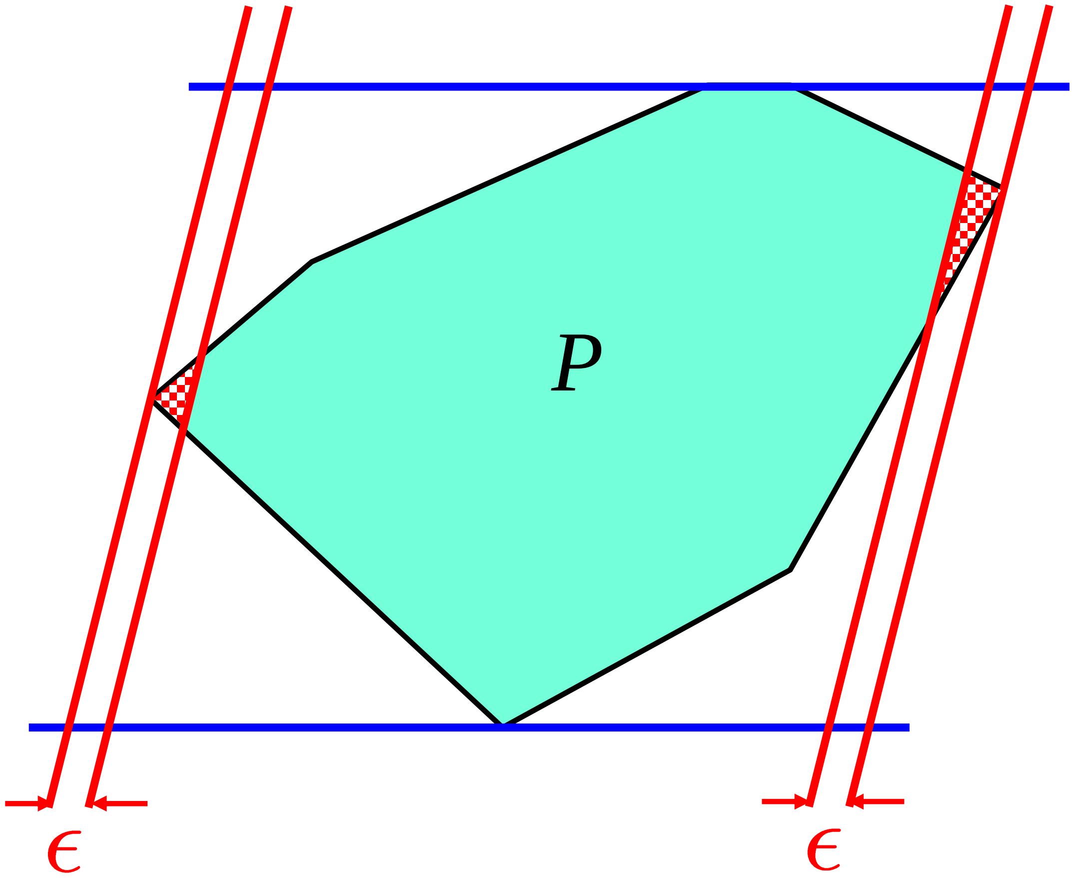

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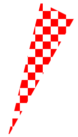
the best line separates  and 

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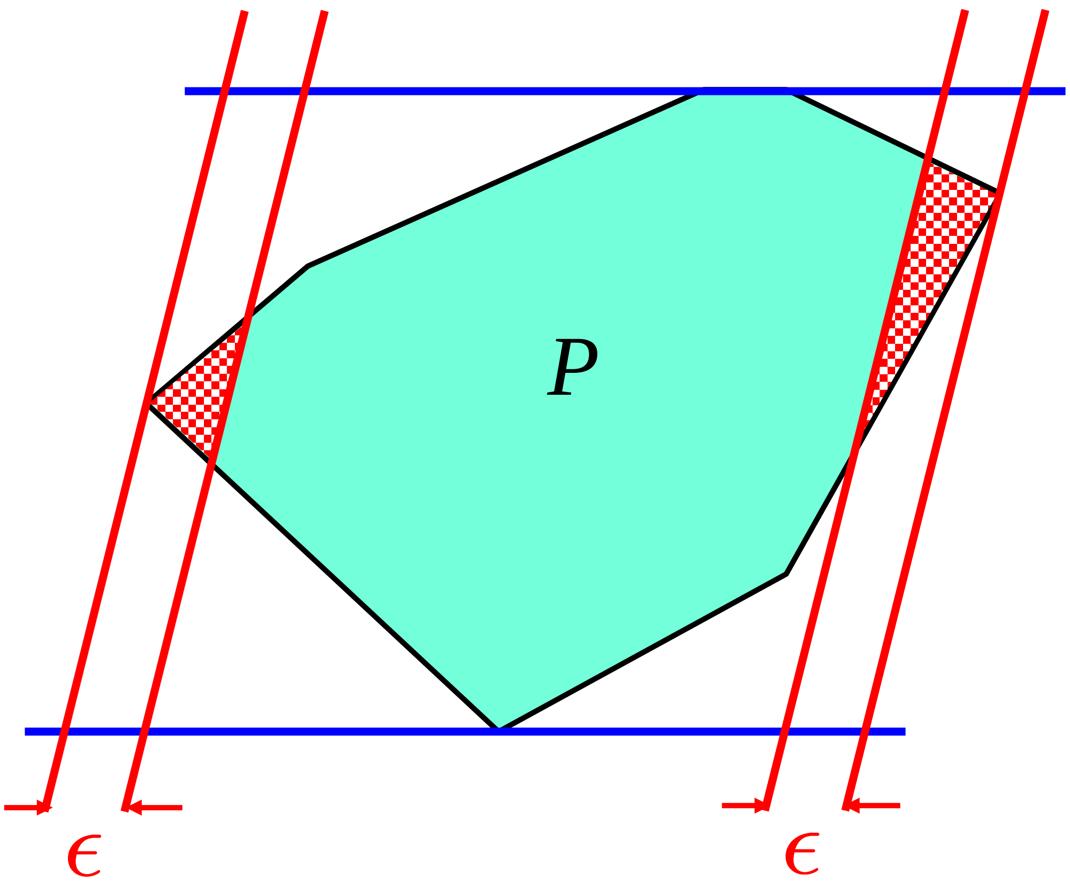

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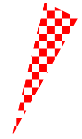
the best line separates  and 

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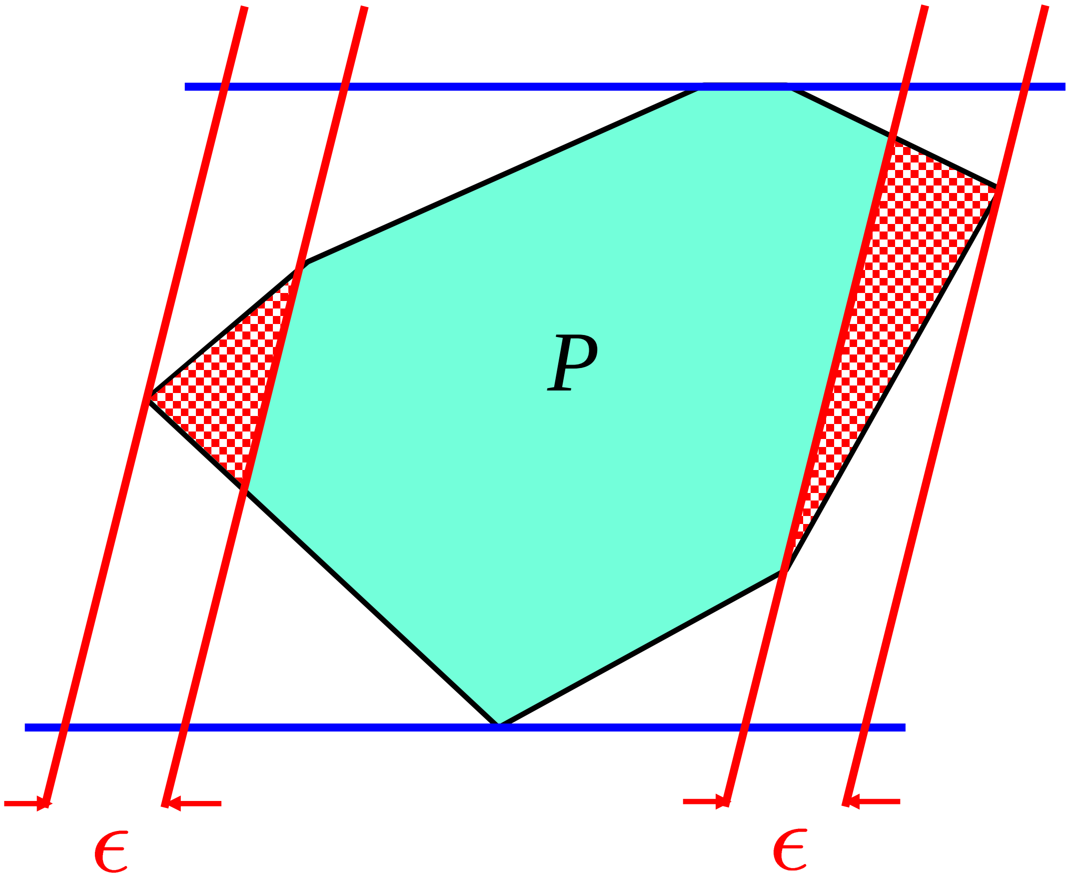

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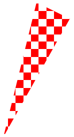
the best line separates  and 

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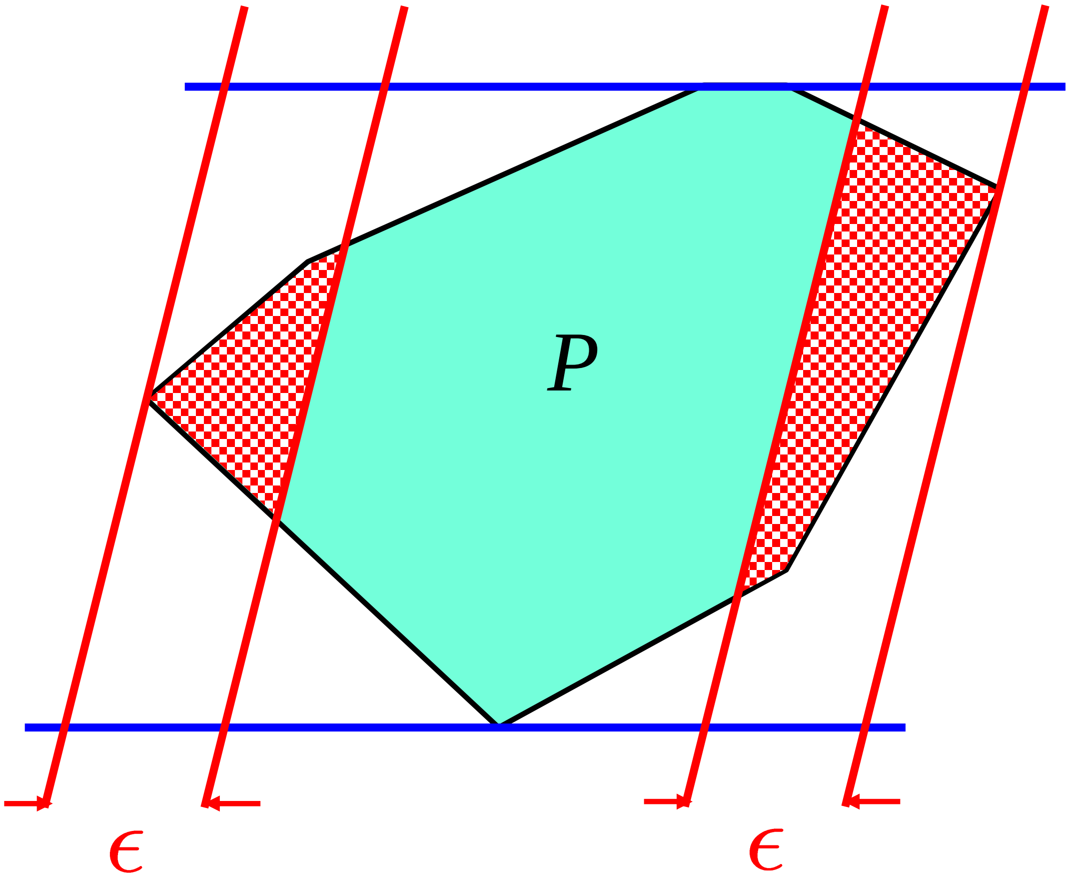

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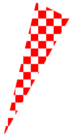
the best line separates  and 

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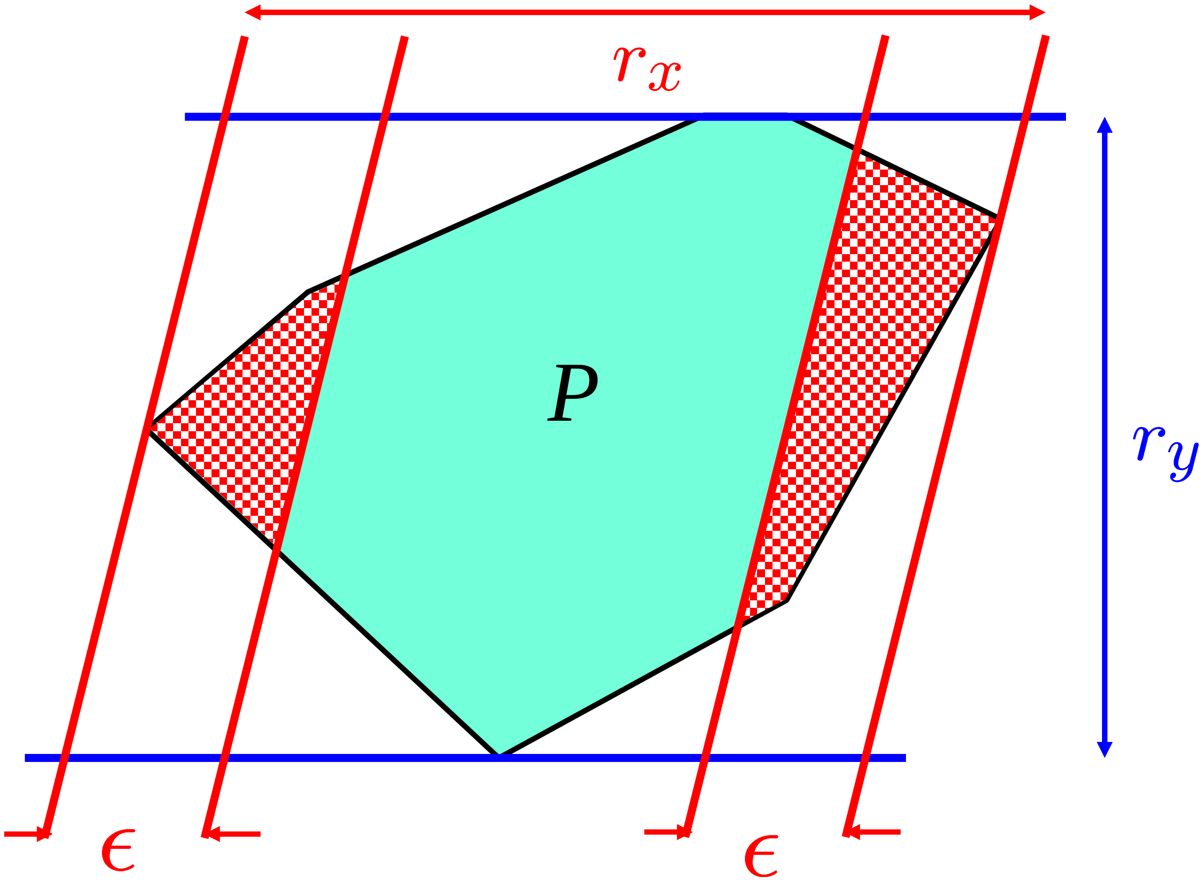

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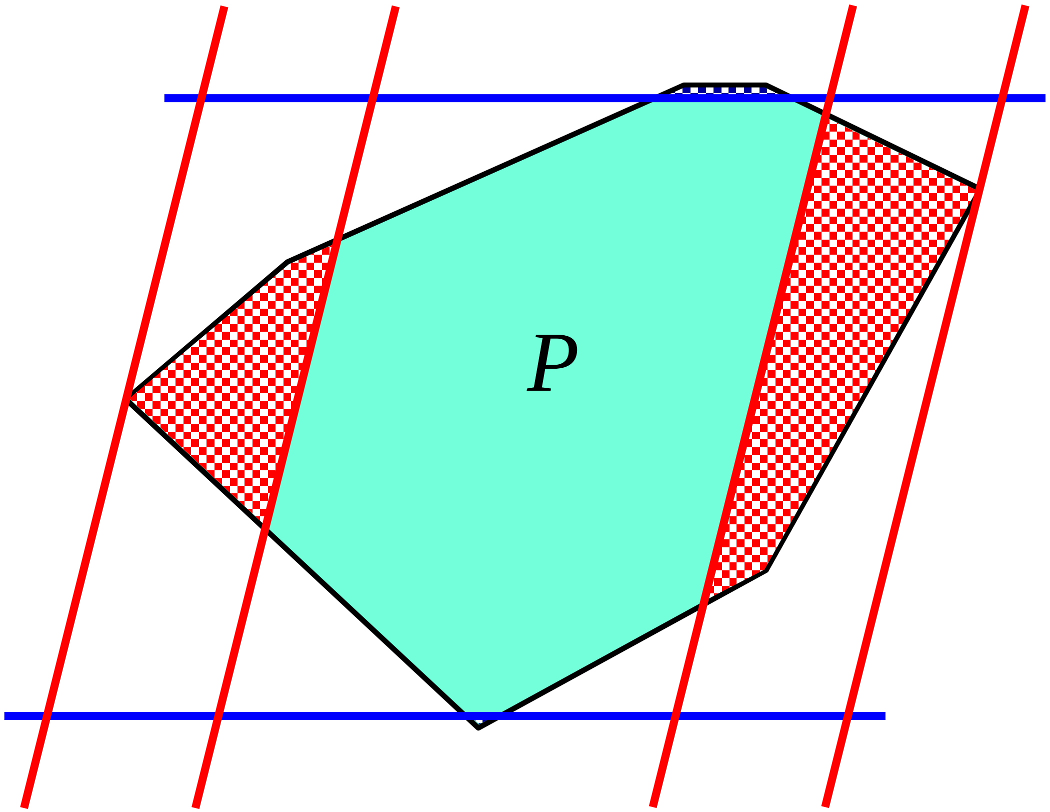
When $\epsilon = r_x - r_y$

Example



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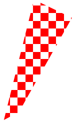
When $\epsilon = r_x - r_y$ the best line cut must separate both  from  and  from 

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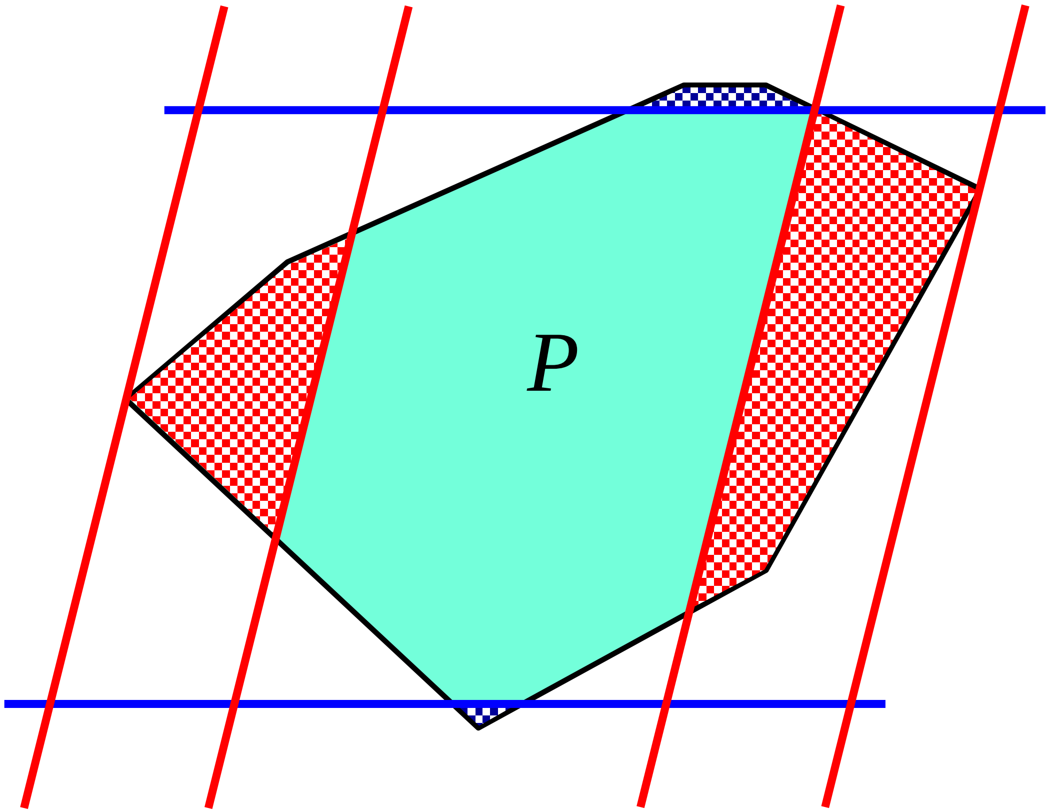

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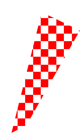
 from  and  from 

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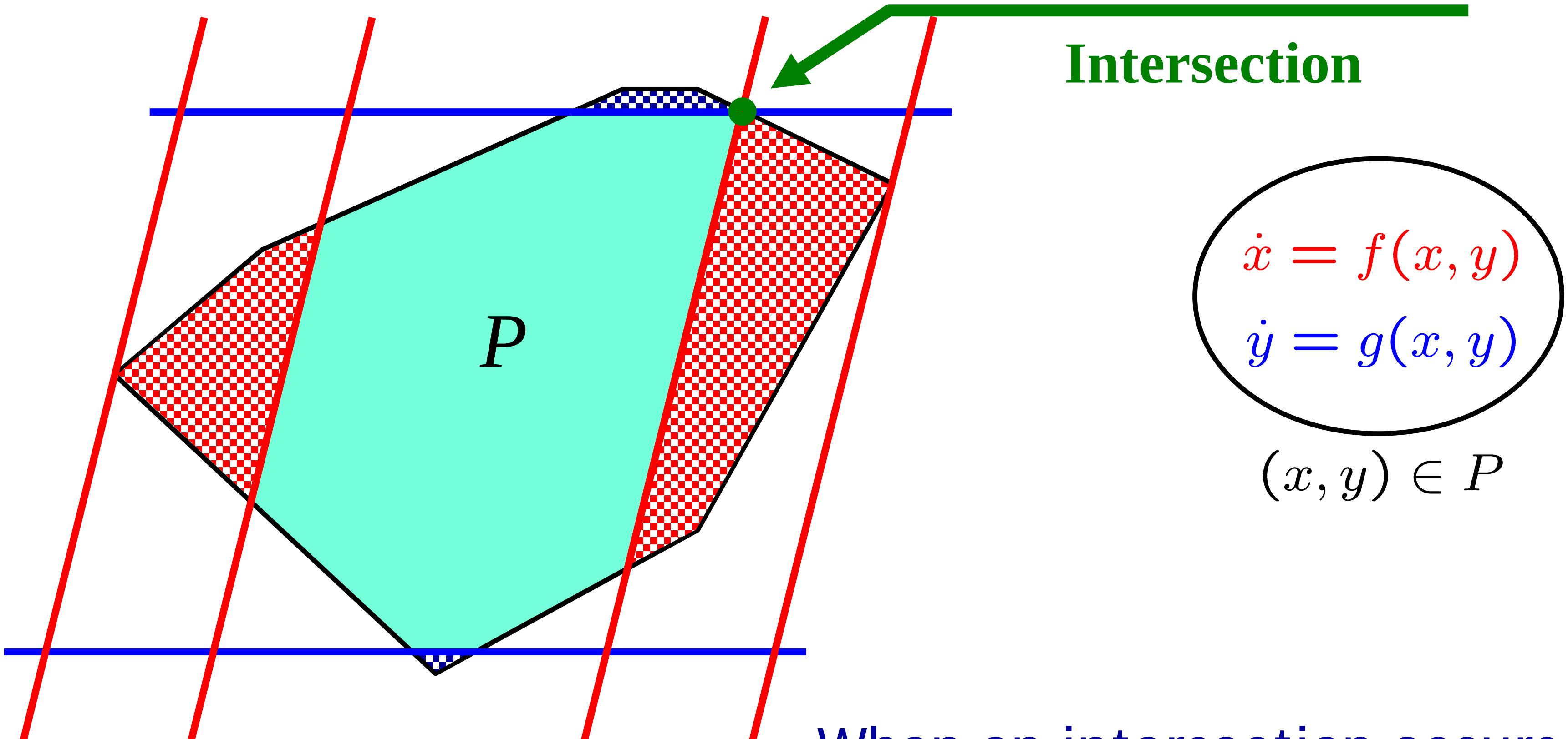
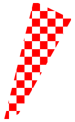


When an intersection occurs...

The process continues because it is still possible to separate

both  from  and  from 

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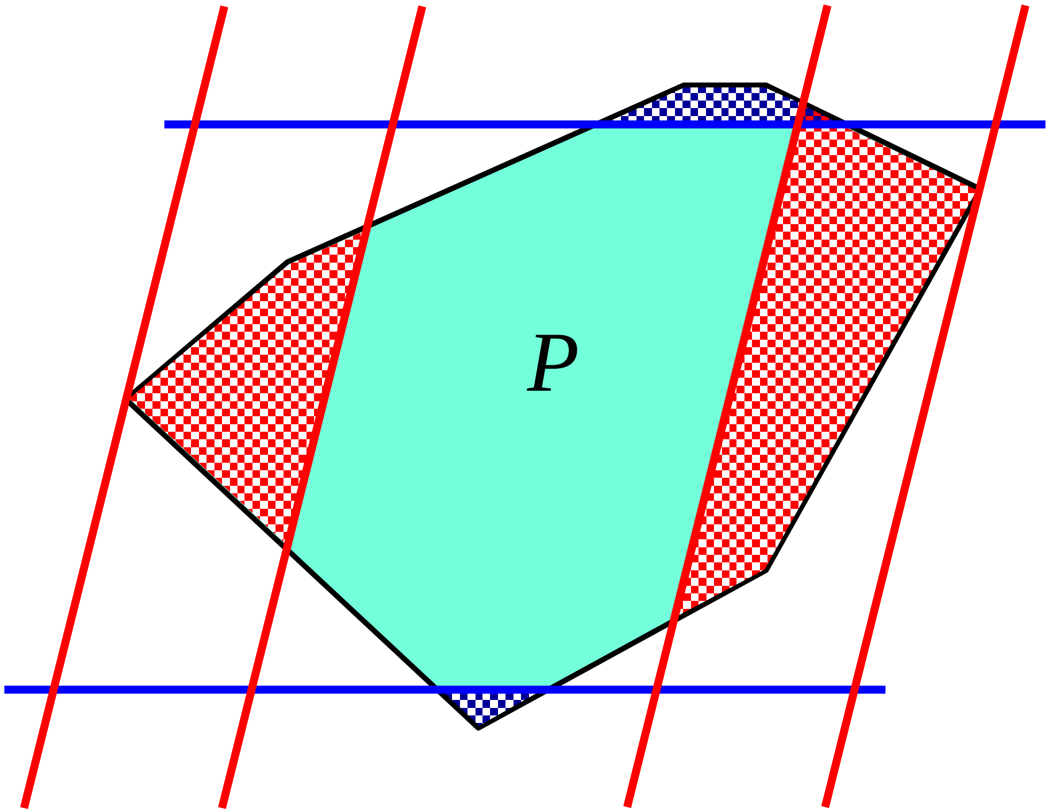

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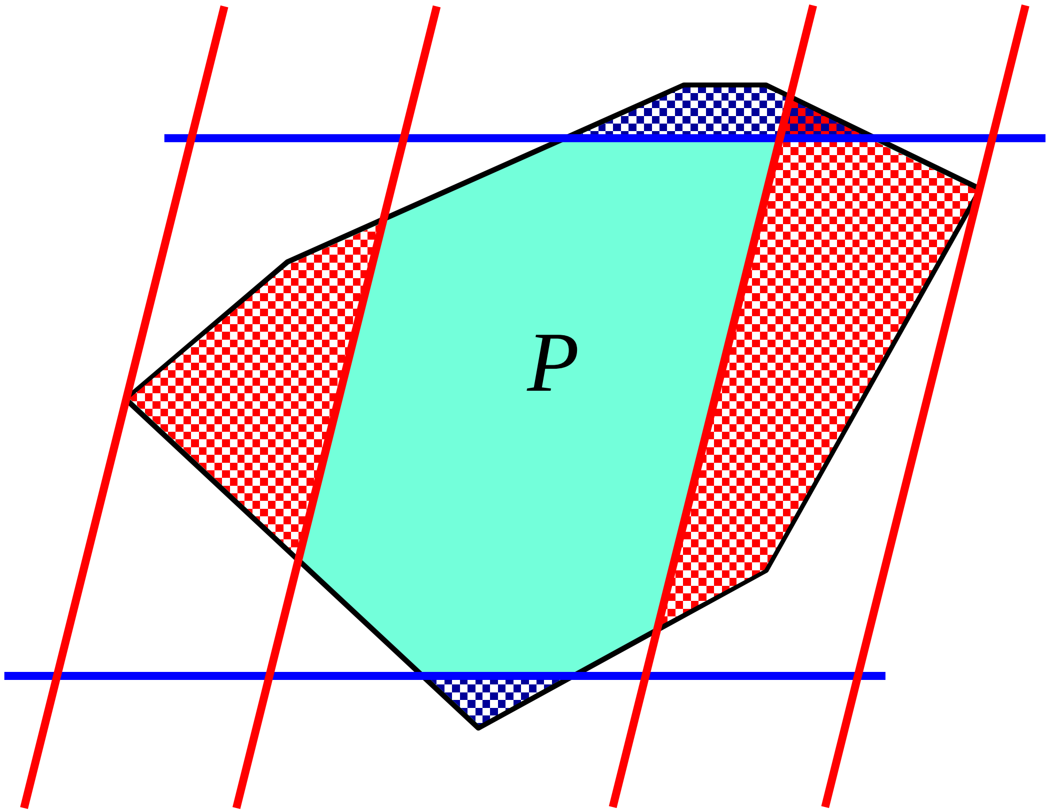

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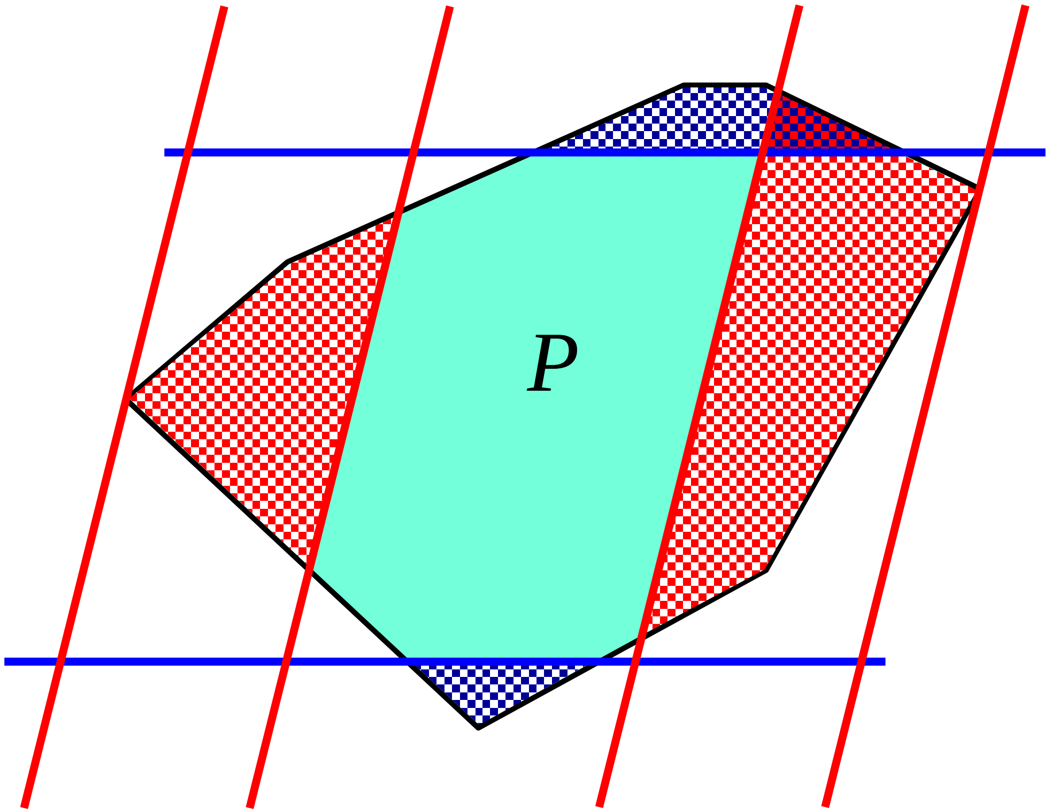

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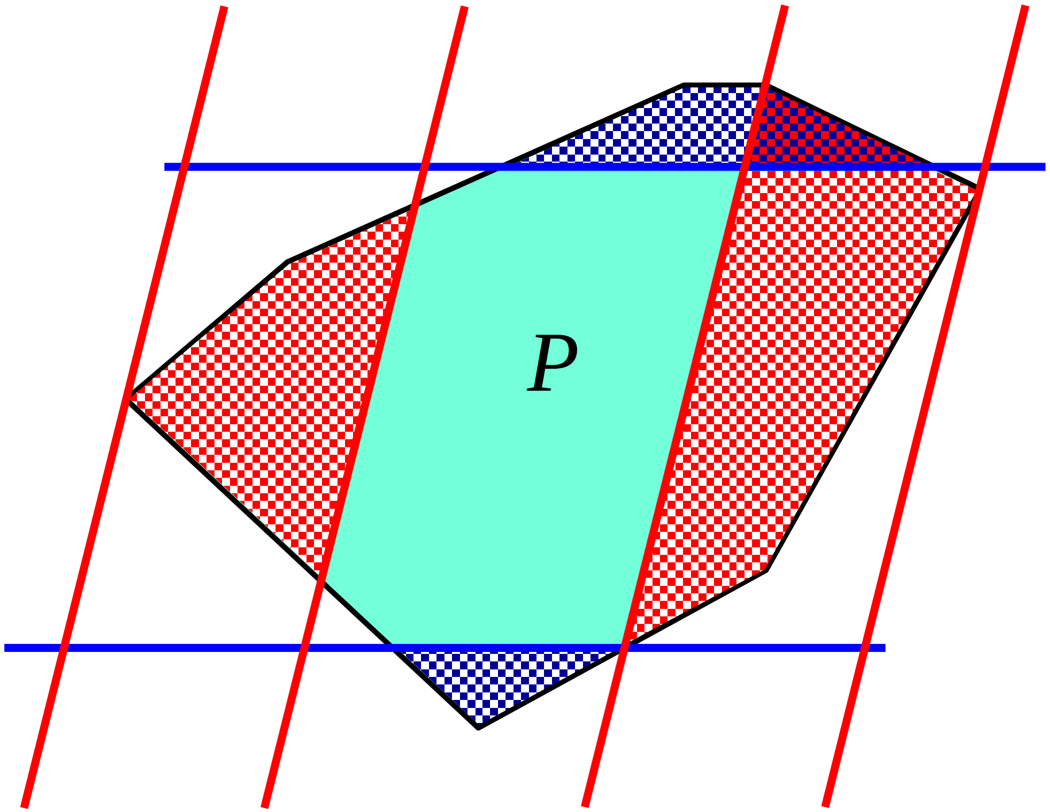

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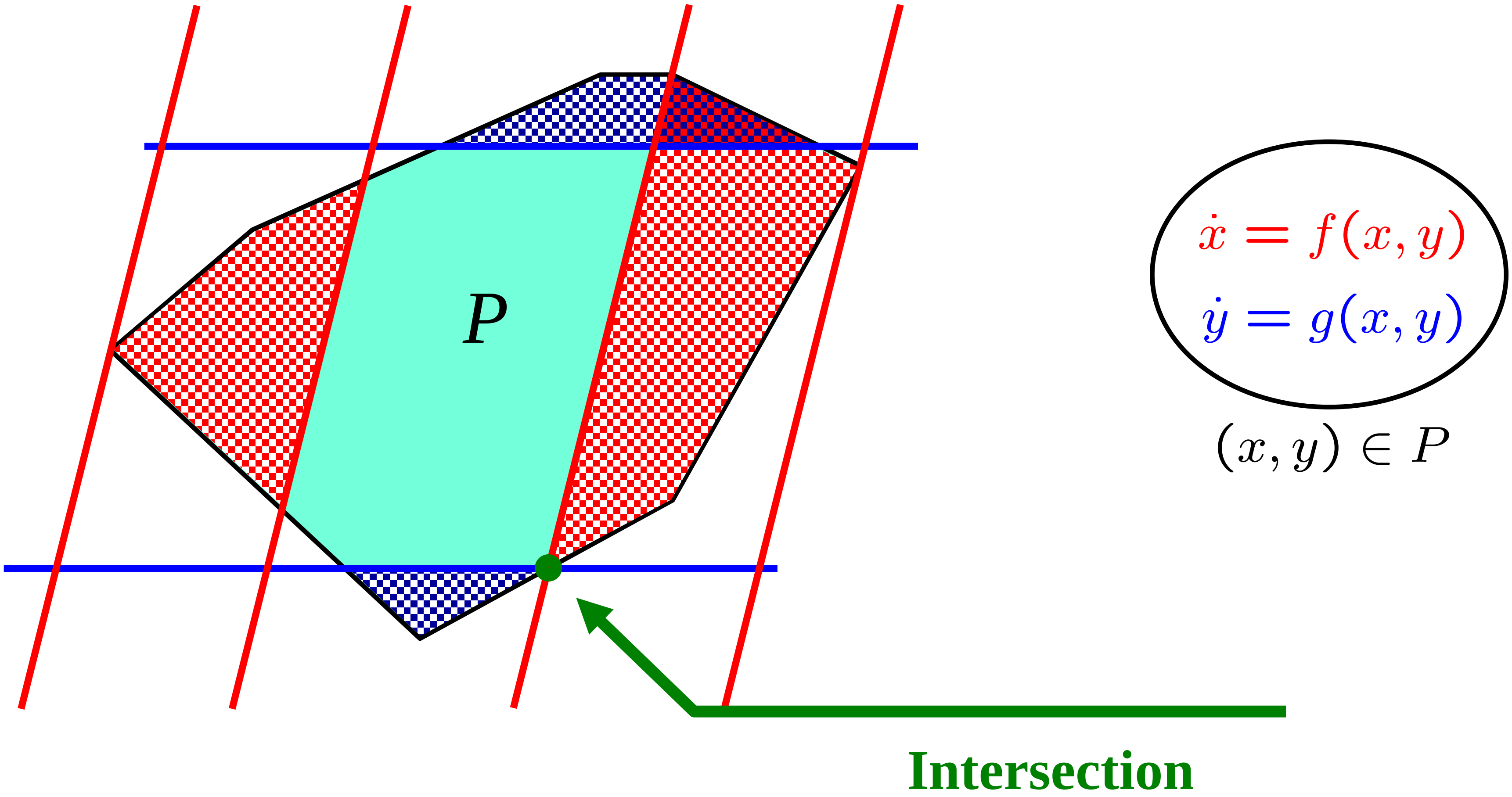
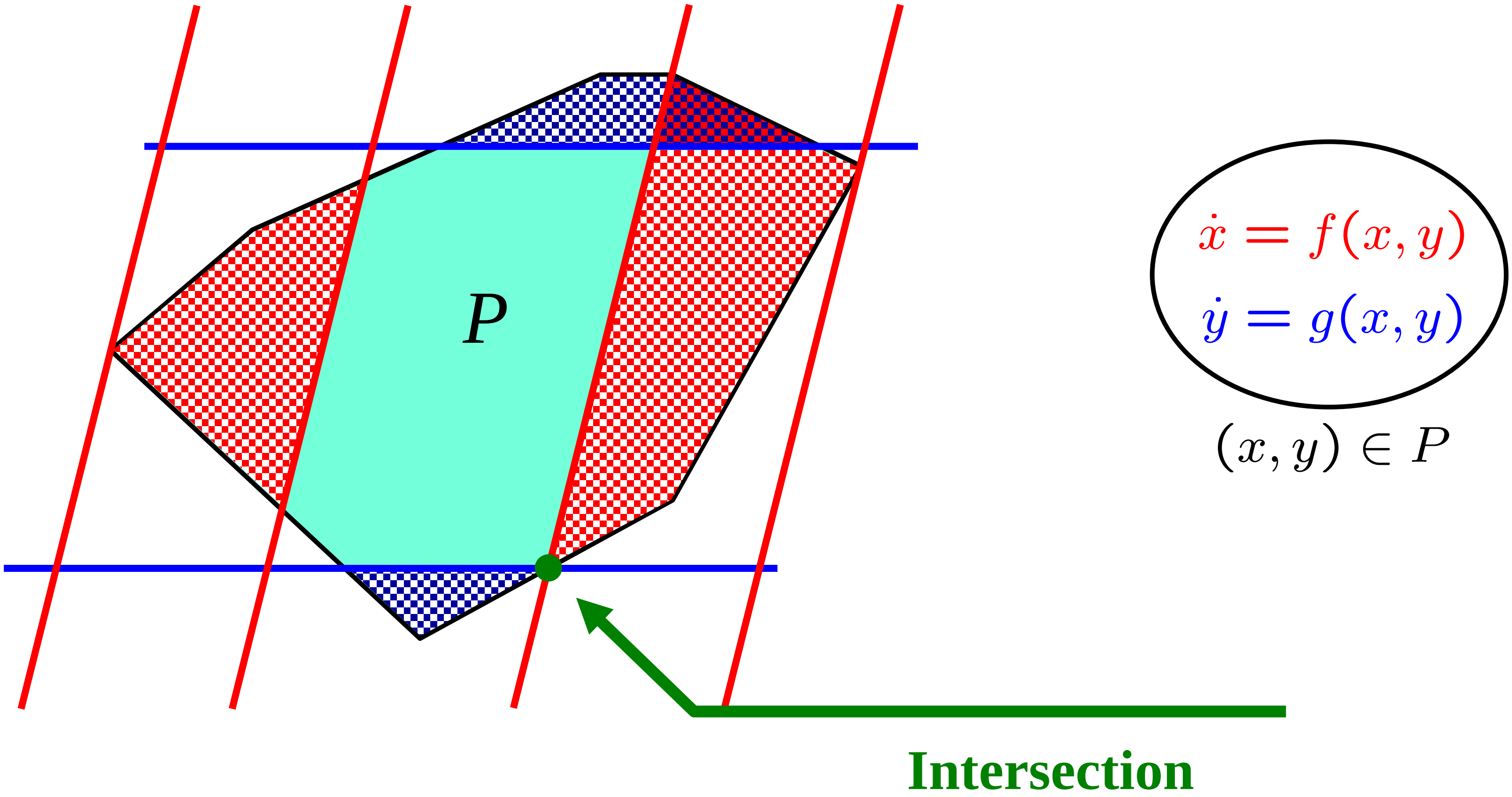


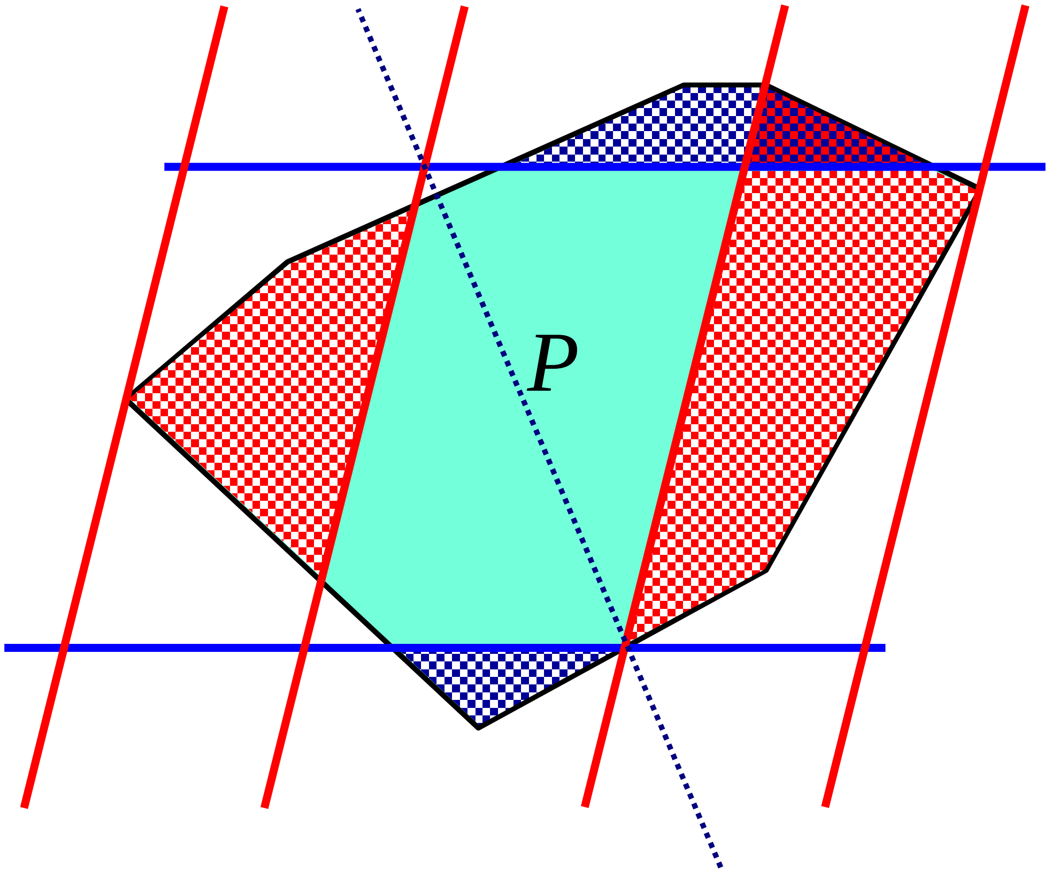
When a second intersection occurs...

Example



In this case, we have reached the "*limit of separability*"

Example

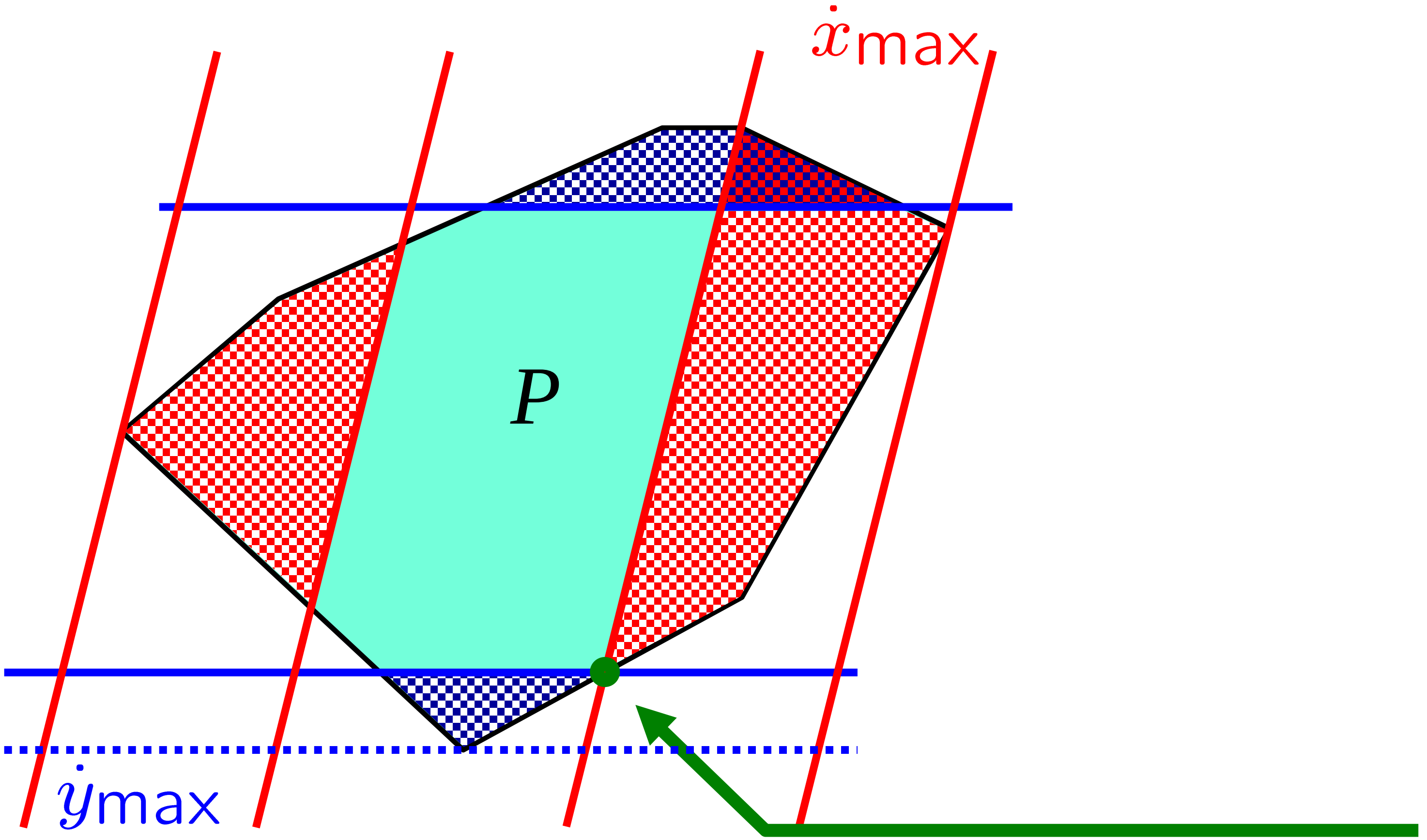


$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

$$(x, y) \in P$$

An optimal cut

How to compute the intersection ?



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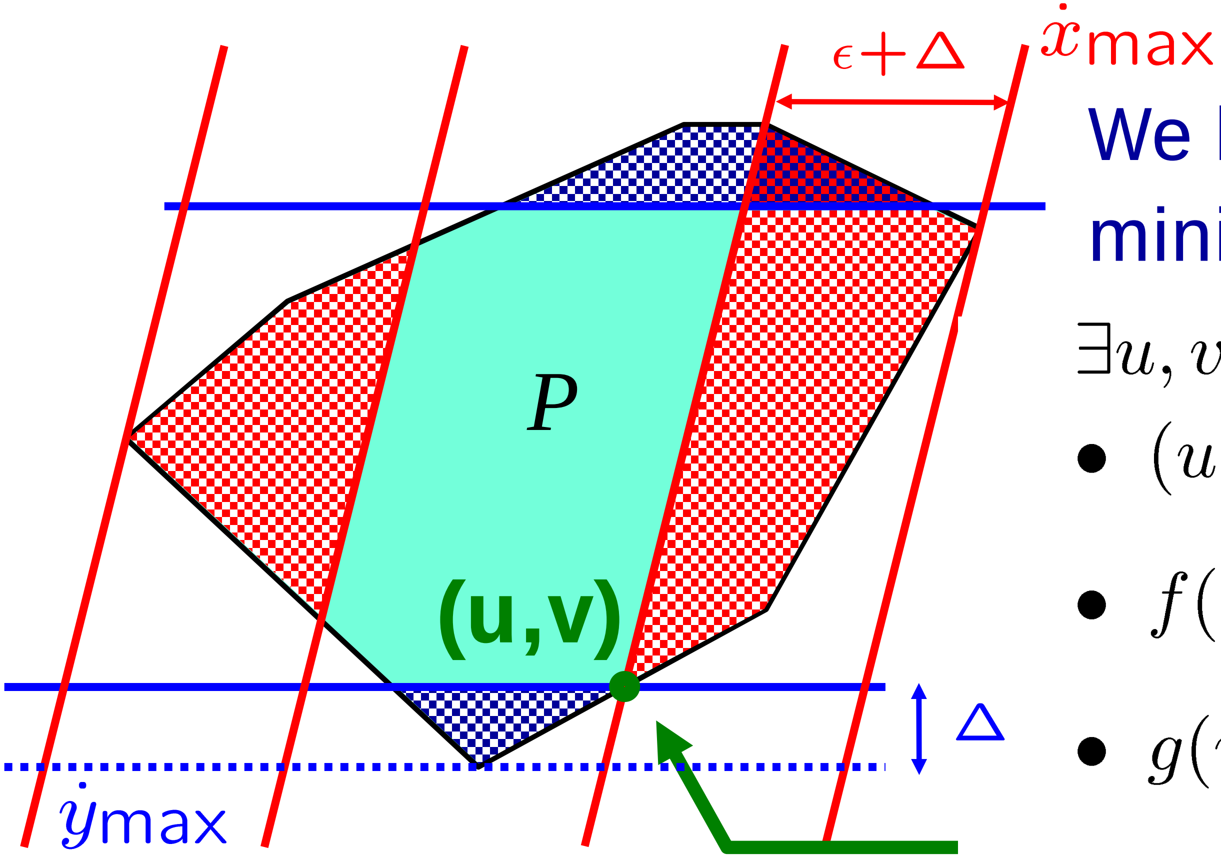

We have to find the minimal Δ such that:

$\exists u, v \in \mathbb{R}$:

- $(u, v) \in P$
- $f(u, v) = \dot{x}_{\max} - \epsilon - \Delta$
- $g(u, v) = \dot{y}_{\max} - \Delta$

where $\epsilon = r_x - r_y$

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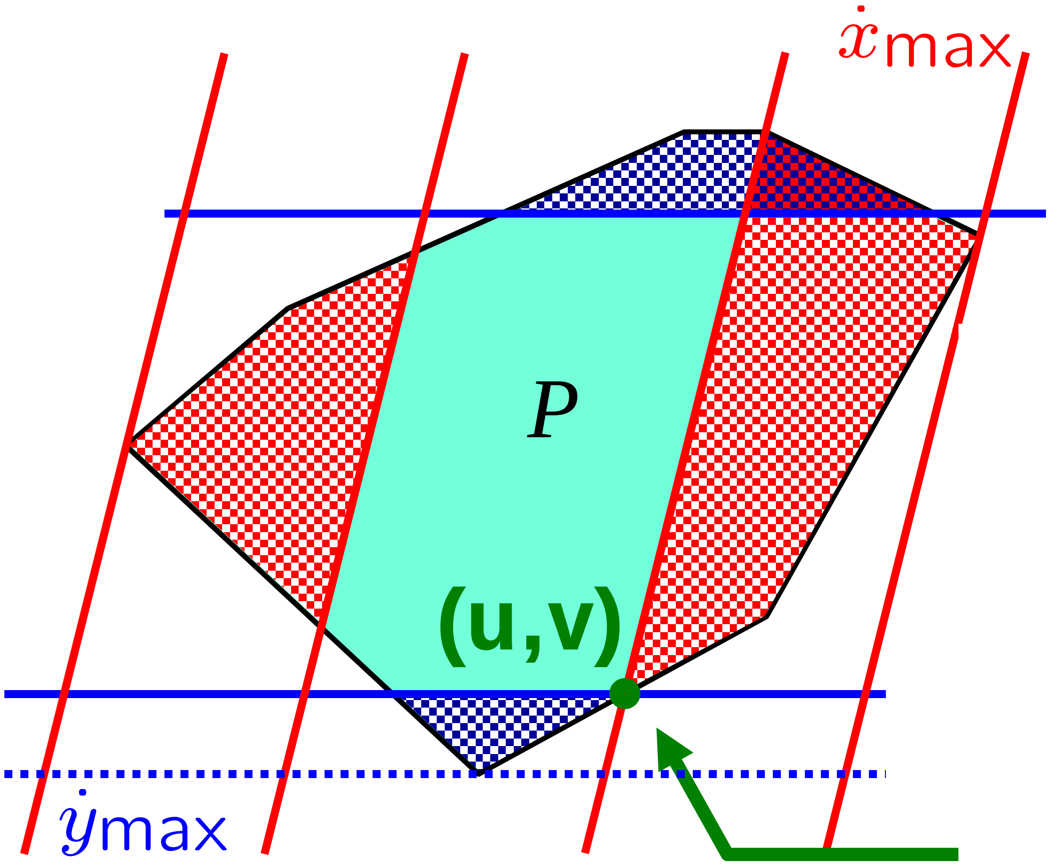

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➡ This is a linear program !

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  - Several particular cases

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  - An option: discretize the problem using a grid
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Algorithm 1: Algorithm OptimalCut for computing the optimal cut in the 2D.

Data : An instance $S = \langle P, F \rangle$ of the optimal cut problem such that $P \subset \mathbb{R}^2$ and $F = \{f_1, f_2\}$.

Result : A line that solves the optimal cut problem for S.

begin

- $[\dot{x}_{\min}, \dot{x}_{\max}] \leftarrow f_1(P)$; $[\dot{y}_{\min}, \dot{y}_{\max}] \leftarrow f_2(P)$;
- $r_x \leftarrow \dot{x}_{\max} - \dot{x}_{\min}$; $r_y \leftarrow \dot{y}_{\max} - \dot{y}_{\min}$;
- Assume wlog. that $r_y \geq r_x$;
- **if** $r_y \geq 2\, r_x$ **then return** $\ell \equiv f_2(x,y) = \dot{y}_{\min} + \frac{r_y}{2}$;
- $\Delta_0 \leftarrow r_y - r_x$;
- Let Δ be a symbolic parameter ;
- $a_\Delta \leftarrow f_2^{-1}(\dot{y}_{\min} + \Delta_0 + \Delta) \cap f_1^{-1}(\dot{x}_{\min} + \Delta)$;
- $b_\Delta \leftarrow f_1^{-1}(\dot{x}_{\min} + \Delta) \cap f_2^{-1}(\dot{y}_{\max} - \Delta_0 - \Delta)$;
- $c_\Delta \leftarrow f_2^{-1}(\dot{y}_{\max} - \Delta_0 - \Delta) \cap f_1^{-1}(\dot{x}_{\max} - \Delta)$;
- $d_\Delta \leftarrow f_1^{-1}(\dot{x}_{\max} - \Delta) \cap f_2^{-1}(\dot{y}_{\min} + \Delta_0 + \Delta)$;
- **for** $z = a$ ***to*** d **do** $\Delta_z \leftarrow \min\{\Delta \mid z_\Delta \in P\}$;
- $\Delta_1 \leftarrow \min(\Delta_a, \Delta_c)$; $\Delta_2 \leftarrow \min(\Delta_b, \Delta_d)$;
- **if** $\Delta_1 \geq \frac{r_y}{2} - \Delta_0 \vee \Delta_2 \geq \frac{r_y}{2} - \Delta_0$ **then return** $\ell \equiv f_2(x,y) = \dot{y}_{\min} + \frac{r_y}{2}$;
- $Q_{\min} \leftarrow P \cap f_1^{-1}(\dot{x}_{\min})$; $Q_{\max} \leftarrow P \cap f_1^{-1}(\dot{x}_{\max})$;
- **if** $f_2(Q_{\min}) \cap [\dot{y}_{\min}, \dot{y}_{\min} + \Delta_0] \neq \varnothing \wedge f_2(Q_{\min}) \cap [\dot{y}_{\max} - \Delta_0, \dot{y}_{\max}] \neq \varnothing$ **then**
  - **return** $\ell \equiv f_2(x,y) = \dot{y}_{\min} + \frac{r_y}{2}$;
- **else if** $f_2(Q_{\min}) \cap [\dot{y}_{\min}, \dot{y}_{\min} + \Delta_0] \neq \varnothing \neq \varnothing$ **then**
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  - **else**
    - **return** $\ell ine(b_{\Delta_2}, d_{\Delta_2})$;
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  - **return** $\ell ine(a_{\Delta_1}, c_{\Delta_1})$;
- **else if** $\Delta_1 > \Delta_2$ **then**
  - **return** $\ell ine(a_{\Delta_1}, c_{\Delta_1})$;
- **else**
  - **return** $\ell inc(b_{\Delta_2}, d_{\Delta_2})$;

end

Navigation benchmark

In each location, the dynamics has the form:

$$\begin{cases} \dot{x} = v \\ \dot{v} = \underbrace{A\,(v - v_d)}_{x_1 \text{ and } x_2 \text{ do not appear...}} \end{cases}$$

→ We cut in the plane v_1-v_2

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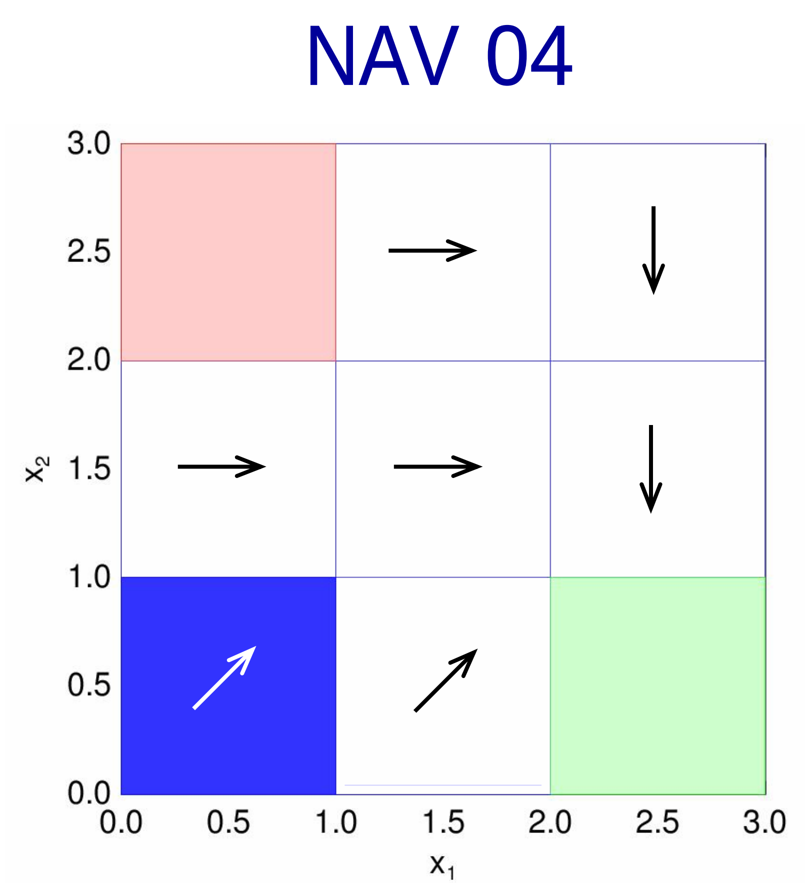
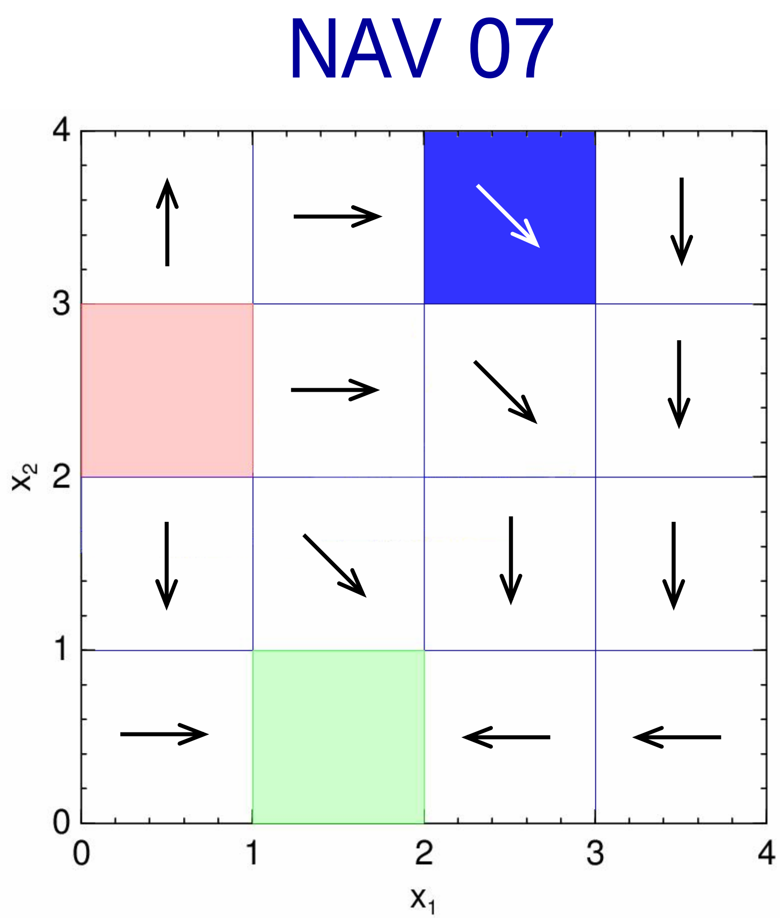
Instance	Grid	Time	(PT)
NAV01	3×3	5s	(35s)
NAV02	3×3	10s	(62s)
NAV03	3×3	10s	(62s)
NAV04	3×3	75s	(225s[i])
NAV07	4×4	11mn	

[i] obtained with a heuristic.

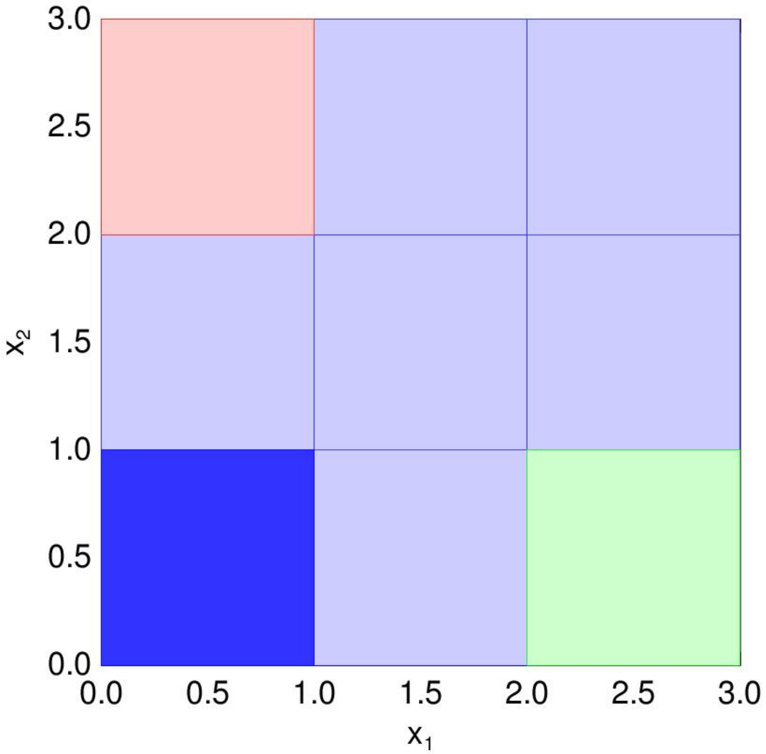
Results



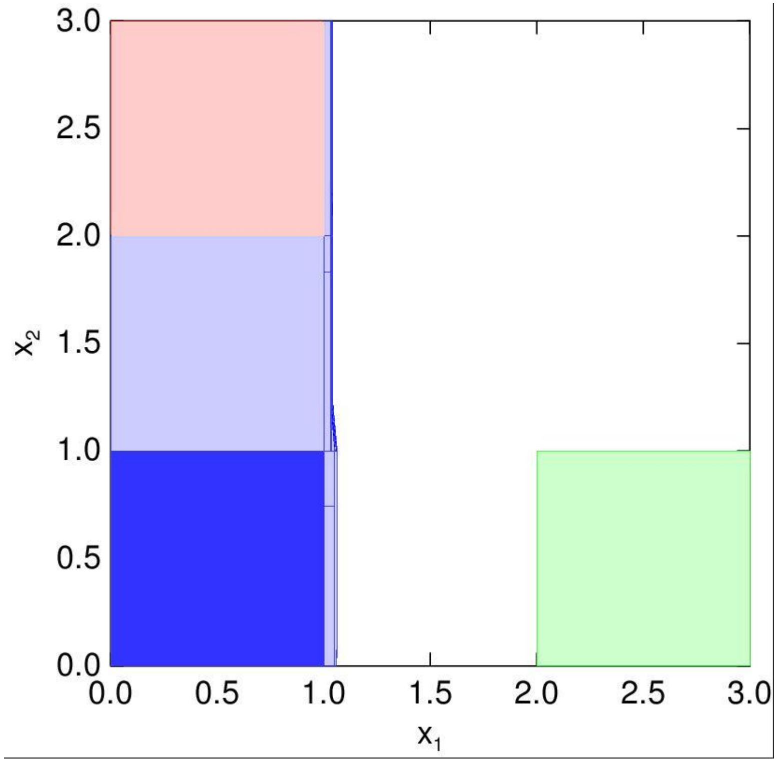


Initial states   Bad states   Good states

Results: NAV 04

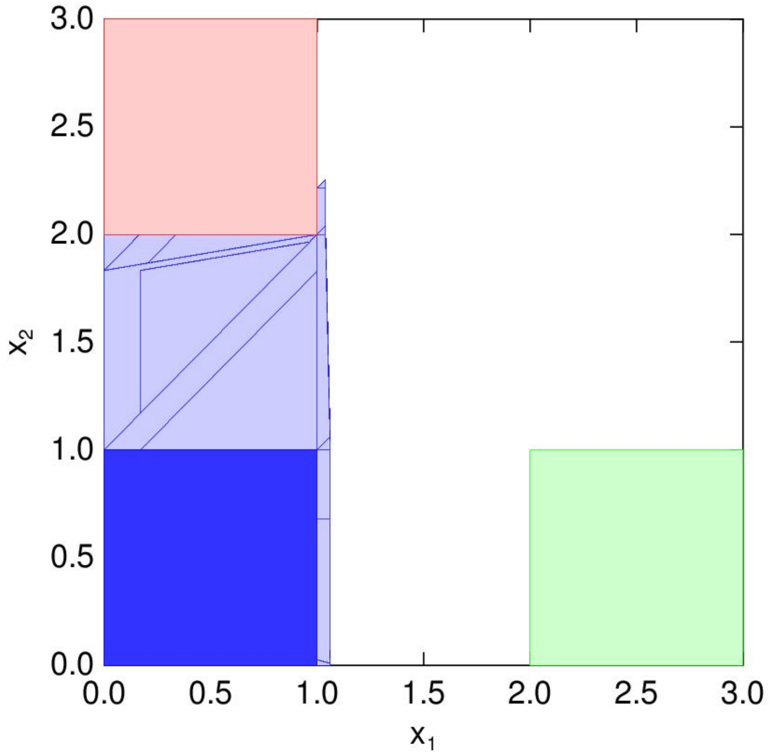


Forward

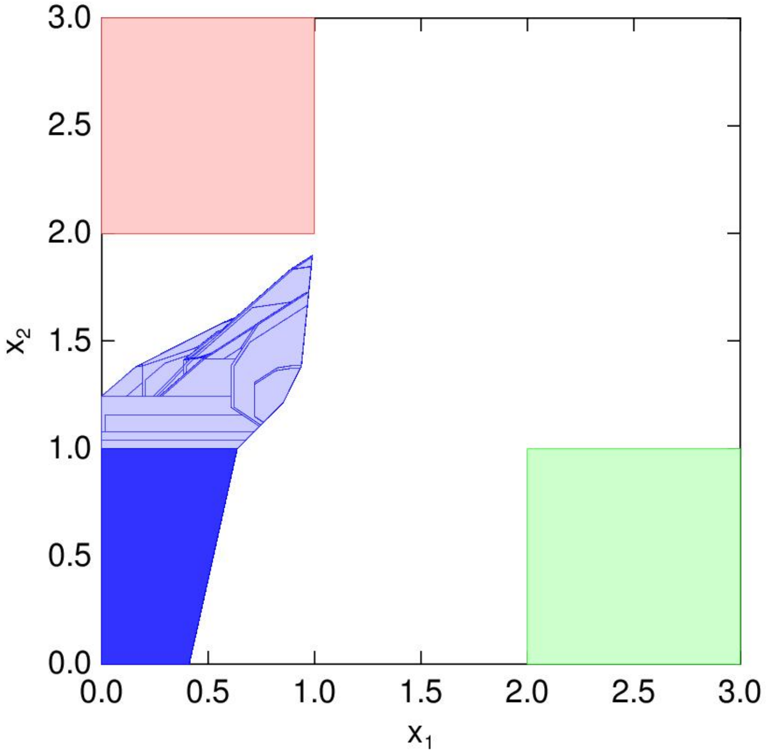


Backward

Results: NAV 04

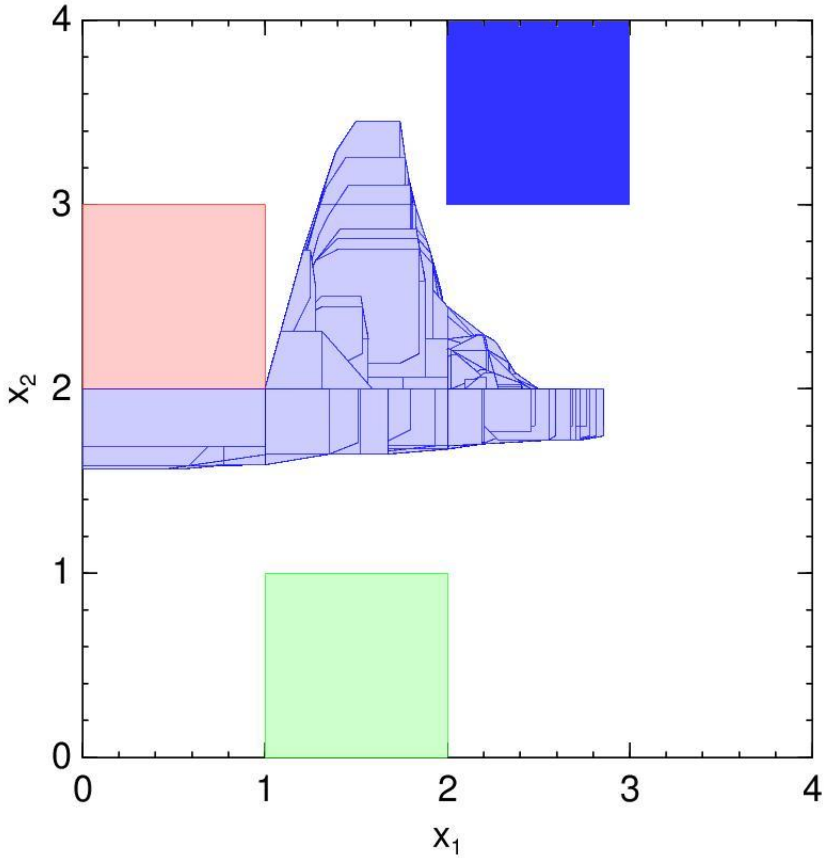


Forward



Forward

Results: NAV 07



Backward

Conclusion

- Approximations
  - Rectangular
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Conclusion

- Approximations
  - Rectangular
  - Over-approximations
- Refinements
  - Automatic
  - Optimal split for some criterion (at least in 2D)
- Possible future work
  - Under-approximations
  - Optimal split for some other criterion
  - Combine with other approaches (barrier certificates, ellipsoïds, ...)

References

- [FI04] A. Fehnker and F. Ivancic. *Benchmarks for hybrid systems verification*. In HSCC 2004, LNCS 2993, pp 326-341.
- [Fre05] G. Frehse. *Phaver: Algorithmic verification of hybrid systems past hytech*. In HSCC 2005, LNCS 3414, pp 258-273.