# Deciding the Winner in Parity Games Is in **UP** ∩ **co-UP**

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**Abstract**

We observe that the problem of *deciding the winner* in *mean payoff games* is in the complexity class **UP** ∩ **co-UP**. We also show a simple reduction from *parity games* to mean payoff games. From this it follows that deciding the winner in parity games and the *modal $\mu$-calculus model checking* are in **UP** ∩ **co-UP**.

**Keywords:** games on graphs, computational complexity.

## 1 Introduction

*Parity games* are infinite duration two-player games played on graphs. From the results of Emerson, Jutla and Sistla [EJS93] it follows that the problem of *deciding the winner* in parity games is equivalent via linear time reductions to the *modal $\mu$-calculus model checking*. (In fact, they show the equivalence of the modal $\mu$-calculus model checking and the *non-emptiness problem for automata on infinite trees* with *parity acceptance conditions* [Mos84, EJ91]. It is an easy exercise to show the equivalence of the non-emptiness problem for parity automata and the problem of deciding the winner in parity games.)

The modal $\mu$-calculus model checking problem is known to be in the complexity class **NP** ∩ **co-NP** [EJS93], so it is considered unlikely to be **NP**-complete. On the other hand no polynomial time algorithms for it have been found so far. The modal $\mu$-calculus is a powerful logic for specifying properties of finite state

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distributed and reactive systems. The question of existence of a polynomial time algorithm for the model checking problem has relevance to the automatic verification of such systems.

A nondeterministic Turing machine is called *unambiguous* if for every input it has at most one accepting computation. The complexity class **UP** is the class of problems recognizable by unambiguous polynomial time nondeterministic Turing machines (*e.g.*, see pages 283–284 in [Pap94]). In other words, **UP** is the class of problems having *unique* short certificates (where *short* means "of length polynomial in the size of the instance of the problem") checkable in polynomial time. In a standard fashion, by **co-UP** we denote the class of problems being the complements of problems in **UP**.

The main result of this note is that the problem of deciding the winner in parity games has *unique short certificates*, *i.e.*, it is in **UP**. To obtain the result claimed in the title we proceed in two steps. First we show a polynomial time reduction of parity games to the *mean payoff games* of Ehrenfeucht and Mycielski [EM79]. A polynomial time reduction of parity games to mean payoff games has been independently obtained by Puri [Pur95] and Jerrum (see [Sti95, Sti96]). Next, using results of Zwick and Paterson [ZP96], we argue that the problem of deciding the winner in mean payoff games is in **UP** $\cap$ **co-UP**.

Before we proceed with the technical part of this note let us comment on the relationship between the complexity classes **UP** $\cap$ **co-UP** and **NP** $\cap$ **co-NP**. Obviously **P** $\subseteq$ **UP** $\cap$ **co-UP** $\subseteq$ **NP** $\cap$ **co-NP**. There are relatively few (natural) problems known, which are in **NP** $\cap$ **co-NP**, and are not known to be in **P**. Interestingly, to our best knowledge, all problems appearing in literature known to be in **NP** $\cap$ **co-NP** are in fact in **UP** $\cap$ **co-UP** as well (*e.g.*, *primality* has been shown to belong to **UP** $\cap$ **co-UP** by Fellows and Koblitz [FK92]).

# 2 Games

We consider infinite duration games played by two players (player 0 and player 1) on finite graphs called arenas. An *arena* $A = (V, V_0, V_1, E)$ is a directed graph $(V, E)$ with a partition $V_0 \cup V_1 = V$ of the set of vertices $V$. We require that the out-degree of each vertex is at least one. This ensures that every finite path in $(V, E)$ can be prolonged.

A *position* in a *game* played on the arena $A$ is a finite path in the underlying graph $(V, E)$. Note that although the arena is finite, the set of positions $\mathrm{Pos}(V) \subseteq V^*$ in the game is infinite, because there are finite paths of arbitrary length in $(V, E)$. The *initial position* of a *play* starting from a vertex $v_0 \in V$ is the path consisting only of the vertex $v_0$. A *move* in a game consists of extending the *current position* by one vertex. The result of a move from a position $\pi_i = \langle v_0, v_1, v_2, \ldots, v_i \rangle$ is a new position $\pi_{i+1} = \langle v_0, v_1, v_2, \ldots, v_i, v_{i+1} \rangle$, *i.e.*, $(v_i, v_{i+1}) \in E$. Let $\pi_i = \langle v_0, v_1, v_2, \ldots, v_i \rangle$ be the current position in a play. If $v_i \in V_0$ then it is the turn of player 0 to make a move, otherwise player 1 moves. The players make their moves indefinitely. Note that a play never gets "stuck", because of our assumption that every vertex has at least one outgoing edge. The *result of a play* in the game is an infinite path $\pi = \langle v_0, v_1, v_2, \dots \rangle$ in the graph $(V, E)$. For brevity, we often just say a *play*, meaning the result of the play.

Formally, a *game* $G = (A, W)$ is an arena $A = (V, V_0, V_1, E)$, together with a *winning condition* $W \subseteq V^\omega$, specifying the set of *winning plays* for player 0. All the other plays are winning for player 1. A *strategy* for player 0 is a function $\zeta : \mathrm{Pos}(V) \to V$, which given the current position in the game specifies what should be the next move to be made by player 0. A strategy is *memoryless* if it does not depend on the whole current position but only on the *current vertex* (*i.e.*, the last vertex in the current position). Memoryless strategies are particularly simple and can be represented as functions $\sigma : V \to V$, *i.e.*, they are finite objects. A *winning strategy* for player 0 from a vertex $v_0 \in V_0$ is a strategy such that every play starting in $v_0$ which is consistent with the strategy is winning for player 0. (A play is *consistent* with a strategy if player 0 makes all her moves along this play according to the strategy.)

The decision problem we are after is: given a game $G$ and a starting vertex $v_0$, decide if player 0 has a winning strategy in $G$ from the vertex $v_0$.

We deal with three types of infinite duration games played on arenas: *parity games*, *mean payoff games* and *discounted mean payoff games*. They differ in the way their winning conditions are represented.

**Definition 1 (Parity game)** A *parity game* [EJ91, Mos91] is a pair $(A, p)$ where $A = (V, V_0, V_1, E)$ is an arena and $p : V \to \{0, 1, \dots, |V|\}$ is a function assigning a *priority* to every vertex of the arena. Let $\pi = \langle v_0, v_1, v_2, \dots \rangle$ be a play in a parity game. We define $\mathrm{Inf}(\pi)$ to be the set of all numbers appearing infinitely often in the sequence $\langle p(v_0), p(v_1), p(v_2), \dots \rangle$. The play $\pi$ is winning for player 0 if $\max\big(\mathrm{Inf}(\pi)\big)$ is even.

**Definition 2 (Mean-payoff game)** A *mean payoff game* [EM79] is a quadruple $(A, \nu, d, w)$, where $A = (V, V_0, V_1, E)$ is an arena, $\nu$ and $d$ are natural numbers, and $w : E \to \{-d, \dots, -1, 0, 1, \dots, d\}$ is a function assigning an integer *weight* to every edge of the arena. A play $\pi = \langle v_0, v_1, v_2, \dots \rangle$ is winning for player 0 if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} w(v_{i-1}, v_i) \geq \nu.$$

**Definition 3 (Discounted mean payoff game)** Let $G = (A, \nu, d, w)$ be a mean payoff game and $0 < \lambda < 1$ be a real number called the *discounting factor*. The *discounted mean payoff game* $G_\lambda$ has the same arena and weight function as $G$. The only difference is the winning condition. A play $\pi = \langle v_0, v_1, v_2, \dots \rangle$ is winning for player 0 in $G_\lambda$ if

$$(1-\lambda) \sum_{i=1}^{\infty} \lambda^i \cdot w(v_{i-1}, v_i) \geq \nu.$$

The main aim of this note is to show that deciding the winner in parity games is in **UP** $\cap$ **co-UP**. We have tried to provide explicit unique short certificates for winning strategies in parity games. One of the very promising candidates seemed to be *canonical signature assignments* studied by Walukiewicz [Wal96]. We have failed, however, to devise a polynomial time algorithm to *check* that a signature assignment is indeed canonical. Instead, we relate parity games to mean payoff games by providing in Section 3 a simple reduction from the former to the latter. Then in Section 4 we argue that winning strategies for (discounted) mean payoff games have unique short certificates.

# 3 The reduction

In this section we show a simple reduction of parity games to mean payoff games. Essentially the same reduction has been given by Puri [Pur95]. A reduction of parity games to mean payoff games has been also obtained by Jerrum [Sti96].

Our reduction relies on *memoryless determinacy* of both parity and mean payoff games. A game is *determined* if for every starting vertex exactly one of the players has a winning strategy. Memoryless determinacy states that if a winning strategy for one of the players exists, then there is also a memoryless winning strategy for her. Memoryless determinacy for parity games was shown by Mostowski [Mos91], and independently by Emerson and Jutla [EJ91], and for mean payoff games by Ehrenfeucht and Mycielski [EM79], and by Gurvich, Karzanov, and Khachiyan [GKK88].

**Theorem 4**
*The problem of deciding the winner in a parity game reduces in polynomial time to the problem of deciding the winner in a mean payoff game.*

**Proof**
Let $G = (A, p)$ be a parity game, where $A = (V, V_0, V_1, E)$ is the arena. Given $G$ and a starting vertex $v_0 \in V_0$ we construct a mean payoff game $H = (A, \nu, d, w)$, such that player 0 has a winning strategy from $v_0$ in $G$ if and only if she has a winning strategy from $v_0$ in $H$. Let $r = \max\{\ p(v) : v \in V\ \}$, and $m = |V|$. To define the mean payoff game $H$ we set $\nu = 0$, $d = m^r$ and for every edge $e = (u, v) \in E$ we let $w(e) = (-m)^{p(u)}$.

Observe that the arenas of the games $G$ and $H$ are the same, hence a (memoryless) strategy in $G$ is at the same time a (memoryless) strategy in $H$ and vice versa. Let $\sigma$ be a memoryless strategy for player 0 from vertex $v_0$ in game $G$ ($H$). By $G_\sigma$ ($H_\sigma$) we denote the subgame of $G$ ($H$) obtained by first removing from the arena $A$ all the edges coming out of vertices in $V_0$ except for those belonging to $\sigma$, and then by removing all the vertices unreachable from $v_0$ via the edges left. Plays in the game $G$ ($H$) starting in $v_0$ and consistent with the strategy $\sigma$ correspond exactly to the set of *all* plays in the game $G_\sigma$ ($H_\sigma$).

Below, by a *simple cycle* we mean a cycle with no repeating vertices. The following lemma is used to prove that the above construction is indeed a reduction from parity games to mean payoff games.

**Lemma 5** Let $\sigma$ be a memoryless strategy for player 0 in $G$ ($H$). Then:

1. for every simple cycle $c$ in $G_\sigma$ the highest priority of a vertex on $c$ is even if and only if the sum of the weights of the edges on $c$ is nonnegative,

2. if $\sigma$ is a winning strategy for player 0 from $v_0$ in $G$ then for every simple cycle $c$ in $G_\sigma$ the highest priority of a vertex appearing on $c$ is even,

3. if $\sigma$ is a winning strategy for player 0 from $v_0$ in $H$ then for every simple cycle $c$ in $H_\sigma$ the sum of the weights of the edges on $c$ is nonnegative.

**Proof:** The first clause is straightforward.

We prove the second clause; the proof of the third one is similar. Suppose that there exists a simple cycle $c$ with the highest priority of a vertex on $c$ being odd. Then player 1 can force the play from $v_0$ to $c$ and also to stay in $c$ indefinitely and thus win. This, however, contradicts our assumption that $\sigma$ is a winning strategy for player 0. □

Suppose that player 0 has a winning strategy from $v_0$ in the game $G$. Due to memoryless determinacy of parity games it follows that there is also a memoryless winning strategy $\sigma$ for player 0 from $v_0$ in $G$. We show that the memoryless strategy $\sigma$ is also a winning strategy for player 0 from $v_0$ in $H$. In order to do that we have to argue that for every play $\langle v_0, v_1, v_2, \dots \rangle$ consistent with the strategy $\sigma$ the following inequality holds:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} w(v_{i-1}, v_i) \geq 0.$$

We show now that every play $\pi = \langle v_0, v_1, v_2, \dots \rangle$ in $H_\sigma$ can be essentially decomposed into simple cycles. This decomposition allows us to apply Lemma 5 and get the above inequality. Consider the following decomposition process of the play $\pi$ in $H_\sigma$. We maintain a stack containing a sequence of distinct nodes forming a finite path $u_0, u_1, \dots, u_h$ in $H_\sigma$, where $h$ is the height of the stack. Whenever the next vertex from the path $\pi$ to be considered happens to be already on the stack, we remove the vertices forming the cycle from the top of the stack. Otherwise we push the new vertex onto the stack. Observe that due to clauses 1 and 2 of Lemma 5 whenever we remove a cycle from the top of the stack the sum of the weights of the edges on the cycle is nonnegative. In this way only the weights of the edges which are on the stack may sum up to a negative value. Hence:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} w(v_{i-1}, v_i) \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{h(n)} w(u_{i-1}, u_i) = 0,$$

because $h(n) \leq |V|$, so the absolute value of the sum $\sum_{i=1}^{h(n)} w(u_{i-1}, u_i)$ is bounded by a constant.

In order to finish the proof of the theorem we have to show the reverse, *i.e.*, if player 0 has a winning strategy from $v_0$ in $H$, then player 0 has also a winning strategy from $v_0$ in $G$. Again due to memoryless determinacy of mean payoff games we can restrict to memoryless strategies. Then it suffices to apply the same technique of decomposing plays in $G_\sigma$ into simple cycles and use clauses 1 and 3 of Lemma 5. $\square$

**Remark:** In the above reduction the weight function $w$ has values exponential with respect to the size of the parity game $G$, but their binary representations are of polynomial size and can be computed in polynomial time. Note that having a reduction with polynomial values of the weight function would imply that parity games are in **P**, as a polynomial time algorithm for mean payoff games with polynomial weights is given by Zwick and Paterson [ZP96].

# 4 The UP $\cap$ co-UP upper bound

Our aim now is to show that deciding the winner in mean payoff games is in **UP** $\cap$ **co-UP**. Let $G = (A, \nu, d, w)$ be a mean payoff game. Ehrenfeucht and Mycielski [EM79] and independently Gurvich, Karzanov, and Khachiyan [GKK88] have shown that for every vertex $v_0$ of the arena $A$ there exists a number $\nu(v_0)$, called the *value of* $G$ *in* $v_0$, such that the following two conditions hold:

1. player 0 has a memoryless strategy such that for every play $\langle v_0, v_1, v_2, \dots \rangle$ consistent with this strategy $\liminf_{n\to\infty} 1/n \sum_{i=1}^{n} w(v_{i-1}, v_i) \geq \nu(v_0)$,
2. player 1 has a memoryless strategy such that for every play $\langle v_0, v_1, v_2, \dots \rangle$ consistent with this strategy $\limsup_{n\to\infty} 1/n \sum_{i=1}^{n} w(v_{i-1}, v_i) \leq \nu(v_0)$.

We call strategies satisfying those conditions *optimal*. Clearly, given the values of a game it is straightforward to decide the winner — it suffices to check whether $\nu(v_0) \geq \nu$. Hence the values of a mean payoff game may seem to be a plausible candidate for unique short certificates. Unfortunately, we have failed to devise a polynomial time algorithm to check whether a vector $\langle x_v \rangle_{v \in V}$ is indeed the vector of the *values* of the game. Moreover, whereas it is straightforward to extract a winning strategy in a parity game from the canonical signature assignment [Wal96], here it is not at all clear to us how to do it efficiently given just the values of the mean payoff game.

In order to get the **UP** $\cap$ **co-UP** upper bound on the complexity of mean payoff games we use the following result of Zwick and Paterson [ZP96].

**Theorem 6 (Zwick and Paterson [ZP96])**
*The problem of deciding the winner in a mean payoff game reduces in polynomial time to the problem of deciding the winner in a discounted mean payoff game.*

Hence it suffices to provide unique short certificates for winning strategies in *discounted* mean payoff games. Zwick and Paterson [ZP96] have shown that *optimal memoryless strategies* for both players exist also in discounted mean payoff games. In the two conditions above one has to replace the term $1/n \sum_{i=1}^{n} w(v_{i-1}, v_i)$ with $(1-\lambda) \sum_{i=1}^{n} \lambda^i \cdot w(v_{i-1}, v_i)$, and the number $\nu(v_0)$

with $\nu_\lambda(v_0)$, which we call the *value of the discounted mean payoff game $G_\lambda$ in $v_0$*.

The vector of the values of a discounted mean payoff game is the unique certificate we are after. The crucial property which allows us to quickly check whether some $\langle x_v \rangle_{v \in V}$ is the vector of the values of a discounted mean payoff game is the following characterisation due to Zwick and Paterson [ZP96].

**Theorem 7 (Zwick and Paterson [ZP96])**
*The vector $\overline{\nu} = \langle \nu_\lambda(v) \rangle_{v \in V}$ of the values of the discounted mean payoff game $G_\lambda$ is the* unique solution *of the following system of equations*

$$x_v = \begin{cases} \max_{(v,u) \in E} \left\{ (1-\lambda) \cdot w(v,u) + \lambda x_u \right\} & \text{if } v \in V_0, \\ \min_{(v,u) \in E} \left\{ (1-\lambda) \cdot w(v,u) + \lambda x_u \right\} & \text{if } v \in V_1. \end{cases}$$

It only remains to show that the values of a discounted mean payoff games are *short*, *i.e.*, can be written using a number of bits polynomial in $N = |G_\lambda|$, the size of the binary representation of the game $G_\lambda$. We prove this by applying a standard technique; the proof sketch below closely resembles, *e.g.*, the proof of Lemma 2 in [Con92]. Let $\overline{x}$ be the only solution of the system of equations from Theorem 7. Then

$$\overline{x} = (1-\lambda) \cdot \overline{w} + \lambda \cdot Q \cdot \overline{x}$$

where $\overline{w}$ is a vector of appropriate weights $w(u,v)$ and $Q$ is a zero-one matrix with exactly one non-zero element in every row. We assume that the constant $\lambda$ is a rational included in the binary representation of the game $G_\lambda$, hence it is a number $a/b$, with $a$ and $b$ being integers satisfying $\log a, \log b < N$. The above equation can be hence rewritten as

$$A \cdot \overline{x} = (b-a) \cdot \overline{w}$$

where $A = b \cdot I - a \cdot Q$, and $I$ is the $|V| \times |V|$ identity matrix. Observe that $A$ is a matrix with at most two non-zero elements in a row, and their absolute values are not bigger than $2^N$. It is easy to show by induction on the size of the matrix that the absolute value of the determinant of $A$ is not bigger than $4^{N|V|}$. According to Cramer's rule we have $x_v = \det A_v / \det A$, where $A_v$ is obtained from $A$ by replacing the $v$-th column with the vector $(b-a) \cdot \overline{w}$. From this it follows that the elements of the vector $\overline{x}$ can be written using a polynomial number of bits.

This settles the **UP** upper bound for the problem of deciding the winner in discounted mean payoff games. The **co-UP** upper bound follows easily, because by the existence of optimal strategies [ZP96] player 0 does *not* have a winning strategy from a vertex $v_0$ in the game $G_\lambda$ if and only if $\nu_\lambda(v_0) < \nu$, so it can be directly read from the vector of the values of the game.

Putting the **UP** $\cap$ **co-UP** upper bound for discounted mean payoff games, together with the reductions of Theorems 4 and 6, we get our main result.

**Theorem 8**
*The problems of deciding the winner in* parity*,* mean payoff *and* discounted mean payoff games *are in* **UP** $\cap$ **co-UP**.

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