Decidability, Introduction Rules, and Automata

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Joint work with Ying Jiang

Proof systems for pushdown systems, CTL...

- Proofs are certificates (that can be independenly rechecked)
- Use general proof-search engines to solve model-checking problems (Ji's thesis)
- Use proof theory tools (cut elimination...) to prove properties (decidability...) of pushdown system reachability, CTL...
- Separate the definition of truth in CTL from algorithms to decide it
- Unify terminology
- New problems and new insight for proof theory

A problem in proof theory

We have a notion of cut for Natural deduction, one for Sequent calculus...

But given a proof system (a set of deduction rules) can we define a notion of cut?

In Natural deduction

$$\frac{A \qquad B}{A \wedge B}\ \wedge\text{-intro} \qquad\qquad \frac{A \wedge B}{A}\ \wedge\text{-elim}$$

$$\dfrac{\dfrac{\dfrac{\pi_1}{A} \qquad \dfrac{\pi_2}{B}}{A \wedge B}\ \wedge\text{-intro}}{A}\ \wedge\text{-elim}$$

reduces to π_1

Introduction rule (in general)

Inference systems on a set S (propositions, sequents, ...)
A well-founded order $\prec$ on S

$$\frac{s_1 \; \dots \; s_n}{s'}$$

is an introduction rule if $s_1 \prec s'$, ..., $s_n \prec s'$

Example

$$\frac{A \qquad B}{A \wedge B} \wedge\text{-intro} \qquad\qquad \frac{A \wedge B}{A} \wedge\text{-elim}$$

Automaton (in general)

An automaton: a (finitely branching) inference system containing introduction rules only

Provability decidable: finite search space

An example

$$\mathit{odd} \xrightarrow{a} \mathit{even} \qquad \mathit{even} \xrightarrow{a} \mathit{odd} \qquad \mathit{even}\ \text{final}$$

$$\frac{\mathit{even}(x)}{\mathit{odd}(a(x))} \qquad \frac{\mathit{odd}(x)}{\mathit{even}(a(x))} \qquad \frac{}{\mathit{even}(\varepsilon)}$$

aaa recognized in *odd*
$\mathit{odd}(a(a(a(\varepsilon))))$ provable

Any finite state automaton

Cut (in general)

$$\dfrac{\dfrac{\cdots}{s_1}\ R_1\ \text{(intro)} \qquad \ldots \qquad \dfrac{\cdots}{s_n}\ R_n\ \text{(intro)}}{s'}\ R'\ \text{(non-intro)}$$

Cut-elimination

π cut-free: no cuts in π

cut-elimination: every proof transformed into a cut-free proof

A theorem: last rule (and all the other rules) property

A cut-free proof has introduction rules only

Induction over proof structure

$$\cfrac{\cfrac{\pi_1}{s_1} \quad \ldots \quad \cfrac{\pi_n}{s_n}}{s'} \; R'$$

π_1, ..., π_n introduction rules only, cut-free: R' introduction rule

Corollary: cut-elimination $\longrightarrow$ automaton $\longrightarrow$ decidability
It's a bit strong, isn't it?

Saturation

$$\frac{r(x)}{q(a(x))}\text{ intro} \qquad \frac{q(x)}{p(x)}\text{ non-intro} \qquad \cfrac{\cfrac{\cfrac{\dots}{r(b(\varepsilon))}}{q(a(b(\varepsilon)))}\text{ intro}}{p(a(b(\varepsilon)))}\text{ non-intro}$$

No cut-elimination?

Add a (derived) rule

$$\frac{r(x)}{p(a(x))}\text{ intro}$$

If saturation terminates: cut-elimination in the resulting system
An example: Kunth-Bendix saturation

Another example: Finite domain logic

Customize Natural deduction to a specific finite model

$$\frac{}{\Gamma \vdash P(c_{i_1}, ..., c_{i_m})} \text{ if } \langle c_{i_1}, ..., c_{i_m} \rangle \in \hat{P}$$

$$\frac{}{\Gamma \vdash \neg P(c_{i_1}, ..., c_{i_m})} \text{ if } \langle c_{i_1}, ..., c_{i_m} \rangle \not\in \hat{P}$$

$$\frac{\Gamma \vdash (c_1/x)A \quad ... \quad \Gamma \vdash (c_n/x)A}{\Gamma \vdash \forall x\ A}$$

Same for $\exists$-elim rule

B has a proof iff B is valid in $\mathcal{M}$ iff B has a cut-free proof

Cut-elimination $\longrightarrow$ automaton (drop the elimation rules) $\longrightarrow$ decidability

Proof search in this system = model checking

$$
\begin{array}{cc}
P(c_{i_1}, ..., c_{i_m}) \hookrightarrow \varnothing \text{ if } ... & A \vee B \hookrightarrow \{A\} \\
\neg P(c_{i_1}, ..., c_{i_m}) \hookrightarrow \varnothing \text{ if } ... & A \vee B \hookrightarrow \{B\} \\
\top \hookrightarrow \varnothing & \forall x\ A \hookrightarrow \{(c_1/x)A, ..., (c_n/x)A\} \\
A \wedge B \hookrightarrow \{A, B\} & \exists x\ A \hookrightarrow \{(c_i/x)A\}
\end{array}
$$

Another example: (alternating) Pushdown systems

$$\frac{q(x)}{p(a(x))}\ \text{pop} \qquad \frac{q(x)}{p(x)}\ \text{neutral} \qquad \frac{q(a(x))}{p(x)}\ \text{push}$$

pop + neutral $\longrightarrow$ pop
pop + push $\longrightarrow$ neutral

Termination of saturation (finite number of possible rules) $\longrightarrow$ cut-elimination $\longrightarrow$ automaton $\longrightarrow$ decidability

Another example: (constructive) Natural deduction

Undecidable: no hope to have a cut-elimination result

Wait, wait, ... Natural deduction does have cut-elimination, right?

(General) introduction rules and cuts in Natural deduction

Introduction rules, elimination rules, and the axiom rule
A new type of cut

$$\cfrac{\cfrac{}{P \wedge Q \vdash P \wedge Q}\ \text{axiom}}{P \wedge Q \vdash P}\ \wedge\text{-elim}$$

All cuts can be eliminated, except the axiom-elim cuts

Cut-free: introduction rules as long as the context is empty

$$\cfrac{\cfrac{\cfrac{\dots}{\vdash A_i}}{\vdash A_1 \vee A_2}\ \vee\text{-intro}}{\vdash \forall x\ (A_1 \vee A_2)}\ \forall\text{-intro}$$

Last rule property, Girard's shocking equality

Saturation?

$$\dfrac{\overline{\Gamma, A \wedge B \vdash A \wedge B}\ \text{axiom}}{\Gamma, A \wedge B \vdash A}\ \wedge\text{-elim}$$

Add a rule (that takes left-hand side into account)

$$\overline{\Gamma, A \wedge B \vdash A}$$

But would not terminate, instead

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

Sequent calculus

Contraction

But an axiom could be used several times: contraction

$$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

Only introduction rules? No: contraction is not
Not possible anyway (undecidability)

More cuts

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma, A \Rightarrow B \vdash A} \qquad \dfrac{\dots}{\Gamma, A \Rightarrow B, B \vdash G}}{\Gamma, A \Rightarrow B, A \Rightarrow B \vdash G}\ \Rightarrow\text{-left}}{\Gamma, A \Rightarrow B \vdash G}\ \text{contraction}$$

and more saturation

$$\dfrac{\Gamma, A \Rightarrow B \vdash A \qquad \Gamma, B \vdash G}{\Gamma, A \Rightarrow B \vdash G}\ \text{contr-}\Rightarrow\text{-left}$$

Kleene style Sequent calculus, but still not an introduction rule

More cuts

$$\dfrac{\dfrac{\Gamma, (C \Rightarrow D) \Rightarrow B, C \vdash D}{\Gamma, (C \Rightarrow D) \Rightarrow B \vdash C \Rightarrow D} \Rightarrow\text{-right} \qquad \Gamma, B \vdash G}{\Gamma, (C \Rightarrow D) \Rightarrow B \vdash G} \text{contr-}\Rightarrow\text{-left}$$

and more saturation

$$\dfrac{\Gamma, (C \Rightarrow D) \Rightarrow B, C \vdash D \quad \Gamma, B \vdash G}{\Gamma, (C \Rightarrow D) \Rightarrow B \vdash G}$$

that simplifies to

$$\dfrac{\Gamma, D \Rightarrow B, C \vdash D \qquad \Gamma, B \vdash G}{\Gamma, (C \Rightarrow D) \Rightarrow B \vdash G}$$

Dyckhoff, *et al.* style Sequent calculus

Decidability

Of larger and larger fragments

- Natural deduction and Gentzen style Sequent calculus: no negative connective and quantifier
- Kleene style: no negative and $\Rightarrow$, $\forall$
- Dyckhoff, *et al.* style: all connectives, shallow universal and existential quantifiers negative existential quantifiers

Conclusion

General notion of introduction rule, automaton, cut, cut-elimination beyond proof theory, automata theory...

Decidability results

The study of inference per se