A constructive proof of Skolem theorem for constructive logic

Joint work with Benjamin Werner

The theorem you want

If f does not occur in Δ and A then

if

$$\Delta, \forall x_1 ... \forall x_n \, (f(x_1, ..., x_n)/y)A \vdash \bot \quad \text{has a proof}$$

then

$$\Delta, \forall x_1 ... \forall x_n \exists y \; A \vdash \bot \quad \text{has a proof}$$

A typical example

$$\forall x \exists y \, (x * y = e)$$

$$\forall x \, (x * inv(x) = e)$$

The theorem you have

If

- f does not occur in Γ, A, and B
- $\Gamma \vdash \forall x_1 ... \forall x_n \exists y\ A$ has a proof
-

$$\Gamma, \forall x_1, ..., \forall x_n\ (f(x_1, ..., x_n)/y)A \vdash B \quad \text{has a proof}$$

then

$$\Gamma \vdash B \quad \text{has a proof}$$

Hence $\Gamma, \forall x_1, ..., \forall x_n\ (f(x_1, ..., x_n)/y)A$ conservative extension of Γ
(that it is an extension is trivial: weakening Lemma)

Apply it to $B = \bot$ and $\Gamma = \Delta, \forall x_1 ... \forall x_n \exists y\ A$

Almost what you wanted

If

- f does not occur in Δ, ~~$\forall x_1 ... \forall x_n \exists y\ A$~~, A, ~~and $\bot$~~
- ~~$\Delta, \forall x_1 ... \forall x_n \exists y\ A \vdash \forall x_1 ... \forall x_n \exists y\ A$ has a proof~~
- $$\Delta, \forall x_1 ... \forall x_n \exists y\ A, \forall x_1, ..., \forall x_n\ (f(x_1, ..., x_n)/y)A \vdash \bot \quad \text{has a proof}$$

then

$$\Delta, \forall x_1 ... \forall x_n \exists y\ A \vdash \bot \quad \text{has a proof}$$

But as $\forall x_1 ... \forall x_n \exists y\ A$ is a consequence of $\forall x_1, ..., \forall x_n\ (f(x_1, ..., x_n)/y)A$ you can drop it
Still an extension (and still conservative)

Why we did not publish the paper

Shoji Maehara published in 1955 a similar method: eliminate Hilbert's choice operator ε

$$\text{Skolem} \simeq \text{Hilbert}$$

Yet:

- our theorem applies constructive rather than classical predicate logic
- our theorem uses natural deduction rather than sequent calculus

minor differences

What is missing in the paper

Motivations

An example

An example

$$\Gamma = \qquad \forall x \exists y\ (x < y), \qquad \forall w\ \neg(g(0) < w)$$

Skolemize

$$\forall x\ (x < f(x))$$

Theorem: if

$$\Gamma, \forall x\ (x < f(x)) \vdash \bot \quad \text{has a proof}$$

then

$$\Gamma \vdash \bot \quad \text{has a proof}$$

Note we keep the (superfluous) $\forall x \exists y\ (x < y)$ in the hypothesis

Transform the proof

$$
\dfrac{
\dfrac{
\dfrac{}{\Gamma, \forall x\, (x < f(x)) \vdash \forall w\, \neg(g(0) < w)}\,\text{ax}
}{
\Gamma, \forall x\, (x < f(x)) \vdash \neg(g(0) < f(g(0)))
}\,\forall\text{-e}
\qquad
\dfrac{
\dfrac{}{\Gamma, \forall x\, (x < f(x)) \vdash \forall x\, (x < f(x))}\,\text{ax}
}{
\Gamma, \forall x\, (x < f(x)) \vdash g(0) < f(g(0))
}\,\forall\text{-e}
}{
\Gamma, \forall x\, (x < f(x)) \vdash \bot
}\,\neg\text{-e}
$$

into

$$
\dfrac{
\dfrac{
\dfrac{}{\Gamma \vdash \forall x \exists y\, (x < y)}\,\text{ax}
}{
\Gamma \vdash \exists y\, (g(0) < y)
}\,\forall\text{-e}
\qquad
\dfrac{
\dfrac{
\dfrac{}{\Gamma, g(0) < y \vdash \forall w\, \neg(g(0) < w)}\,\text{ax}
}{
\Gamma, g(0) < y \vdash \neg(g(0) < y)
}\,\forall\text{-e}
\qquad
\dfrac{}{\Gamma, g(0) < y \vdash g(0) < y}\,\text{ax}
}{
\Gamma, g(0) < y \vdash \bot
}\,\neg\text{-e}
}{
\Gamma \vdash \bot
}\,\exists\text{-e}
$$

Partial instance, total instance, frozen

Partial instance $\forall x_{i+1} ... \forall x_n \; (f(t_1, ..., t_i, x_{i+1}, ..., , x_n)/y)A$
Total instance: $(f(t_1, ..., t_n)/y)A$
u frozen subterm of C if all variable occurrences of u are free in C

Two consequences of the subformula property

- If all f-terms are frozen in $\Gamma \vdash B$, then all f-terms are also frozen in a cut free proof of this sequent
- If all f-terms are frozen in Γ and B, then in a cut free proof of $\Gamma, \forall x_1 ... \forall x_n \; (f(x_1, ..., x_n)/y)A \vdash B$, any proposition C is either a partial instance or all f-terms are frozen in C

Transform the proof

By induction over proof structure
But to do what exactly?

- You start with a proof of

$$\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B$$

  where in Γ and B all the f-terms are frozen and $f(\overline{u}_1)$, ..., $f(\overline{u}_k)$ are the f-terms occurring in Γ and B
- You end with a proof of $\Gamma, \Delta \vdash B$ where Δ is $(\overline{u}_1/\overline{x}, f(\overline{u}_1)/y)A, \ldots (\overline{u}_k/\overline{x}, f(\overline{u}_k)/y)A$

The example, step by step

$$\dfrac{\dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash \neg(g(0) < f(g(0)))}\ \forall\text{-e} \qquad \dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall x\,(x < f(x))}\ \text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash g(0) < f(g(0))}\ \forall\text{-e}}{\Gamma, \forall x\,(x < f(x)) \vdash \bot}\ \neg\text{-e}$$

Step 1:

$$\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}$$

transforms into

$$\overline{\Gamma \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}$$

The example, step by step

$$\dfrac{\dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash \neg(g(0) < f(g(0)))}\ \forall\text{-e} \qquad \dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall x\,(x < f(x))}\ \text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash g(0) < f(g(0))}\ \forall\text{-e}}{\Gamma, \forall x\,(x < f(x)) \vdash \bot}\ \neg\text{-e}$$

Step 2:

$$\dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash \neg(g(0) < f(g(0)))}\ \forall\text{-e}$$

transforms into

$$\dfrac{\overline{\Gamma, g(0) < f(g(0)) \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}}{\Gamma, g(0) < f(g(0)) \vdash \neg(g(0) < f(g(0)))}\ \forall\text{-e}$$

Note the use of weakening

The example, step by step

$$\dfrac{\dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall w\,\neg(g(0) < w)}\;\text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash \neg(g(0) < f(g(0)))}\;\forall\text{-e} \qquad \dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall x\,(x < f(x))}\;\text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash g(0) < f(g(0))}\;\forall\text{-e}}{\Gamma, \forall x\,(x < f(x)) \vdash \bot}\;\neg\text{-e}$$

Step 3 (Focus: the only non trivial step):

$$\dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall x\,(x < f(x))}\;\text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash g(0) < f(g(0))}\;\forall\text{-e}$$

We want a proof of $\Gamma, g(0) < f(g(0)) \vdash g(0) < f(g(0))$

Just use the axiom rule

Works because the conclusion is a total instance of the Skolemized axiom $\forall \overline{x}\,(f(\overline{x})/y)A$

Uses the fact that the proof is cut free, the subformula property and Lemma 3

The example, step by step

$$\dfrac{\dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash \neg(g(0) < f(g(0)))}\ \forall\text{-e} \qquad \dfrac{\overline{\Gamma, \forall x\,(x < f(x)) \vdash \forall x\,(x < f(x))}\ \text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash g(0) < f(g(0))}\ \forall\text{-e}}{\Gamma, \forall x\,(x < f(x)) \vdash \bot}\ \neg\text{-e}$$

Step 4: the full proof

$$\dfrac{\dfrac{\overline{\Gamma, g(0) < f(g(0)) \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}}{\Gamma, g(0) < f(g(0)) \vdash \neg(g(0) < f(g(0)))}\ \forall\text{-e} \qquad \overline{\Gamma, g(0) < f(g(0)) \vdash g(0) < f(g(0))}\ \text{ax}}{\Gamma, g(0) < f(g(0)) \vdash \bot}\ \neg\text{-e}$$

Trivial: just apply the same $\neg$-e rule

But an unwanted hypothesis $g(0) < f(g(0))$ (because $f(g(0))$ is neither in Γ nor in B)

Need to eliminate it

Eliminating extra hypotheses

$$\dfrac{\dfrac{\overline{\Gamma, g(0) < f(g(0)) \vdash \forall w\, \neg(g(0) < w)}\;\text{ax}}{\Gamma, g(0) < f(g(0)) \vdash \neg(g(0) < f(g(0)))}\;\forall\text{-e} \qquad \overline{\Gamma, g(0) < f(g(0)) \vdash g(0) < f(g(0))}\;\text{ax}}{\Gamma, g(0) < f(g(0)) \vdash \bot}\;\neg\text{-e}$$

Remember frozen terms are frozen (Lemma 1)

$$\dfrac{\dfrac{\overline{\Gamma, g(0) < y \vdash \forall w\, \neg(g(0) < w)}\;\text{ax}}{\Gamma, g(0) < y \vdash \neg(g(0) < y)}\;\forall\text{-e} \qquad \overline{\Gamma, g(0) < y \vdash g(0) < y}\;\text{ax}}{\Gamma, g(0) < y \vdash \bot}\;\neg\text{-e}$$

Independently

Using the non-Skolemized axiom

$$\dfrac{\dfrac{}{\Gamma \vdash \forall x \exists y\, (x < y)}\ \text{ax}}{\Gamma \vdash \exists y\, (g(0) < y)}\ \forall\text{-e}$$

Conclude with an $\exists$-e rule (Lemma 2)

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x \exists y\, (x < y)}\ \text{ax}}{\Gamma \vdash \exists y\, (g(0) < y)}\ \forall\text{-e} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, g(0) < y \vdash \forall w\, \neg(g(0) < w)}\ \text{ax}}{\Gamma, g(0) < y \vdash \neg(g(0) < y)}\ \forall\text{-e} \qquad \dfrac{}{\Gamma, g(0) < y \vdash g(0) < y}\ \text{ax}}{\Gamma, g(0) < y \vdash \bot}\ \neg\text{-e}}{\Gamma \vdash \bot}\ \exists\text{-e}$$

The importance of cut-freeness and of the subformula property

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\Gamma, \forall x\,(x < f(x)) \vdash \forall x\,(x < f(x))}\;\text{ax}}{\Gamma, \forall x\,(x < f(x)) \vdash g(x') < f(g(x'))}\;\forall\text{-e}}{\Gamma, \forall x\,(x < f(x)) \vdash {\color{red}\forall x'\,(g(x') < f(g(x')))}}\;\forall\text{-i}}{\Gamma, \forall x\,(x < f(x)) \vdash (g(0) < f(g(0)))}\;\forall\text{-e}$$

$\forall x'\,(g(x') < f(g(x')))$ not a partial instance of $\forall x\,(x < f(x))$ and $f(g(x'))$ is not frozen

Do not know how to translate it

The general case

The last rule derives $\Gamma, \forall x\ (f(x)/y)A \vdash B$ from premises $\Gamma_1, \forall x\ (f(x)/y)A \vdash B_1$, ..., $\Gamma_p, \forall x\ (f(x)/y)A \vdash B_p$.

- Either one of the B_i's is a partial instance of the Skolemized axiom (like in step 3) and the B is a total instance of the Skolemized axiom. Hence $\Delta \vdash B$ has a proof built with the axiom rule
- Or (like in step 1, 2 and 4), all all the f-terms are frozen in all the B_i's. We apply the induction hypothesis and the same rule as in the original proof. Then we adjust the hypotheses with the weakening lemma (like in step 2) and the $\exists$-e rule (like in step 4)