Ecumenism: logical constants an beyond

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La mathématique est nécessairement toujours en crise, et toujours en train de la résoudre.

Michel Serres, *La Communication*

The universality of mathematical truth

The truth conditions of a mathematical statement must be the object of unanimous agreement

Constitutive of the notion of mathematical truth itself
Yet, constantly jeopardized
When mathematicians disagree on the truth of some statements: a crisis of the universality of mathematical truth

In the past

- The incommensurability of the diagonal and side of a square

$$\exists x\ (x \textit{ is a number} \wedge x^2 = 2)$$

- The introduction of infinite series

$$\sum_n \frac{1}{2^n} = 2 \qquad \sum_n (-1)^n = 0$$

- The non-Euclidean geometries

The sum of the angles in a triangle equals the straight angle

- The independence of the axiom of choice

Every vector space has a basis

- Constructivity

If $A \cup B$ infinite, then A infinite or B infinite

All these crises have been resolved

Yet another crisis: computerized proof systems

Coq, Isabelle/HOL, PVS, HOL Light, Lean...
A major step forward in the quest of mathematical rigor

But jeopardizes, once again, the universality of mathematical truth

A proof of Fermat's little theorem $\longrightarrow$ a Coq proof of Fermat's little theorem, a PVS proofs of Fermat's little theorem...

Each proof system: its own language and its own truth conditions

Solutions

The incommensurability of the diagonal and side of a square:
rational numbers and real numbers

Infinite series: limits

Non-Euclidean geometries: several solutions

- Different spaces: truth of

  On a space of zero curvature, the sum of the angles in a triangle equals the straight angle

  but not of

  On a space of negative curvature, the sum of the angles in a triangle equals the straight angle

- Axiomatic theories: E and H. Truth of

  $E \vdash$ *the sum of the angles in a triangle equals the straight angle*

  but not of

  $H \vdash$ *the sum of the angles in a triangle equals the straight angle*

Equivalent (soundness and completeness)

The second solution

- A true $\longrightarrow \Gamma \vdash A$ true
- Truth conditions: for the statements of geometry $\longrightarrow$ for arbitrary sequents
- Separation between the definition of the truth conditions of a sequent: the logical framework and the definition of the various geometries as theories
- The various geometries defined in the same logical framework
- A logical framework: Predicate logic

The axiom of choice

First solution: truth of

In a model of ZFC, every vector space has a basis

but not of

In a model of ZF, every vector space has a basis

Second: *Every vector space has a basis* consequence of the axiom of choice

First solution does not work:
- Too far from the original formulation
- Problem of the "absolute" theory in which this should be proved

Thus, second chosen, paving the way to reverse mathematics

Constructivity

First solution: truth of

In a model valued in a Boolean algebra, if $A \cup B$ infinite, then A infinite or B infinite

but not of

In a model valued in a Heyting algebra, if $A \cup B$ infinite, then A infinite or B infinite

Again, too far from the original formulation and question of the "absolute" theory

Second: *if $A \cup B$ infinite, then A infinite or B infinite* consequence of the excluded middle

Ecumenism

Changing the axioms while keeping the same symbols?

Axioms express the meaning of the symbols: different axioms $\longrightarrow$ different meanings $\longrightarrow$ different symbols

The only "mistake" is not to accept or to reject the excluded middle, but to use the same symbol for $\vee$ and $\vee_c$

Nothing prevents from using them both

A third solution to the crisis of constructivity: truth of

$$\mathit{Infinite}(A \cup B) \Rightarrow_c \mathit{Infinite}(A) \vee_c \mathit{Infinite}(B)$$

but not of

$$\mathit{Infinite}(A \cup B) \Rightarrow \mathit{Infinite}(A) \vee \mathit{Infinite}(B)$$

Several symbols for greater precision

A very common practise in science: *mass* and *weight*...

Solution of the crisis of constructivity

But also of the crisis of $\sqrt{2}$: *rational number* and *real number*

But also the crisis of non Euclidean geometries: *Euclidean space* and *Hyperbolic space* but also possibly different words for the *line*, *triangle*, *angle*...

Already (partially) the case: the geodesics and the triangles of the sphere already called *great circles* and *spherical triangles*

Axiom of choice? Not quite (two notions of vector space? e.g. vector space vs. well-ordered vector space?)

Independent definitions vs. holistic meaning

Meaning of a symbol defined not only by its own deduction rules, but also by the other deduction rules

Peirce's law: changing the meaning of implication, but also that of disjunction, negation...

In the same way

Axiom of choice introduces a choice symbol ε and a statement defining its meaning

$$(\exists x\ A) \Rightarrow ((\varepsilon x\ A)/x)A$$

Ths modifies the provability of

Every vector space has a basis

that does not contain ε

Belnap: the addition of symbol (and its axioms / rules) should be conservative: not often the case

Does adding the excluded middle modifies the meaning of logical constants only or also that of predicate symbols? Sometimes it does (Prawitz15).

The diversity of proof systems

Pieces of software that can check, build, transform... proofs.

Started in the middle of the 20^{th} century, long after design of set theory, simple type theory...

Two intertwined research streams, just like thermodynamics and the construction of steam engines

- Set theory not been designed to be eventually implemented on a computer and used in practice. Not fitted to this goal (no terms, no computation, no cut elimination...)
- Best way to experiment with a new innovative formalism: implement it on a computer
- Computer science natural affinities with type theory and with constructivity: programming languages are often typed, constructive proofs have an algorithmic content...

- Few systems: Mizar and B implement set theory (even then deviant)
- Several systems: HOL4, Isabelle/HOL, HOL Light implement classical Simple type theory
- Several: Coq, Lean, Matita, Agda, various forms of constructive type theory
- PVS implements classical Simple type theory, with predicate subtyping
- Several others systems implement newer formalisms

More than twenty proof systems, each defining its own language and its own truth conditions jeopardizing, once again, the universality of mathematical truth

Difficult to use formal proofs to teach mathematics: always a risk that a statement has a proof in some systems, but not in others

Logical frameworks

Universality of mathematical truth compatible with the use of various axioms for geometry

But requires to express the axioms of the various geometries in a common logical framework: Predicate logic

One way to resolve this crisis of the universality of the mathematical truth initiated with the development of proof systems

- Express the formalisms implemented in these systems in Predicate logic
- Identify the common axioms and the differentiating ones (should have a large number of axioms in common and a few axioms differentiating them)
- Analyze which proof uses which axiom

But too naive: Predicate logic not the good logical framework

Simple type theory can indeed be expressed in Predicate logic, with some distortions

Expressing Martin-Löf's type theory or the Calculus of constructions requires even stronger distortions

A history of logical frameworks

- Bound variables: λ-Prolog, Isabelle, $\lambda\Pi$
- Proof terms: $\lambda\Pi$
- Computation: Deduction modulo theory
- Cut elimination: Deduction modulo theory
- Constructive proofs: Ecumenical logic

All together: DEDUKTI

Expressing terms and propositions

DEDUKTI weaker than Predicate logic: no primitive notion of term and proposition
Constructing these notions

$$I : \mathrm{TYPE}$$

$$Prop : \mathrm{TYPE}$$

And, for instance

$$0 : I$$

$$succ : I \to I$$

$$= : I \to I \to Prop$$

$$\Rightarrow : Prop \to Prop \to Prop$$

$$\forall : (I \to Prop) \to Prop$$

Expressing proofs

Brouwer-Heyting-Kolmogorov, Curry-de Bruijn-Howard
A constant *Prf* that maps propositions to the types of their proofs

$$Prf : Prop \to \mathsf{TYPE}$$

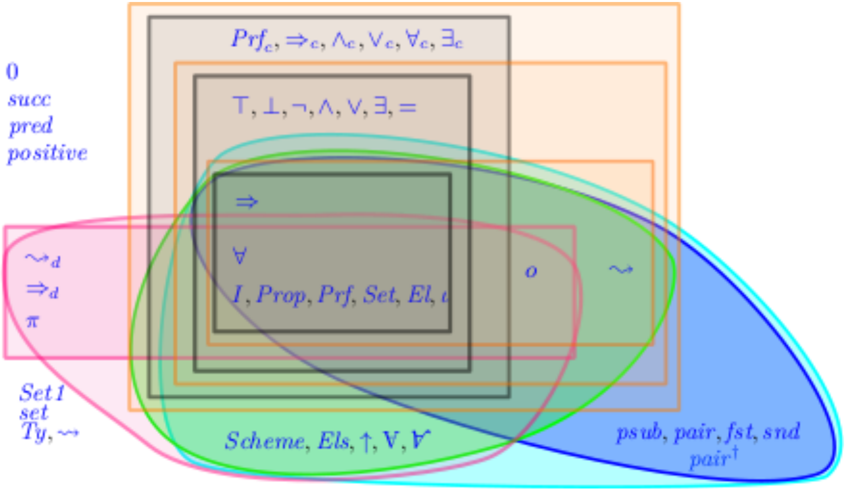
and computation rules express that the proofs of a proposition $(\Rightarrow x\ y)$ are the functions mapping proofs of x to proofs of y

$$(Prf\ (\Rightarrow x\ y)) \longrightarrow (Prf\ x) \to (Prf\ y)$$

And similarly for $\forall$

$$\lambda x : (Prf\ P)\ x : Prf(P \Rightarrow P)$$

More axioms


$Prf_c, \Rightarrow_c, \wedge_c, \vee_c, \forall_c, \exists_c$
0
$succ$
$pred$
$positive$
$\top, \bot, \neg, \wedge, \vee, \exists, =$
$\Rightarrow$
$\forall$
$I, Prop, Prf, Set, El, \iota$
o
$\rightsquigarrow$
$\rightsquigarrow_d$
$\Rightarrow_d$
π
$Set1$
set
$Ty, \rightsquigarrow$
$Scheme, Els, \uparrow, \forall, \forall$
$psub, pair, fst, snd$
$pair^\dagger$

$\rightsquigarrow$ and $\rightsquigarrow_d$

When several sorts of objects, instead of I: I_1 and I_2 several $\forall$...
But when a lot, a type *Set* for the codes of sorts, $\iota_1 : Set$, $\iota_2 : Set$, and an embedding *El* of type $Set \to$ TYPE
In Simple type theory a symbol $\rightsquigarrow$ of type Set $\to$ Set $\to$ Set and a rule

$$El(x \rightsquigarrow y) \longrightarrow El(x) \to El(y)$$

But in Martin-Löf type theory and in the Calculus of constructions, a symbol $\rightsquigarrow_d$ of type $\Pi x : \mathrm{Set}((El(x) \to \mathrm{Set}) \longrightarrow \mathrm{Set})$ and a rule

$$El(x \rightsquigarrow_d y) \longrightarrow \Pi z : El(x) \to El(y\ z)$$

Which one to chose? Both
The same holds for $\Rightarrow$ and $\Rightarrow_d$

Various forms of Ecumenism

In DEDUKTI: both $\rightsquigarrow$ and $\rightsquigarrow_d$, both $\Rightarrow$ and $\Rightarrow_d$, and of course logical constants

In older mathematics: rational numbers and real numbers, lines and great circles, inclusive and exclusive disjunction

A general method to solves crises of the universality of mathematical truth