Simulation of physical phenomena with cellular automata

Gilles Dowek

Joint work with Pablo Arrighi

I. From Gandy's hypotheses to digital physics

Gandy's hypotheses

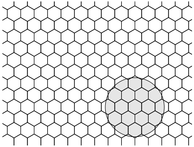
Homogeneity of time and space

Bounded velocity of information 

Bounded density of information 

If Gandy's hypotheses are verified...

...any system (can be simulated by | is) a cellular automaton



- Each cell has a finite state space (bounded density)
- State of a cell depends on state of a finite number of cells at previous time step (bounded velocity)
- Local evolution function the same everywhere and everywhen (homogeneity)

Bounded density of information

The amount of information that can be stored in a region of space of radius R is bounded

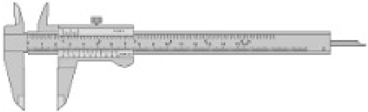
Robin Gandy (1980): the three hypotheses together imply the physical Church-Turing thesis (any physical system (can be simulated by | is) a cellular automaton)

Jacob Bekenstein (1981): $I_{max} = \frac{1}{4\ln(2)} \frac{c^3}{\hbar \mathcal{G}} 4\pi R^2$
R^2 and not R^3
Bekenstein's constant ($1.4\ 10^{69}\ \mathrm{m}^{-2}$)

Real numbers

An infinite sequence of bits: the digits of a real number

encoded in the distance between the jaws of a



But hundredth digit of magnitude: no physical meaning

Digital physics

A theoretical result: any system (can be simulated by | is) a cellular automaton (computable universe, Galileo's thesis...)

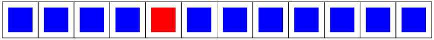
But also: model phenomena with ~~real numbers and differential equations~~ cellular automata

Chapter by chapter
This talk: gravitation

Motion in a cellular automaton

States: $\{q, ..., \}$

All cells are quiescent (q) except one: the particle
E.g.



Evolution rules preserve this invariant

II. A digression: what is the Planck constant the magnitude of?

c

Not just a constant in some equation: the speed of something
$\hbar$: the action of what?

No name: ~~The Rømer constant~~, ~~The Einstein constant~~: the speed of light
The Planck constant

Just one c
but h, $\hbar$...

Units

Often unit system such that $c = 1$, $\mathcal{G} = 1$, $\hbar = 1$
A unique unit system for distance, time, mass

Here, just $c = 1$ and $\mathcal{G} = 1$, keep distances in m
Time and mass are distances (everything in m)

t in s $\longrightarrow$ ct in m
m in kg $\longrightarrow$ $(\mathcal{G}/c^2)m$ in m   half of Schwarzschild's radius

c : m s^{-1} and $\mathcal{G}/c^2$: m kg^{-1} unit conversion constants (like 0.0254 m in^{-1})

Units

velocity:	m s^{-1}	$\longrightarrow$	m^0 (scalar)
momentum:	kg m s^{-1}	$\longrightarrow$	m
acceleration:	m s^{-2}	$\longrightarrow$	m^{-1}
force:	kg m s^{-2}	$\longrightarrow$	m^0
energy:	kg m^2 s^{-2}	$\longrightarrow$	m
action:	kg m^2 s^{-1}	$\longrightarrow$	m^2

The Planck constant

$\hbar = 1.054\ 10^{-34}$ m^2 kg s^{-1} homogeneous to m^2

The Planck constant in m^2: $a_P = \hbar(\mathcal{G}/c^2)(1/c) = 2.61\ 10^{-70}$ m^2
The Planck area

$$I_{max} = \frac{1}{4\ln(2)}\frac{c^3}{\hbar\mathcal{G}}4\pi R^2 = \frac{1}{4\ln(2)a_P}4\pi R^2$$

$4\ln(2)a_P = 7.2\ 10^{-70}$ m^2 is the area of one bit

Not h, $\hbar$, or a_P: $4\ln(2)a_P$: the area of one bit

III. A first example: free fall in Newtonian physics

What is free fall?

Gravitational attraction

Approximated (force variation neglected)

Constant force

Newton's law

$$f = \frac{mm'}{(R-z)^2}$$

Approximation:

$$f = \frac{mm'}{R^2} = m'g$$

where $g = m/R^2$

Earth: $g = 1.09\ 10^{-16}\ \mathrm{m}^{-1}$ ($gc^2 = 9.81\ \mathrm{m\ s}^{-2}$)

Trajectory

$$f = m'a$$

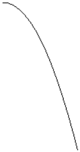
$$a = g$$

$$v = gt$$

$$z = \frac{1}{2}gt^2$$

Parabolic motion

Time-space 1D diagram



Velocity unbounded
No bounded velocity of information

Moreover: velocity part of the internal state: no bounded density of information (even if finite precision)

No cellular automaton

IV. Free fall in Special Relativity

Momentum increases linearly with time

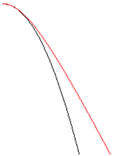
$$m'g = \frac{d}{dt}(m'\frac{1}{\sqrt{1-v^2}}v) = m'\frac{1}{\sqrt{1-v^2}^3}\frac{dv}{dt}$$

$$\frac{dv}{dt} = g\sqrt{1-v^2}^3$$

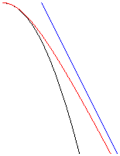
$$v = \frac{gt}{\sqrt{1+(gt)^2}}$$

$$z = \frac{1}{g}(\sqrt{1+(gt)^2} - 1)$$

Hyperbolic motion



Asymptote $z' = t - (1/g)$



When $t \geq \theta = \frac{1-(g\Delta)^2}{2g^2\Delta}$ indistinguishable: $|z - z'| \leq \Delta$

After θ: momentum still increases linearly, but little impact on velocity

Discrete motion

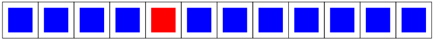
Space and time coordinates in $\Delta\mathbb{Z}$

$\tilde{y}$ discretization of y

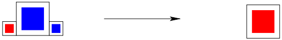
Number of time steps before curve indistinguishable from asymptote: $L = \lceil \theta/\Delta \rceil$

A cellular automaton

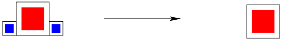
States: $\{q, 0, ..., L-1, \infty\}$ $\quad$ 0, ..., $L-1$: clock, momentum



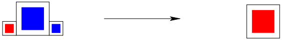
In state ∞, motion at velocity c:



in state k



or



depending on $\tilde{y}((k+1)\Delta) = \tilde{y}(k\Delta)$ or $\tilde{y}((k+1)\Delta) = \tilde{y}(k\Delta) + \Delta$

Number of states / amount of information in the particle

If Δ fixed in advance

Number of states

$$2 + \frac{\theta}{\Delta} = \frac{1}{2g^2\Delta^2} + \frac{3}{2}$$

Amount of information

$$I = \log_2\left(\frac{1}{2g^2\Delta^2} + \frac{3}{2}\right)$$

Better...

Fix the size ρ of one bit
Size of the particle at least

$$\rho I = \rho \log_2\left(\frac{1}{2g^2\Delta^2} + \frac{3}{2}\right)$$

Δ smallest size such that

$$\rho \log_2\left(\frac{1}{2g^2\Delta^2} + \frac{3}{2}\right) \leq \Delta$$

$\Delta - \rho \log_2(\frac{1}{2g^2\Delta^2} + \frac{3}{2})$ monotonic in Δ: numerical solution

$\rho = 1.6\ 10^{-35}$ m: 320 bits, $\Delta = 5.1\ 10^{-33}$ m
($\rho = 2.7\ 10^{-35}$ m: 319 bits, $\Delta = 8.6\ 10^{-33}$ m)

Towards an uncertainty principle

Internal state: clock, but also momentum $p = m'gt = m'gk\Delta$
Uncertainty on the momentum $P = m'g\Delta$
Size of the particle (uncertainty on the position)

$$\Delta = \rho \log_2\left(\frac{1}{2(P/m')^2} + \frac{3}{2}\right)$$

More accurate the momentum, less accurate the position
Qualitative uncertainty principle (rather than quantitative

$$2^{\Delta/\rho}(P/m')^2 = 1/2 \quad ?!)$$

V. Free fall in General Relativity

Forces and metrics

General relativity: "force" replaced by "metric"
Special relativity: when changing reference frame $x^2 + y^2$ not preserved, t^2 not preserved
But "distance" $t^2 - x^2 - y^2$ preserved

$$\begin{pmatrix} t & x & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \end{pmatrix}$$

Metric tensor

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Trajectories: geodesics: spacetime straight lines: rectilinear motion

Schwarzschild's metric

Gravitational attraction of a star: not a force, but a modification of the metric tensor, at $\langle t, x, y \rangle$

$$\begin{pmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & -\frac{x^2}{r(r-2m)}-\frac{y^2}{r^2} & -\frac{2mxy}{r^2(r-2m)} \\ 0 & -\frac{2mxy}{r^2(r-2m)} & -\frac{x^2}{r^2}-\frac{y^2}{r(r-2m)} \end{pmatrix}$$

where $r = \sqrt{x^2 + y^2}$, m mass of the star in m, i.e. $\frac{\mathcal{G}}{c^2}M$

Motion of a particle: geodesic $t(\tau), x(\tau), y(\tau)$ for this metric

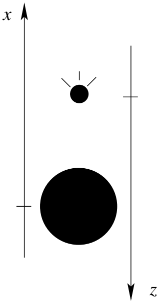
Free fall: 1D

Drop the last line and column. And $y = 0$, and $x > 0$ in the others

$$\begin{pmatrix} g_{tt} & 0 \\ 0 & -\frac{1}{g_{tt}} \end{pmatrix}$$

where $g_{tt} = 1 - 2m/x$

Constant force in General Relativity



$x = R - z$

$$g_{tt} = 1 - \frac{2m}{R - z}$$

First-order expansion

$$g_{tt} = 1 - \frac{2m}{R} - \frac{2m}{R^2} z = 1 - \frac{2m}{R} - 2gz = 2g(z_1 - z)$$

where $z_1 = (1 - 2m/R)/(2g)$

The equations of the motion

$$\frac{d^2t}{d\tau^2} = -\frac{1}{g_{tt}}\frac{dg_{tt}}{dz}\frac{dt}{d\tau}\frac{dz}{d\tau}$$

$$\frac{d^2z}{d\tau^2} = -\frac{1}{2}\frac{dg_{tt}}{dz}(g_{tt}(\frac{dt}{d\tau})^2 - \frac{1}{g_{tt}}(\frac{dz}{d\tau})^2)$$

$$g_{tt}(\frac{dt}{d\tau})^2 - \frac{1}{g_{tt}}(\frac{dz}{d\tau})^2 = 1$$

Trajectory

$$z = z_1 \tanh^2(gt)$$

Limit point

Special Relativity: velocity goes to infinity when τ does
but mapping from t to τ slows down
thus velocity bounded

General Relativity: velocity goes to infinity when τ does
but mapping from t to τ slows down (infinite t: finite τ)
particle has a limit position

Even better than an asymptote: a limit point

Cellular automaton

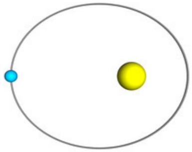
Final state: the particle stays at the same place

Number of states / amount of information in the particle:

$\rho = 1.6\ 10^{-35}$ m: 168 bits, $\Delta = 2.6\ 10^{-33}$ m

($\rho = 2.710^{-35}$ m: 167 bits, $\Delta = 4.5\ 10^{-33}$ m)

V. With no approximation



Geodesics of

$$\begin{pmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & -\frac{x^2}{r(r-2m)}-\frac{y^2}{r^2} & -\frac{2mxy}{r^2(r-2m)} \\ 0 & -\frac{2mxy}{r^2(r-2m)} & -\frac{x^2}{r^2}-\frac{y^2}{r(r-2m)} \end{pmatrix}$$

Bounded density of information

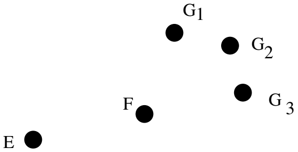
Coordinates in $\Delta\mathbb{Z}$

What is a geodesic in a discrete spacetime?

Geodesics

Geodesic: shortest and straightest path between two points

w function such that $w(E, F, G)$ measures how E, F, G deviates from going straight ahead



G chosen to minimize (external) angle in F
Discrete-time continuous-space spacetime:
$E_0, E_1, E_2, E_3, \ldots$ geodesic if for all i, $w(E_{i-1}, E_i, E_{i+1}) = 0$

From the continuous to the discrete

Too strong in discrete space

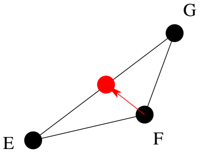
$w(E_{i-1}, E_i, E_{i+1})$ minimum for local variations of E_{i+1}: for any spatial neighbor G of E_{i+1}

$$w(E_{i-1}, E_i, G) \geq w(E_{i-1}, E_i, E_{i+1})$$

Algorithm:
E_{i-1}, E_i given
Pick random E_{i+1}
If $w(E_{i-1}, E_i, E_{i+1})$ not local minimum replace E_{i+1} by a better neighbor and iterate

Metrics and deviations

$$l(E, F, G) = d(E, F) + d(F, G)$$



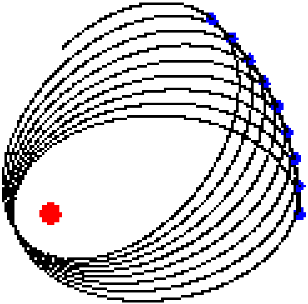
In the continuous case

$w(E, \langle t, x, y\rangle, G) =$
$(\partial_t l(E, \langle t, x, y\rangle, G))^2 + (\partial_x l(E, \langle t, x, y\rangle, G))^2 + (\partial_y l(E, \langle t, x, y\rangle, G))^2$

In the discrete case: replace derivatives with finite differences

Schwarzschild's metric $\longrightarrow$ distance $\longrightarrow$ deviation function $\longrightarrow$ cellular automaton to compute geodesics

Experimental results



Almost only integers in the program

A fake planet very close to the Sun (maximize relativistic effects)

Perihelion shift 6.27°, expected 6.17°

No (t yet) similar results for Mercury: very small shift

Theoretical results

Continuous trajectories when step $\to 0$

Unlike other discrete formulations of General relativity

Conclusions

Physics can be reformulated in a discrete (and computational) way (in the case of free fall and motion of a planet)

Free fall: for (Special and General) Relativity, but not for Newtonian physics

Free fall: exact up to Δ: no accumulation of rounding errors

Small amount of information: 319 / 167 bits

General Relativity simpler than Special Relativity (simpler than Newtonian physics)

Falsifiability

"The world is a cellular automaton": a falsifiable hypothesis

Not all motions possible

$x = t^2$ contradicts bounded velocity and bounded density

$x = \sqrt{t+1}$ would not contradict bounded velocity but would still contradict bounded density

But still a lot to be done

An existence proof, rather than a satisfactory automaton

Position: on the grid
momentum / velocity: internal state
metric tensor: internal state

Momentum: in a fixed coordinate system

Metric tensor: internal state
The particle knows the position of the star
Calls for more natural automata: e.g. messenger particles

The particle "knows" the laws of physics: an accurate simulation rather than an accurate description of the physical phenomenon

For free fall: exact up to Δ, only a limit for the motion of a planet

Mercury?

Drop the rigid grid: causal graph dynamics