Curry-de Bruijn-Howard correspondence, introduction rules, and cuts, in general

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(joint work with Ying Jiang)

What is the generality of

- the Curry-de Bruijn-Howard correspondence
- the notion of introduction rule
- the notion of cut

?

Curry-de Bruijn-Howard: Natural deduction, then other systems (e.g. Sequent calculi)

Introduction rule: Natural deduction

Cut: Natural deduction, then other systems (e.g. Sequent calculi) but with a very different meaning

A anti-/pro-Natural deduction talk

I. The Curry-de Bruijn-Howard correspondence, in general

An inference system

$$\frac{P \quad Q}{R}\, f \qquad \frac{R}{P}\, g \qquad \frac{}{P}\, a \qquad \frac{}{Q}\, b$$

Proofs

$$\cfrac{\cfrac{\cfrac{\cfrac{}{P}\,a \qquad \cfrac{}{Q}\,b}{R}\,f}{P}\,g \qquad \cfrac{}{Q}\,b}{R}\,f$$

Proofs

$$\dfrac{\dfrac{}{P} \qquad\qquad \dfrac{}{Q}}{R}$$

$$\dfrac{\dfrac{}{P} \qquad\qquad \dfrac{}{Q}}{R}$$

Correctness checking algorithm

Proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P}\,a \qquad \dfrac{}{Q}\,b}{R}\,f}{P}\,g \qquad \dfrac{}{Q}\,b}{R}\,f$$

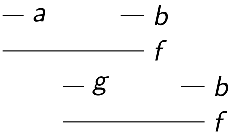
Proofs

$$\cfrac{\overline{}\,a \qquad \overline{}\,b}{}\,f$$

$$\cfrac{\overline{}\,g \qquad \overline{}\,b}{}\,f$$

Conclusion inference algorithm, conclusion checking algorithm

A linear notation for trees



$f(g(f(a,b)),b)$
Proof-term

Conclusion checking algorithm: set of pairs $\pi : A$ such that π ~~has type~~ is a linear representation of a proof of A decidable

Why?

Transform each rule

$$\frac{s_1 \; \ldots \; s_n}{s'} \; h$$

into

$$\frac{\pi_1 : s_1 \; \ldots \; \pi_n : s_n}{h(\pi_1, \ldots, \pi_n) : s'}$$

An inference system that proves exactly the pairs $\pi : A$ such that π is a linear representation of a proof of A

Type-assignment inference system of the first inference system

Always decidable

$\prec$ strict subterm relation on proof-terms (well-founded)
extends to a well-founded relation on pairs

$$(\pi : A) \prec (\pi' : B) \text{ if and only if } \pi \prec \pi'$$

$$\frac{\pi_1 : s_1 \ \ldots \ \pi_n : s_n}{h(\pi_1, \ldots, \pi_n) : s'}$$

All premises smaller than the conclusion: finite search space

Inductively defined set: projection of a decidable set

An example

$$\frac{A \wedge B}{A}\,fst \qquad \frac{A \wedge B}{B}\,snd \qquad \frac{A \Rightarrow B \quad A}{B}\,app \qquad \frac{}{P \wedge (P \Rightarrow Q)}\,c$$

$$\frac{\pi : A \wedge B}{fst(\pi) : A} \qquad \frac{\pi : A \wedge B}{snd(\pi) : B} \qquad \frac{\pi_1 : A \Rightarrow B \quad \pi_2 : A}{app(\pi_1, \pi_2) : B} \qquad \frac{}{c : P \wedge (P \Rightarrow Q)}$$

$$\dfrac{\dfrac{\dfrac{}{P \wedge (P \Rightarrow Q)}\,c}{P \Rightarrow Q}\,snd \qquad \dfrac{\dfrac{}{P \wedge (P \Rightarrow Q)}\,c}{P}\,fst}{Q}\,app$$

$$app(snd(c), fst(c)) : Q$$

An extension of the notion of inference system

$$\frac{s_1 \ \dots \ s_n}{s'} \ h$$

To prove s' prove $s_1, \dots, s_n$
To prove s_i, and only there, you can use an additional (nullary)

rule $\overline{P}^{\,x}$
h must introduce the name x, in use only in its i-th argument
Bound variable
Example:

$$\frac{B}{A \Rightarrow B} \ \lambda \ (*)$$

$(*)$ $\overline{A}^{\,x}$ may be used in the proof of B

$$\frac{\pi : B}{\lambda x : A \ \pi : A \Rightarrow B} \ (*)$$

$(*)$ $x : A$ may occur in π

An example

$$\frac{A \wedge B}{A}\ \mathit{fst} \qquad \frac{A \wedge B}{B}\ \mathit{snd} \qquad \frac{A \Rightarrow B \quad A}{B}\ \mathit{app} \qquad \frac{B}{A \Rightarrow B}\ \lambda\ (*)$$

$$\frac{\pi : A \wedge B}{\mathit{fst}(\pi) : A} \qquad \frac{\pi : A \wedge B}{\mathit{snd}(\pi) : B} \qquad \frac{\pi_1 : A \Rightarrow B \quad \pi_2 : A}{\mathit{app}(\pi_1, \pi_2) : B} \qquad \frac{\pi : B}{\lambda x : A\ \pi : A \Rightarrow B}\ (*)$$

$$\cfrac{\cfrac{\cfrac{}{P \wedge (P \Rightarrow Q)}\ x}{P \Rightarrow Q}\ \mathit{snd} \qquad \cfrac{\cfrac{}{P \wedge (P \Rightarrow Q)}\ x}{P}\ \mathit{fst}}{\cfrac{Q}{(P \wedge (P \Rightarrow Q)) \Rightarrow Q}\ \lambda}\ \mathit{app}$$

$$\lambda x : P \wedge (P \Rightarrow Q)\ \mathit{app}(\mathit{snd}(x), \mathit{fst}(x)) : (P \wedge (P \Rightarrow Q)) \Rightarrow Q$$

With / without Brouwer-Heyting-Kolmogorov interpretation

II. The case of a finite set of propositions

An example

Two rules

$$\frac{even}{odd}\,a \qquad \frac{odd}{even}\,a \qquad \frac{}{even}\,\varepsilon$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{even}\,\varepsilon}{odd}\,a}{even}\,a}{odd}\,a$$

An example

$$\frac{\mathit{even}}{\mathit{odd}}\ a \qquad \frac{\mathit{odd}}{\mathit{even}}\ a \qquad \frac{}{\mathit{even}}\ \varepsilon$$

$$\frac{\pi : \mathit{even}}{a(\pi) : \mathit{odd}} \qquad \frac{\pi : \mathit{odd}}{a(\pi) : \mathit{even}} \qquad \frac{}{\varepsilon : \mathit{even}}$$

$a(a(a(\varepsilon))) : \mathit{odd}$ provable

An example

$$\frac{even}{odd}\, a \qquad \frac{odd}{even}\, a \qquad \frac{}{even}\, \varepsilon$$

$$odd \xrightarrow{a} even \qquad even \xrightarrow{a} odd \qquad even \text{ final}$$

The word $a(a(a(\varepsilon)))$ (a.k.a. aaa) recognized in *odd*

Unary rules: word automata, n-ary rules: tree automata

Finite state automata

state: proposition (type)

word: proof (term)

reachability: provability (inhabitation)

π recognized in A: π proof of A (π term of type A)

III. Introduction rules, in general

Introduction rule

Given a well-founded relation $\prec$

$$\frac{s_1 \; \dots \; s_n}{s'}$$

an introduction rule if $s_1 \prec s'$, ..., $s_n \prec s'$

Examples

$$\frac{A \quad B}{A \wedge B}\ \wedge\text{-intro}$$

$$\frac{A \wedge B}{A}\ \wedge\text{-elim}$$

$$\frac{\pi_1 : s_1 \ \dots \ \pi_n : s_n}{h(\pi_1, \dots, \pi_n) : s'}$$

$$\frac{x : even}{a(x) : odd}$$

$$\frac{even(x)}{odd(a(x))}$$

Automaton

A (finite in conclusions) inference system containing introduction rules only

Provability decidable: finite search space

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

$odd(a(a(a(\varepsilon))))$ provable
$even(a(a(a(\varepsilon))))$ not provable

Representing words

$$\frac{odd(x)}{even(a(x))}$$

Representation of words as first order terms: "aab" $= a(a(b(\varepsilon)))$

The benefit of generalization

Inference systems need not operate on terms / propositions of predicate logic

$$\frac{odd(x)}{even(ax)}$$

ax concatenation of a to x (independent of any representation)

E.g. ^ and ε, quotient by associativity and neutral : free monoid

Introduction rule because for all words w, w shorter than aw

Another example

$$\frac{}{L(ab)} \qquad\qquad \frac{L(x)}{L(axb)}$$

Introduction rules

$$\cfrac{\cfrac{\cfrac{\overline{L(ab)}}{L(aabb)}}{L(aaabbb)}}{L(aaaabbbb)}$$

$$\begin{array}{lcl} L & \longrightarrow & ab \\ L & \longrightarrow & aLb \end{array}$$

Any context free grammar as a (generalized) automaton

Decidability of context free grammars

IV. Cuts, in general

(General) cuts

$$\dfrac{\dfrac{\cdots}{s_1}\, R_1 \text{ (intro)} \quad \ldots \quad \dfrac{\cdots}{s_n}\, R_n \text{ (intro)}}{s'}\, R' \text{ (non-intro)}$$

π cut-free: no cuts in π

An inference system has the cut-elimination property if every proof can be transformed into a cut-free proof

A theorem

A cut-free proof contains introduction rules only

Generalizes the last rule property and Girard's shocking equalities:
a cut free proof of $\forall x\ (A \vee B)$ ends with two introduction rules

Induction over proof structure

$$\cfrac{\cfrac{\pi_1}{s_1} \quad \ldots \quad \cfrac{\pi_n}{s_n}}{s'} \; R'$$

π_1, ..., π_n: introduction rules only
π_1, ..., π_n end with introduction rules
R': introduction rule

A corollary

In an inference system that has the cut-elimination property provability is decidable

Drop the non-introduction rules, preserving provability

An automaton

V. Finite domain logic

Finite domain logic

Natural deduction tailored to prove the propositions valid in a given finite model $\mathcal{M}$

- a constant for each element in the model (and no other function symbols)
- $A \Rightarrow B$ abbreviation for $\neg A \vee B$
- negation pushed to atomic propositions (using de Morgan's laws)

- $\forall$-intro and $\exists$-elim rules replaced by enumeration rules (ω-rules?)

$$\frac{\Gamma \vdash (c_1/x)A \quad \dots \quad \Gamma \vdash (c_n/x)A}{\Gamma \vdash \forall x\ A}\ \forall\text{-intro}$$

$$\frac{\Gamma \vdash \exists x\ A \quad \Gamma, (c_1/x)A \vdash C \quad \dots \quad \Gamma, (c_n/x)A \vdash C}{\Gamma \vdash C}\ \exists\text{-elim}$$

- atom rule

$$\frac{}{\Gamma \vdash L}\ \text{atom if } L \in \mathcal{P}$$

$\mathcal{P}$ finite set containing P or $\neg P$, for each closed atomic P

- no rules for implication and negation

$$\frac{}{\Gamma, A \vdash A}\text{ axiom}$$

$$\frac{}{\Gamma \vdash L}\text{ atom if } L \in \mathcal{P}$$

$$\frac{}{\Gamma \vdash \top}\ \top\text{-intro}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ \bot\text{-elim}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \vee\text{-elim}$$

$$\frac{\Gamma \vdash (c_1/x)A \quad \ldots \quad \Gamma \vdash (c_n/x)A}{\Gamma \vdash \forall x\ A}\ \forall\text{-intro}$$

$$\frac{\Gamma \vdash \forall x\ A}{\Gamma \vdash (c_i/x)A}\ \forall\text{-elim}$$

$$\frac{\Gamma \vdash (c_i/x)A}{\Gamma \vdash \exists x\ A}\ \exists\text{-intro}$$

$$\frac{\Gamma \vdash \exists x\ A \quad \Gamma, (c_1/x)A \vdash C \quad \ldots \quad \Gamma, (c_n/x)A \vdash C}{\Gamma \vdash C}\ \exists\text{-elim}$$

Cut elimination

Business as usual
Drop all the elimination rules
An automaton

Drop the axiom rule (context always empty)

An automaton

$$\frac{}{\vdash L} \text{ atom if } L \in \mathcal{P}$$

$$\frac{}{\vdash \top} \top\text{-intro}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-intro}$$

$$\frac{\vdash A}{\vdash A \vee B} \vee\text{-intro}$$

$$\frac{\vdash B}{\vdash A \vee B} \vee\text{-intro}$$

$$\frac{\vdash (c_1/x)A \quad \dots \quad \vdash (c_n/x)A}{\vdash \forall x\ A} \forall\text{-intro}$$

$$\frac{\vdash (c_i/x)A}{\vdash \exists x\ A} \exists\text{-intro}$$

Proof search or model checking?

Proving implication

Additive rules

$$\frac{\vdash B}{\vdash A \Rightarrow B}$$

$$\frac{A \vdash}{\vdash A \Rightarrow B}$$

instead of the multiplicative

$$\frac{A \vdash B}{\vdash A \Rightarrow B}$$

How do you know that "if the Sun shines, then the Sun shines"?
A different proof every talk
No need for an axiom rule

The axiom rule

Not needed in finite domain logic

Comes from the will to reason generically on elements of an infinite domain

$$\forall x\ (P(x) \Rightarrow P(x))$$

VI. (constructive) Natural deduction

Natural deduction (for Predicate logic)

Undecidable

Can it have the cut elimination property?

No

Natural deduction (for Predicate logic)

Introduction rules: $\Rightarrow$-intro, $\forall$-intro, $\wedge$-intro... and axiom

$$\frac{}{\Gamma, A \vdash A}\text{ axiom}$$

Non-introduction rules: $\Rightarrow$-elim, $\forall$-elim, $\wedge$-elim...

$$\dfrac{\dfrac{}{P \wedge Q \vdash P \wedge Q}\text{ axiom}}{P \wedge Q \vdash P}\ \wedge\text{-elim}$$

is a (general) cut but not a (specific) cut
Remember finite domain logic: no axiom rule

VII. Saturating inference systems (and attempting to do so)

When an inference system does not have the cut-elimination property

$$\frac{}{R(a(x))} \qquad \frac{R(x)}{Q(a(b(x)))} \qquad \frac{Q(a(x))}{P(x)}$$

$$\dfrac{\dfrac{\overline{R(a(b(\varepsilon)))}}{Q(a(b(a(b(\varepsilon)))))}}{P(b(a(b(\varepsilon))))}$$

Transform it in such a way it does

Saturation

Add the derived rule

$$\frac{R(x)}{P(b(x))}$$

The proof

$$\cfrac{\cfrac{\cfrac{}{R(a(b(\varepsilon)))}}{Q(a(b(a(b(\varepsilon)))))}}{P(b(a(b(\varepsilon))))}$$

then reduces to

$$\cfrac{\cfrac{}{R(a(b(\varepsilon)))}}{P(b(a(b(\varepsilon))))}$$

Saturation(?) $\longrightarrow$ cut elimination $\longrightarrow$ automaton $\longrightarrow$ decidability

Attempting to saturate Natural deduction

$$\dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\text{ axiom}}{A \wedge B \vdash A}\ \wedge\text{-elim}$$

Add the derived rule

$$\dfrac{}{\Gamma, A \wedge B \vdash A}$$

Conditions of application of the rule: conjunction on the left
But would generate an infinite number of rules
Instead

$$\dfrac{\Gamma, A, B \vdash D}{\Gamma, A \wedge B \vdash D}$$

$+$ contraction
Gentzen style sequent calculus

Attempting to saturate Gentzen style sequent calculus

A non-introduction rule: contraction

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, A, B, A \wedge B \vdash D}}{\Gamma, A \wedge B, A \wedge B \vdash D}\,\wedge\text{-left}}{\Gamma, A \wedge B \vdash D}\,\text{contraction}$$

general cut

But can be eliminated: $\Gamma, A, B \vdash D$ is enough

Attempting to saturate Gentzen style sequent calculus

In contrast

$$\dfrac{\dfrac{\dfrac{\ldots}{\Gamma, \forall x\ A, (t/x)A \vdash B}}{\Gamma, \forall x\ A, \forall x\ A \vdash B}\ \forall\text{-left}}{\Gamma, \forall x\ A \vdash B}\ \text{contraction}$$

general cut

Derivable rule

$$\dfrac{\Gamma, \forall x\ A, (t/x)A \vdash B}{\Gamma, \forall x\ A \vdash B}\ \text{contr-}\forall\text{-left}$$

Kleene style sequent calculus

Attempting to saturate Kleene style sequent calculus

Still two non-introduction rules

$$\frac{\Gamma, A \Rightarrow B \vdash A \quad \Gamma, B \vdash G}{\Gamma, A \Rightarrow B \vdash G} \text{ contr-}\Rightarrow\text{-left}$$

$$\frac{\Gamma, \forall x\ A, (t/x)A \vdash G}{\Gamma, \forall x\ A \vdash G} \text{ contr-}\forall\text{-left}$$

$$\frac{\dfrac{\dfrac{\dots}{\Gamma, (C \wedge D) \Rightarrow B \vdash C} \qquad \dfrac{\dots}{\Gamma, (C \wedge D) \Rightarrow B \vdash D}}{\Gamma, (C \wedge D) \Rightarrow B \vdash C \wedge D} \qquad \dfrac{\dots}{\Gamma, B \vdash G}}{\Gamma, (C \wedge D) \Rightarrow B \vdash G}$$

general cut

Derivable rule

$$\frac{\Gamma, (C \wedge D) \Rightarrow B \vdash C \qquad \Gamma, (C \wedge D) \Rightarrow B \vdash D \qquad \Gamma, B \vdash G}{\Gamma, (C \wedge D) \Rightarrow B \vdash G}$$

Simplified to

$$\frac{\Gamma, C \Rightarrow B \vdash C \qquad \Gamma, D \Rightarrow B \vdash D \qquad \Gamma, B \vdash G}{\Gamma, (C \wedge D) \Rightarrow B \vdash G}$$

Vorob'ev-Hudelmaier-Dyckhoff-Negri style sequent calculus

A hierarchy of calculi

?
Vorob'ev-Hudelmaier-Dyckhoff-Negri style Sequent calculus
Kleene style Sequent calculus
Gentzen style Sequent calculus
Natural deduction

Decidability

When saturation succeeds: decidability

Here it cannot succeed (undecidable)
But each step proved the decidability of a larger fragment

Fragments

Natural Deduction and Gentzen style sequent calculus: fragment where no connective or quantifier has a negative occurrence

Kleene: fragment where the implication and the universal quantifier have no negative occurrences

Vorob'ev-Hudelmaier-Dyckhoff-Negri: fragment containing all connectives, shallow universal and existential quantifiers, and negative existential quantifiers (contains the prenex fragment that contains the propositional fragment)