Geodesics in a discrete-time discrete-space frame

Gilles Dowek

Joint work with Pablo Arrighi

How can we formulate physical laws in a discrete-time discrete-space frame ?

The example of the notion of geodesic

I. From Gandy's hypotheses to digital physics

Gandy's hypotheses

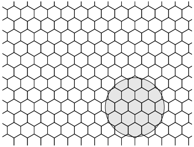
Homogeneity of time and space

Bounded velocity of information

Bounded density of information

If Gandy's hypotheses are verified...

...any system (can be simulated by | is) a cellular automaton



- Each cell has a finite state space (bounded density)
- State of a cell depends on state of a finite number of cells at previous time step (bounded velocity)
- Local evolution function the same everywhere and everywhen (homogeneity)

Bounded density of information

The amount of information that can be stored in a region of space of radius R is bounded

Robin Gandy (1980): the three hypotheses together imply the physical Church-Turing thesis (any physical system (can be simulated by | is) a cellular automaton)

Jacob Bekenstein (1981): $I_{max} = \frac{1}{4\ln(2)} \frac{c^3}{\hbar \mathcal{G}} 4\pi R^2$

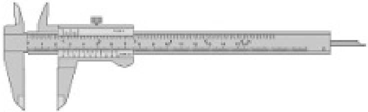
R^2 and not R^3

Bekenstein's constant ($1.4\ 10^{69}\ \mathrm{m}^{-2}$)

Real numbers

An infinite sequence of bits: the digits of a real number

encoded in the distance between the jaws of a



But hundredth digit of magnitude: no physical meaning

Digital physics

A theoretical result: any system (can be simulated by | is) a cellular automaton (computable universe, Galileo's thesis...)

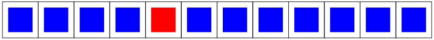
But also: model phenomena with ~~real numbers and differential equations~~ cellular automata

Chapter by chapter
This talk: gravitation

Motion in a cellular automaton

States: $\{q, \ldots, \}$

All cells are quiescent (q) except one: the particle
E.g.



Evolution rules preserve this invariant

II. A digression: what is the Planck constant the magnitude of?

c

Not just a constant in some equation: the speed of something
$\hbar$: the action of what?

No name: ~~The Rømer constant~~, ~~The Einstein constant~~: the speed of light
The Planck constant

Just one c
but h, $\hbar$...

Units

Often unit system such that $c = 1$, $\mathcal{G} = 1$, $\hbar = 1$
A unique unit system for distance, time, mass

Here, just $c = 1$ and $\mathcal{G} = 1$, keep distances in m
Time and mass are distances (everything in m)

t in s $\longrightarrow$ ct in m
m in kg $\longrightarrow$ $(\mathcal{G}/c^2)m$ in m half of Schwarzschild's radius

c : m s^{-1} and $\mathcal{G}/c^2$: m kg^{-1} unit conversion constants (like 0.0254 m in^{-1})

Units

velocity:	m s^{-1}	$\longrightarrow$	m^0 (scalar)
momentum:	kg m s^{-1}	$\longrightarrow$	m
acceleration:	m s^{-2}	$\longrightarrow$	m^{-1}
force:	kg m s^{-2}	$\longrightarrow$	m^0
energy:	kg m^2 s^{-2}	$\longrightarrow$	m
action:	kg m^2 s^{-1}	$\longrightarrow$	m^2

The Planck constant

$\hbar = 1.054\ 10^{-34}$ m^2 kg s^{-1} homogeneous to m^2

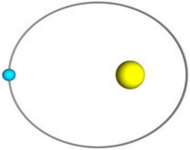
The Planck constant in m^2: $a_P = \hbar(\mathcal{G}/c^2)(1/c) = 2.61\ 10^{-70}$ m^2
The Planck area

$$I_{max} = \frac{1}{4\ln(2)}\frac{c^3}{\hbar\mathcal{G}}4\pi R^2 = \frac{1}{4\ln(2)a_P}4\pi R^2$$

$4\ln(2)a_P = 7.2\ 10^{-70}$ m^2 is the area of one bit

Not h, $\hbar$, or a_P: $4\ln(2)a_P$: the area of one bit

III. Geodesics



Forces and metrics

General relativity: “force” replaced by “metric”
Special relativity: when changing reference frame $x^2 + y^2$ not preserved, t^2 not preserved
But “distance” $t^2 - x^2 - y^2$ preserved

$$\begin{pmatrix} t & x & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \end{pmatrix}$$

Metric tensor

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Trajectories: geodesics: spacetime straight lines: rectilinear motion

Geodesics of

$$\begin{pmatrix} 1-\frac{2m}{r} & 0 & 0 \\ 0 & -\frac{x^2}{r(r-2m)}-\frac{y^2}{r^2} & -\frac{2mxy}{r^2(r-2m)} \\ 0 & -\frac{2mxy}{r^2(r-2m)} & -\frac{x^2}{r^2}-\frac{y^2}{r(r-2m)} \end{pmatrix}$$

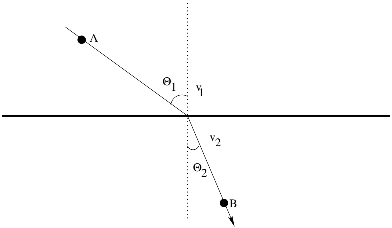
Bounded density of information

Coordinates in $\Delta\mathbb{Z}$

How can we formulate the laws of physics?

The laws of physics

Derivative $\longrightarrow$ finite differences
Formulations of the form $a = 0$ vs. u minimum / maximum
E.g., Snell-Descartes law for refraction :



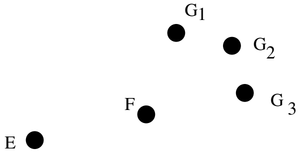
$$\sin\theta_1/v_1 = \sin\theta_2/v_2$$

vs. from Fermat's principle of least time
Variational principles more robust than equations

What is a geodesic in a discrete spacetime?

Geodesic: shortest and straightest path between two points

w function such that $w(E, F, G)$ measures how E, F, G deviates from going straight ahead



G chosen to minimize (external) angle in F

Discrete-time continuous-space spacetime:

$E_0, E_1, E_2, E_3, \ldots$ geodesic if for all i, $w(E_{i-1}, E_i, E_{i+1}) = 0$

From the continuous to the discrete

Too strong in discrete space

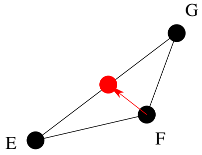
$w(E_{i-1}, E_i, E_{i+1})$ minimum for local variations of E_{i+1}: for any spatial neighbor G of E_{i+1}

$$w(E_{i-1}, E_i, G) \geq w(E_{i-1}, E_i, E_{i+1})$$

Algorithm:
E_{i-1}, E_i given
Pick random E_{i+1}
If $w(E_{i-1}, E_i, E_{i+1})$ not local minimum replace E_{i+1} by a better neighbor and iterate

Metrics and deviations

$$l(E, F, G) = d(E, F) + d(F, G)$$



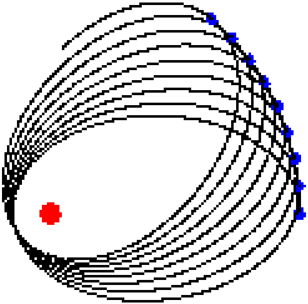
In the continuous case

$$w(E, \langle t, x, y \rangle, G) =$$
$$(\partial_t l(E, \langle t, x, y \rangle, G))^2 + (\partial_x l(E, \langle t, x, y \rangle, G))^2 + (\partial_y l(E, \langle t, x, y \rangle, G))^2$$

In the discrete case: replace derivatives with finite differences

Schwarzschild's metric $\longrightarrow$ distance $\longrightarrow$ deviation function $\longrightarrow$ cellular automaton to compute geodesics

Experimental results



Almost only integers in the program

A fake planet very close to the Sun (maximize relativistic effects)

Perihelion shift 6.27°, expected 6.17°

No (t yet) similar results for Mercury: very small shift

Theoretical results

Continuous trajectories when step $\to 0$

Unlike other discrete formulations of General relativity

Falsifiability

"The world is a cellular automaton": a falsifiable hypothesis

Not all motions possible

$x = t^2$ contradicts bounded velocity and bounded density

$x = \sqrt{t+1}$ would not contradict bounded velocity but would still contradict bounded density

But still a lot to be done

Position: on the grid
momentum / velocity: internal state
metric tensor: internal state

Momentum: in a fixed coordinate system

Metric tensor: internal state
The particle knows the position of the star
Calls for more natural automata: e.g. messenger particles

Mercury?

Drop the rigid grid: causal graph dynamics